# Solution of Homework #8 (ASEN5022, Spring 2005)

8.1 **Analysis and Design of a Shock Absorber by Two-DOF Model**

Consider the system illustrated in Fig. 22.6 and perform the following.

8.1.1 Obtain the governing equations of motion for the proposed model.

8.1.2 Obtain the frequency response function, $X_1/x_g$.

8.1.3 Carry out non-dimensionalization of the frequency response or transfer function.

8.1.4 Initially by setting $K_1/M_1 = K_2/M_2$, plot the FRF vs. the driving frequency ($\omega$), which is a road profile in this instance by varying the system damping. Does the model exhibit invariant points? If so, describe how you may utilize them for a good suspension system.

8.1.5 Carry out, either analytically or relying on simulations, an optimum set of model paramters in terms of mass ratio ($\mu$) and damping ratio $\zeta$.

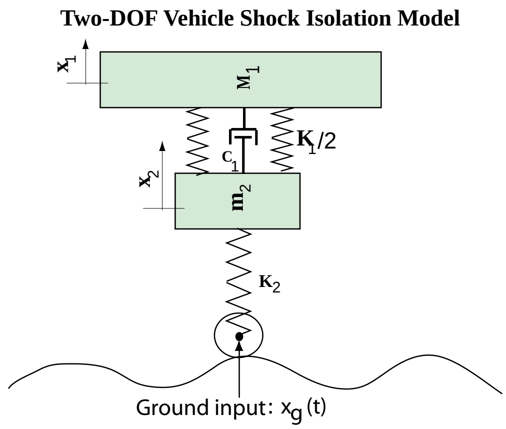


**Figure 1 Two-DOF Model for a Suspension System**

The governing equations are given by

$$\begin{aligned} &M_1\ddot{x}_1 + c(\dot{x}_1 - \dot{x}_2) + K_1(x_1 - x_2) = 0 \\ &M_2\ddot{x}_2 + c(\dot{x}_2 - \dot{x}_1) - K_1(x_1 - x_2) + K_2 x_2 = K_2\bar{x}_g \end{aligned} \tag{1}$$

Assuming that the ground input $x_g$ can be characterized by

$$x_g(t) = X_g e^{j\omega t} \tag{2}$$

equation(1) can be expressed in frequency domain as

$$\begin{aligned} &(K_1 + jc\omega - \omega^2 M_1)X_1 - (K_2 + jc\omega)X_2 = 0 \\ &-(K_2 + jc\omega)X_1 + (K_1 + K_2 + jc\omega - \omega^2 M_2)X_2 = X_g \end{aligned} \tag{3}$$

Let's introduce the following nondimensional parameters:

$$\Omega_1^2 = K_1/M_1,\ \omega_2^2 = K_2/M_2,\ s = \omega/\Omega_1,\ c_c = 2M_1\Omega_1,\ \ f = \Omega_1/\omega_2,\ \mu = M_1/M_2 > 1 \tag{4}$$

Note that the definitions of $f$ and $\mu$ are reversed from the den Hartog shock absorber model.

The nondimensionalized transfer function is shown to be

$$
\begin{aligned}
|\frac{X_1(s)}{X_g}| &= [\frac{(\frac{c}{c_c})^2 A + B}{(\frac{c}{c_c})^2 C + D}]^{\frac{1}{2}} \\
A &= 4s^2, \quad B = 1 \\
C &= 4s^2[1-(1+\mu) f^2 s^2]^2 \\
D &= [s^2(f^2 s^2 - 1) + (1-(1+\mu) f^2 s^2)]^2
\end{aligned} \tag{5}
$$

As in the case of den Hartog absorber, we seek two invariant points by the following relation:

$$
\frac{A}{C} = \frac{B}{D} \tag{6}
$$

which gives the following equation for the two invariant points:

$$
s^2(f^2 s^2 - 1) + (1-(1+\mu) f^2 s^2) = \pm[1-(1+\mu) f^2 s^2] \tag{7}
$$

First, by taking the plus sign, we find the two invariant points:

$$
\begin{aligned}
&(s = \omega/\Omega_1 = 0, \quad |\frac{X_1(s)}{X_g}| = 1\ ) \\
&(s = \omega/\Omega_1 = \frac{1}{f}, \quad |\frac{X_1(s)}{X_g}| = \frac{1}{\mu}\ )
\end{aligned} \tag{8}
$$

