Solution of Homework #5 (ASEN5022, Spring 2005)

5.1 Vibrations of Beam Bending with Uncertain Boundary Conditions

Vibration experiments were conducted on a beam in a laboratory setting and yielded the following results:

- The measured first mode was: $\beta L = 3.75$.
- The peak amplitude of its mode shape occurred at the beam span $x = 0.54L$, measured from the left end.

Determine the boundary spring constants by modifying Matlab code provided.

Hint: The first modes of classical boundary cases are given by:

$$\begin{aligned} \textit{Simply supported at both ends:} &\quad \beta L = \pi \\ \textit{One end fixed and the other free (cantlievered beam):} &\quad \beta L = 1.875 \\ \textit{Both ends fixed:} &\quad \beta L = 4.730 \\ \textit{One end fixed and other end simply supported :} &\quad \beta L = 3.937 \end{aligned}$$

Solution: Since the frequency ($\beta L = 3.75$) is higher than the case of simply supported at both ends ($\beta L = 3.75$) and lower than the fixed at one end and simply surpported at the other end ($\beta L = 3.937$), we conclude that the observed case would fall into partially moment-resisting ends.

To this end, let's recall the characteristicc equation (27) of the lecture notes (Lecture 13: 01 March) for this case:

Simply Supported and Fixed-Fixed Beams $(w(0) = w(L) = 0$ or $k_{w1} = k_{w2} \to \infty)$

$$\begin{bmatrix} 0 & -1 & 0 & -1 \\ -\bar{k}_{\theta 1} & -\bar{\beta} & -\bar{k}_{\theta 1} & \bar{\beta} \\ -\sin\bar{\beta} & -\cos\bar{\beta} & -\sinh\bar{\beta} & -\cosh\bar{\beta} \\ \begin{matrix}\bar{\beta}\sin\bar{\beta} \\ -\bar{k}_{\theta 2}\cos\bar{\beta}\end{matrix} & \begin{matrix}\bar{\beta}\ \cos\bar{\beta} \\ +\bar{k}_{\theta 2}\sin\bar{\beta}\end{matrix} & \begin{matrix}-\bar{\beta}\ \sinh\bar{\beta} \\ -\bar{k}_{\theta 2}\cosh\bar{\beta}\end{matrix} & \begin{matrix}-\bar{\beta}\ \cosh\bar{\beta} \\ -\bar{k}_{\theta 2}\sinh\bar{\beta}\end{matrix} \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix} = 0 \tag{1}$$

which yields the following characteristic equation:

$$\begin{aligned} 2\bar{\beta}^2 \sin\bar{\beta}\sinh\bar{\beta} + (\bar{k}_{\theta 1} + \bar{k}_{\theta 2})\ \bar{\beta}\ (\sin\bar{\beta}\cosh\bar{\beta} - \cos\bar{\beta}\sinh\bar{\beta}) \\ + \bar{k}_{\theta 1}\ \bar{k}_{\theta 2}\ (1 - \cos\bar{\beta}\cosh\bar{\beta}) = 0 \end{aligned} \tag{2}$$

In addition, at $x = 0.54$ the slope of the mode shape must be zero:

$$W_{x\, x_m = 0/54} = \{(c_1\cos\beta x_m - c_2\sin\beta x_m + c_3\cosh\beta x_m + c_4\sinh\beta x_m)\}|_{x_m = 0.54} = 0 \tag{3}$$

By solving for (c_1, c_2, c_3) in terms of c_4 from (1), we obtain (Mathematica here!):

$$\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \frac{1}{(\sinh\bar{\beta} - \sin\bar{\beta})\ k_{\theta 1}} \begin{Bmatrix} 2\bar{\beta}\ \sinh\bar{\beta} - (\cos\bar{\beta} - \cosh\bar{\beta})\ k_{\theta 1} \\ -(\sinh\bar{\beta} - \sin\bar{\beta})\ k_{\theta 1} \\ -2\bar{\beta}\ \sin\bar{\beta} + (\cos\beta - \cosh\bar{\beta})\ k_{\theta 1} \end{Bmatrix} c_4 \tag{4}$$