Second, with the minus sign, we find the following equation:

$$
as^4 - bs^2 + c = 0, \quad (a = f^2, \quad b = [1 + 2(1+\mu) f^2], \quad c = 2) \tag{9}
$$

which has two positive real roots, $s_1$ and $s_2$, resulting in the following third and fourth invariant points:

$$
\begin{aligned}
&P = (s_1, |\frac{X_1(s)}{X_g}| = \frac{1}{1-(1+\mu) f^2 s_1^2}), \quad Q = (s_2, |\frac{X_1(s)}{X_g}| = \frac{1}{(1+\mu) f^2 s_2^2 - 1}) \\
&s_1^2 = \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}, \quad s_2^2 = \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}
\end{aligned} \tag{10}
$$

Let's now examine which of the two magnitudes (the vertical axis of P and Q points) is bigger. To this end, let's rearrange (9) as

$$
s^2(f^2 s^2 - 1) - 2[(1+\mu) f^2 s^2 - 1] = 0 \tag{11}
$$

Now, for the larger root $s_2$ we have asymptotically

$$
\begin{aligned}
&s_1^2\ (f^2 s_2^2 - 1) > 0 \\
&\Downarrow \\
&\boxed{\frac{1}{f^2} < s_2^2}
\end{aligned} \tag{12}
$$

For the smaller root $s_1$, we have

$$
\begin{aligned}
&[(1+\mu) f^2 s_1^2 - 1] < 0 \\
&\Downarrow \\
&\boxed{\frac{1}{(1+\mu) f^2} < s_1^2}
\end{aligned} \tag{13}
$$

The above inequalities enable us to assess the bounds of magnitude of $|\frac{X_1(s)}{X_g}|$ at the two invariant points:

$$|\frac{X_1(s)}{X_g}|_{s_1} = (\frac{1}{1-(1+\mu)f^2 s_1^2})|_{(s_1^2=\frac{1}{(1+\mu)f^2})} > 1$$

$$|\frac{X_1(s)}{X_g}|_{s_2} = (\frac{1}{(1+\mu)f^2 s_1^2 - 1})|_{(s_2^2=\frac{1}{f^2})} < \frac{1}{\mu} < 1 \tag{14}$$

This clearly shows that one needs to minimize the amplitude at $s_1$. Observe also that at $s_2$ the dynamic amplitude is less than unity, viz., less than the static amplitude.