If $c_4 = (\sinh\bar{\beta} - \sin\bar{\beta})\ k_{\theta 1}$, then we obtain:

$$\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix} = \begin{Bmatrix} 2\bar{\beta}\ \sinh\bar{\beta} - (\cos\bar{\beta} - \cosh\bar{\beta})\ k_{\theta 1} \\ -(\sinh\bar{\beta} - \sin\bar{\beta})\ k_{\theta 1} \\ -2\bar{\beta}\ \sin\bar{\beta} + (\cos\beta - \cosh\bar{\beta})\ k_{\theta 1} \\ (\sinh\bar{\beta} - \sin\bar{\beta})\ k_{\theta 1} \end{Bmatrix} \tag{5}$$

Substituting this into (3) and after some arrangements, $k_{\theta 1}$ is found as

$$k_{\theta 1} = \frac{(\cos \beta x_m) * (2\bar{\beta} \sinh \bar{\beta}) - \cosh \beta x_m * (2\bar{\beta} \sin \bar{\beta})}{(\cosh \beta x_m - \cos \beta x_m) * (\cosh \bar{\beta} - \cos \bar{\beta}) + (\sin \beta x_m + \sinh \beta x_m) * (\sin \bar{\beta} - \sinh \bar{\beta})} \tag{6}$$

Once ($k_{\theta 1}$) is computed, then ($\theta_{\theta 2}$) can be computed from (2):

$$k_{\theta 2} = \frac{2(\bar{\beta})^2 \sin \bar{\beta} \sinh \bar{\beta} + k_{\theta 1}\bar{\beta} * (\sin \bar{\beta} \cosh \bar{\beta} - \cos \bar{\beta} \sinh \bar{\beta})}{-k_{\theta 1} * (1 - \cos \bar{\beta} \cosh \bar{\beta}) + \bar{\beta} \cos \bar{\beta} \sinh \bar{\beta}} \tag{7}$$

```
**************Mathematica code to obtain various expressions**************************
(* Frequency Analysis of a partially constrained beam *)
Cbeam = {  {0, -1,  0, -1},
                        {-k1, -x, -k1,  x},
                       {-ss, -cc, -sh, -ch},
       { x*ss - k2*cc,   x*cc + k2*ss, - x*sh - k2*ch, -x*ch - k2*sh} } ;
Print["Cbeam=", Cbeam // MatrixForm];
Cbeamdet = Simplify[Det[Cbeam]]; Print["
        Characteristic Function = ", Cbeamdet];

B1 = {Take[Cbeam[[1]], 3], Take[Cbeam[[2]], 3], Take[Cbeam[[3]], 3] };

Print["B1=", B1 // MatrixForm];
Cbeamt  = Transpose[Cbeam];
 R1 = -Take[Cbeamt[[4]], 3] ;

Print["R1=", R1 // MatrixForm];

Mshape  = Simplify[LinearSolve[B1, R1]];
Print[" (c_1, c_2, c_3) = ", Mshape // MatrixForm];
AppendTo[Mshape, 1];
Print["(c_1, c_2, c_3, c_4) = ", Mshape // MatrixForm];
```

```
C4 = Det[B1]; Mshape = Simplify[C4*Mshape];
Print[" Normalized (c_1, c_2, c_3, c_4) =  ", Mshape // MatrixForm];
```

************Matlab code to compute the two spring constants *******

```
%
%Analytical solution of Homework 5.1
%

betaL = 3.75;  L = 1.00;

x = betaL;

xL = 0.54*betaL;

double(x);
double(xL);

c  = cos(x);   s = sin(x);  ch = cosh(x);   sh = sinh(x);
cx = cos(xL); sx = sin(xL);  chx = cosh(xL); shx = sinh(xL);

nom1 = -cx*(2*x*sh) + chx*(2*x*s);
denom1 = (chx-cx)*(c -ch)+(shx+sx)*(sh-s);

% k1 is   k_(theta 1)
k1 = nom1/denom1;

% compute k_theta 2

nom2 = 2.0*x^2*s*sh + k1*x*(s*ch-c*sh);
```