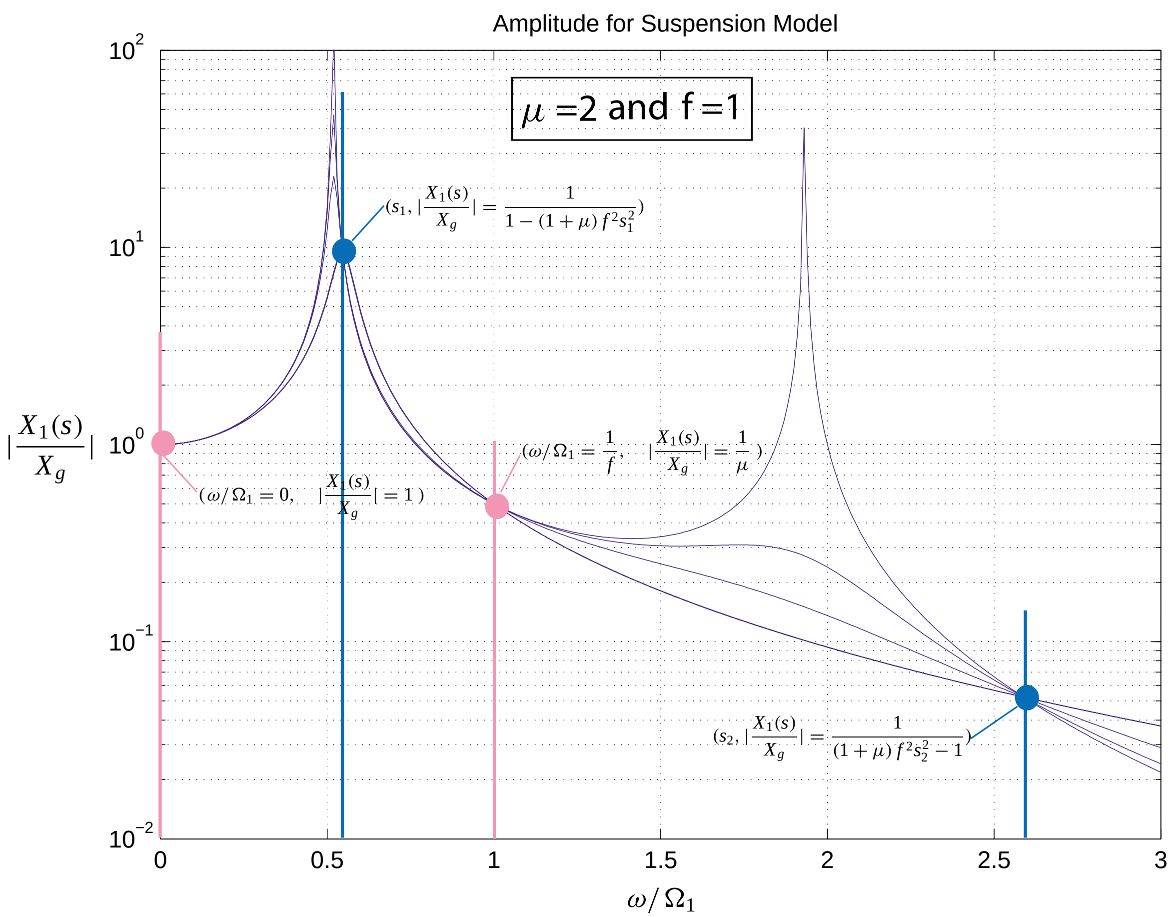


**Figure 2. Four Invariant Points in Two-DOF Model for a Suspension System**

The first two invariant points given by (8) is marked with pink circles, and the third and fourth ones as given by (10) are marked in blue in Figure 2. Note that the magnitude at the invariant point $(s_1, |\frac{X_1(s)}{X_g}| = \frac{1}{1-(1+\mu)f^2 s_1^2})$ is clearly the one should be minimized.

The damping ratio tht would make $|\frac{X_1(s)}{X_g}|_{s_1}$ to be flat is coded by modifying the Mathematica code distributed as follows:

```
(* compute the optimum damping ratio for  2-
    dof suspension    system  *)
```

```
Clear[f2, h2,pi,pr,mu,s2]
(*  x = s^2 in the lecture notes*)
A1 =4*x
B1 = 1
C1 =4*x *(1 -(1+mu)*x*f2 )^2
D1 = ( x *(x*f2-1)+( 1-(1+mu)*x*f2) )^2
a4 = f2
 b4 =  (1+ 2*(1+mu)*f2)
c4 = 2
pr = b4/(2*a4)
 pi  =Sqrt[ b4^2 - 4*a4*c4]/(2*a4)
s2 = pr - pi
A1d = D[A1,x]
B1d =D[B1,x]
C1d  =D[C1,x]
D1d  =D[D1,x]

(*        parameter set*)
h2 =1/(1 - (1+mu)*f2*s2)
A1d  =Simplify[A1d /.x\[Rule]s2]
B1d  =Simplify[B1d /.x\[Rule]s2]
C1d  =Simplify[C1d /.x\[Rule]s2]
D1d  =Simplify[D1d /.x\[Rule]s2]

nomin =Simplify[ (B1d  - h2^2*D1d )]
denom = Simplify[(h2^2*C1d  - A1d )]
damping = Simplify[nomin/denom]
Numdamping =Simplify[damping /. f2\[Rule]1]
finaldamping=Simplify[Numdamping /.mu\[Rule]2]
N[finaldamping]
ClearAll
```

The results is obtained as:

$$(\frac{c}{c_c})^2 = \frac{(1+\mu)f^2 - \sqrt{(1+2(1+\mu)f^2)^2 - 8f^2}}{2(1+\mu)[-1-2(1+\mu)f^2 + \sqrt{(1+2(1+\mu)f^2)^2 - 8f^2}]}$$

$$\Downarrow$$

$$(\frac{c}{c_c})^2 = 0.95026 \quad \Rightarrow |\frac{X_1(s)}{X_g}|_{s_1} = 9.55234 \quad \text{with} \quad (f = \frac{\Omega_1}{\omega_2} = 1, \quad \mu = \frac{M_1}{M_2} = 2) \tag{15}$$

$$(\frac{c}{c_c})^2 = 0.49856 \quad \Rightarrow |\frac{X_1(s)}{X_g}|_{s_1} = 2.9216 \quad \text{with} \quad (f = \frac{\Omega_1}{\omega_2} = 0.25, \quad \mu = \frac{M_1}{M_2} = 2)$$

$$(\frac{c}{c_c})^2 = 0.610448 \quad \Rightarrow |\frac{X_1(s)}{X_g}|_{s_1} = 1.90803 \quad \text{with} \quad (f = \frac{\Omega_1}{\omega_2} = 0.1, \quad \mu = \frac{M_1}{M_2} = 3)$$

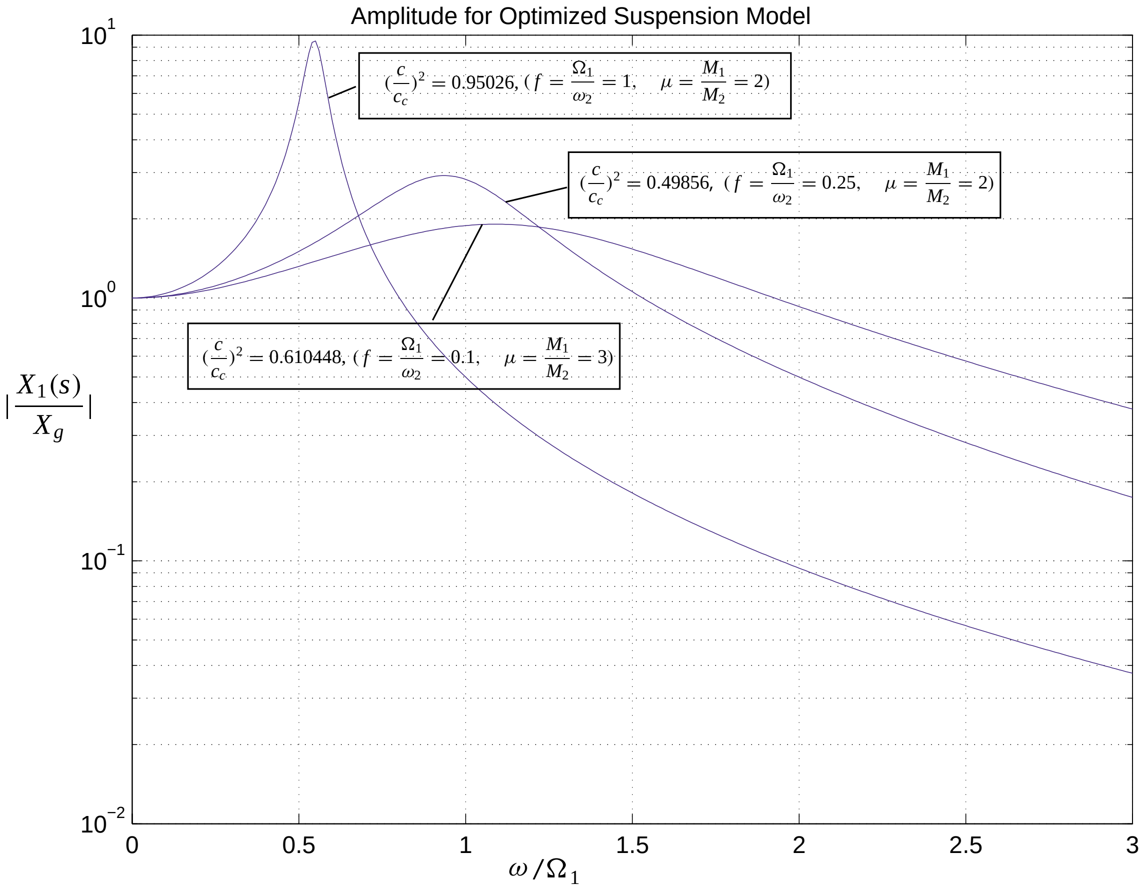


**Figure 3. Four Invariant Points in Two-DOF Model for a Suspension System**

The smaller the frequency ratio, the smaller the maximum amplification factor is, thus yield a better suspension system. Also, note that the mass with the suspension mass is smaller than the vehicle mass, indicating that the two-DOF model leads to somewhat realistic design parameters.