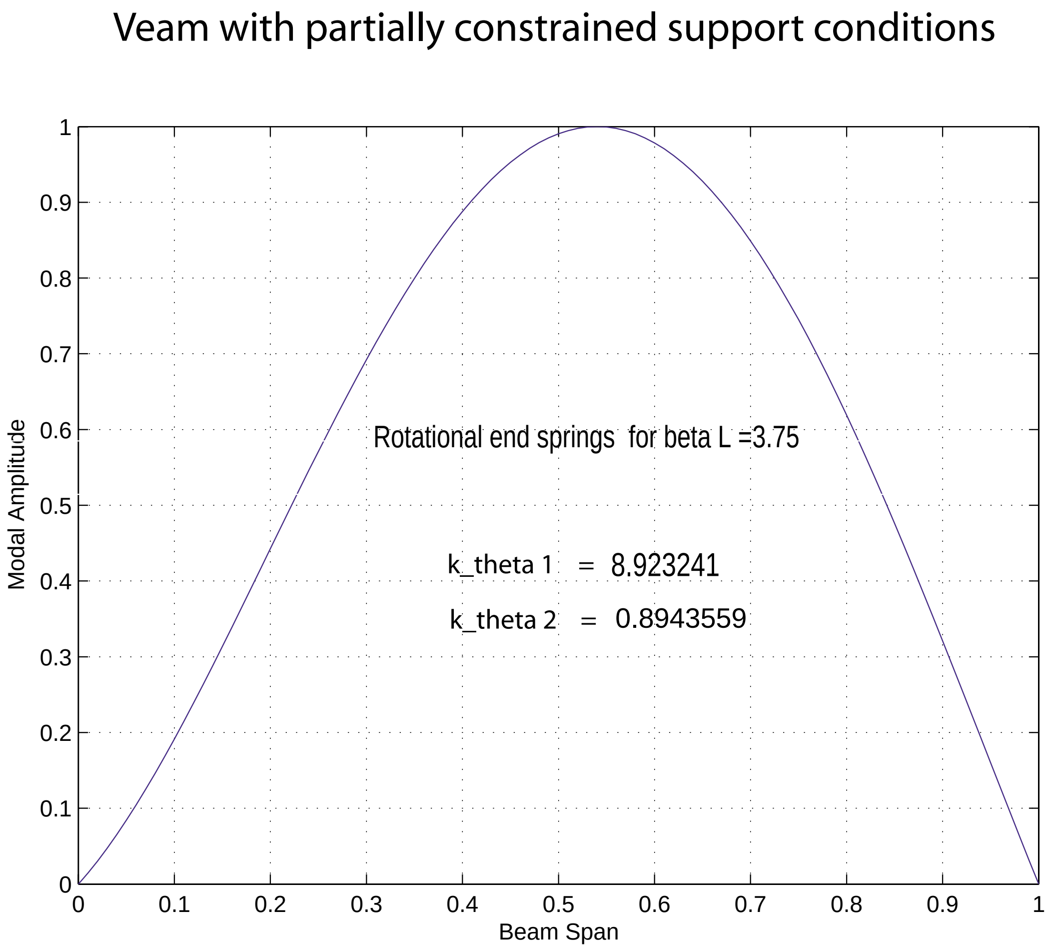
```
denom2 = k1*(1.0-c*ch) + x*(s*ch-c*sh);
k2 = -nom2/denom2;

% determinant (root accuracy) test

det_check = k2*k1*(1.0 -c*ch)    +( k1 + k1)*x*(s*ch - c*sh)  +2*x^2*s*sh;

disp([’The left-end rotational spring, k_theta 1  =  ’,  num2str(k1,7)]);
disp([’The right-end rotational spring, k_theta 2  =  ’,  num2str(k2,7)]);
disp([’Characteristic root accuracy test, det(A)  =   ’,  num2str(det_check,7)]);

% plot the mode shape

c1 =  2*x*sh    + (ch-c)*k1;
c2 = (s-sh)*k1;
c3 = -2*x*s -(ch-c)*k1;
c4 = (sh-s)*k1;

ell = 0:0.01:1;

W = c1*sin(x*ell) + c2*cos(x*ell) + c3*sinh(x*ell) + c4*cosh(x*ell);
W = W/ max((W));
figure(1);
plot(ell, W);
xlabel(’Beam Span’);
ylabel(’Modal Amplitude’);
legend([’   Rotational end springs’,  ’ k_theta1 =  ’, num2str(k1,7),  ...
’k_theta2 = ’, num2str(k2,7)]);
grid on;
```

****

The results are:

$$\boxed{\begin{aligned}\kappa_{\theta 1} &= 8.9232414\\ \kappa_{\theta 2} &= 0.8943559\end{aligned}} \tag{8}$$

These values together with $\kappa_{w1} = \kappa_{w2} = 10^8$ were used to confirm the βL value as well as the mode shape by running a general Matlab codehanded out. They indeed match the theoretical values.