8.2 **Analysis and Design of a Resonator**

8.2.1 Carry out the double-beam resonator performance by varying the offset, $1/16 \le e \le 1/2$, hence varying the offset factor $a$. Which offset gives a minimum frequency separation between the first peak and the second?

8.2.2 Now vary the dimensions of the link beam, $(L_s, b_s)$ to see if the resulting FRF can be improved. Keep in mind this will change $k_s$ and $m_s$, hence changing $M_1 = M_2$ as well as $K_{12}$. What is your finding?

**I am skipping the above two sub-problems as they are straightforward**

*8.2.3 Now, imagine that one could come up with a resonator that can be modeled as the system shown in Fig. 22.1. Would you be able to use the model and analysis procedures discussed in Lecture 22 to maximize the peak response at the invariant points? Show your answer by your analysis.*

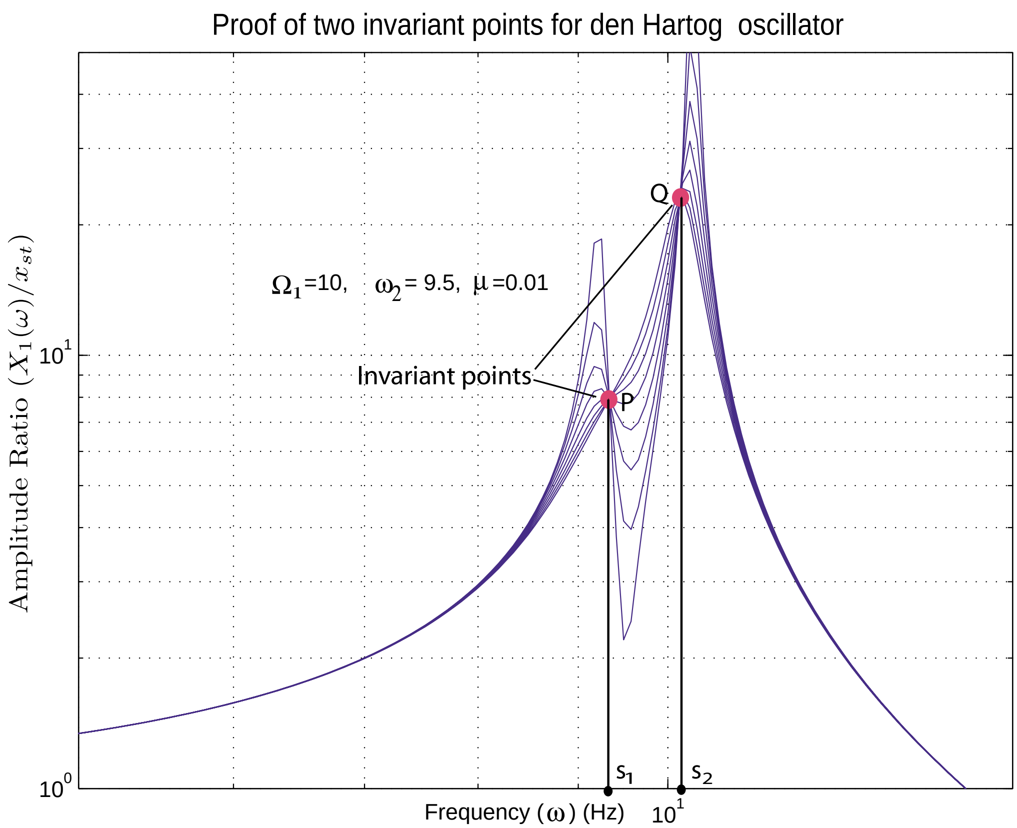


**Figure 4. Invariant Points in Den Hartog Two-DOF Model**

Recalling the two invariant points in the den Hartog model, it may be possible to maximize the magnitude at $P$ point. This may be realized by pushing the $s_1$ as close as possible to the first resonant peak and concurrently to put more energy into the first mode, and at the same time bringing the two points closer.

*8.2.4 Repeat Problem 8.2.3 by employing the model shown in Fig. 22.6 where $x_g$ now acts as the substrate wave motion.*

By examining the suspension system model as shown in Figure 2, except at $s_1$ point, the magnitude at remaining two invariant points ($s = 1/f$ and $s_2$)are much lower than unity. Note that the invariant point ($s = 0$) corresponds to the zero frequency and as such it can't be used for resonance design! Therefore, we conclude the two-DOF suspension model we have studies cannot be used for designing a resonator.