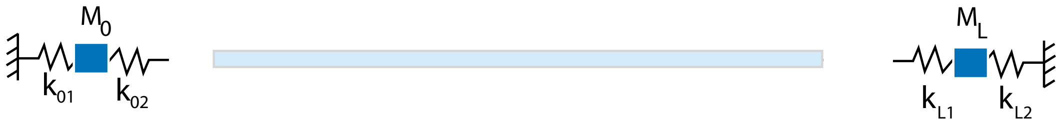


5.2 FEM Modeling of Vibration of Bars

Consider a bar whose ends are constrained by springs and masses shown below as discussed in the class. For your convenience, you may utilize the non-dimensional parameters given by

$$\kappa_{01} = k_{01}L/EA, \quad \kappa_{02} = k_{02}L/EA, \quad \kappa_{L1} = k_{L1}L/EA, \quad \kappa_{L2} = k_{L2}L/EA$$
$$\mu_0 = M_0/\rho L, \quad \mu_L = M_L/\rho L$$

Unassembled Model (hardware)



Unassembled Model (software)

$$(\mathbf{m}_0 \quad \mathbf{k}_0) \qquad (\mathbf{M}_c \quad \mathbf{K}_c) \qquad (\mathbf{m}_L \quad \mathbf{k}_L)$$

Figure for Problem 5.2

5.2.1 Perform an FEM vibration analysis by gradually increasing the number of bar elements for the case of *free-free bar*. This condition can be realized in your FEM code by setting all of the end masses and springs to be zero. Determine the number elements needed to obtain the FEM computed two lowest frequencies to be within 0.1 % of the continuum results.

5.2.2 Repeat Problem 5.2.1 with your choice of *soft support conditions* by selecting appropriate end masses and strpings.

5.2.3 Repeat Problem 5.2.1 with your choice of *hard support conditions* by selecting appropriate end masses and strpings.

Discuss your FEM analysis results as compared to the analytical results (the continuum solution). What have you learned?

Theoretical Basis: Same as the continuum bar model!

Kinetic energy of the contimuun bar:

$$T_{bar} = \tfrac{1}{2} \int_0^L m(x)\ u_t(x,t)^2 dx, \quad u_t(x,t) = \frac{\partial u(x,t)}{\partial t} \tag{9}$$

Potential energy of the continuum bar:

$$\begin{aligned} V_{bar} &= \tfrac{1}{2} \int_0^L \{\ EA(x)\ u_x(x,t)^2\ \}\ dx \\ u_x(x,t) &= \frac{\partial u(x,t)}{\partial x} \end{aligned} \tag{10}$$

External energy due to distributed applied force $\mathbf{f}(x,t)$:

$$\delta W_{bar} = \int_0^L \mathbf{f}(x,t)\ \delta u(x,t)\ dx \tag{11}$$

Kinetic energy of the discrete masses :

$$T_s = \tfrac{1}{2} M_0\ \dot{u}_0^2(t) + \tfrac{1}{2} M_L\ \dot{u}_L^2(t) \tag{12}$$

Potential energy of four springs:

$$\begin{aligned} V_s = \tfrac{1}{2}\{\ & k_{01}\ u_0^2(t) + k_{02}\ [u_0(t) - u(0,t)]^2 \\ & + k_{L2}\ u_L^2(t) + k_{L1}\ [u_L(t) - u(L,t)]^2 \end{aligned} \tag{13}$$

Hamilton's principle

$$\int_{t_1}^{t_2} [\delta T_{total} - \delta V_{bar} - \delta V_s + \delta W_{bar}\ ]\ dt = 0, \quad T_{total} = T_{bar} + T_s \tag{14}$$

Hamilton's principle for bar with end masses and end springs:

$$
\begin{aligned}
&\int_{t_1}^{t_2} \langle \int_0^L \{[-\rho(x)\ddot{u}(x,t) \; + EA(x)\, u_{xx}(x,t) + \mathbf{f}(t)]\; \delta u(x,t)\} dx \\
&- \{M_0\; \ddot{u}_0(t) + K_{01}\; u_0(t) + K_{02}\; [u_0(t) - u(L,t)]\}\; \delta u_0(t) \\
&- \{M_L\; \ddot{u}_L(t) + K_{L2}\; u_0(t) + K_{L1}\; [u_L(t) - u(L,t)]\}\; \delta u_L(t) \\
&- \{(EA(x)\; u_x(x,t) + K_{L1}\; [u(L,t) - u_L(t)]\;\}\; \delta u(x,t)|_{x=L} \\
&- \{-(EA(x)\; u_x(x,t) + K_{02}\; [u(0,t) - u_0(t)]\;\}\; \delta u(x,t)|_{x=0}\; \rangle \; dt = 0
\end{aligned}
\tag{15}
$$

The governing equation of motion for the continuum bar:

$$\boxed{\rho(x)\ddot{u}(x,t) = EA(x)\, u_{xx}(x,t) + \mathbf{f}(t)} \tag{16}$$

The governing equation of motion for the left-side spring mass:

$$\boxed{M_0\, \ddot{u}_0(t) + K_{01}\, u_0(t) + K_{02}\, [u_0(t) - u(0,t)] = 0} \tag{17}$$

The governing equation of motion for the right-side spring mass:

$$\boxed{M_L\, \ddot{u}_L(t) + K_{L2}\, u_0(t) + K_{L1}\, [u_L(t) - u(L,t)] = 0} \tag{18}$$

The boundary conditions:

$$\boxed{\begin{aligned} EA(L)\, u_x(L,t) + K_{L1}\, [u(L,t) - u_L(t)] \ &= 0 \\ -\, EA(0)\, u_x(0,t) + K_{02}\, [u(0,t) - u_0(t)] &= 0 \end{aligned}} \tag{19}$$

Derivation of Characteristic Equation

With $f(t) = 0$, the general solution of the continuum equation (16) assumes the form of

$$u(x,t) = \{C_1 \sin\beta x + C_2 \cos\beta x\}\, F(t), \qquad F(t) = e^{j\omega t}, \quad \beta = \omega\sqrt{\frac{\rho}{EA}} \tag{20}$$

Similary, the discrete displacements, $(u_0(t), u_L(t))$, for (17) and (18) are assumed to have their solutions in the form:

$$\begin{aligned} u_0(t) &= \bar{u}_0\, e^{j\omega t} \\ u_L(t) &= \bar{u}_L\, e^{j\omega t} \end{aligned} \tag{21}$$

Substituting the preceding two solution forms into the two discrete mass equations (17) and (18), and the two boundary conditions (19), we obtain the following characteristic equation

$$\begin{bmatrix} (K_{01}+K_{02}-\omega^2 M_0) & 0 & 0 & -K_{02} \\ 0 & (K_{L1}+K_{L2}-\omega^2 M_L) & -K_{L1}\sin\beta L & -K_{L1}\cos\beta L \\ 0 & -K_{L1} & (K_{L1}\sin\beta L + EA\beta\cos\beta L) & (K_{L1}\cos\beta L - EA\beta\sin\beta L) \\ -K_{02} & 0 & -EA\beta & K_{02} \end{bmatrix} \begin{Bmatrix} u_0(t) \\ u_L(t) \\ C_1(t) \\ C_2(t) \end{Bmatrix} = \mathbf{0} \tag{22}$$

For the ease of algebra, we multiply all the rows of the above equation by (L/EA) and simply by introducing the following non-dimensional parameters:

$$\begin{aligned} &\kappa_{01} = \frac{K_{01}L}{EA},\ \kappa_{02} = \frac{K_{02}L}{EA},\ \kappa_{L1} = \frac{K_{L1}L}{EA},\ \kappa_{L2} = \frac{K_{L2}L}{EA}, \\ &\omega^2 M_0 L/EA = (\beta^2\frac{EA}{\rho})(M_0 L/EA) = (\beta L)^2\frac{M_0}{\rho L} = (\beta L)^2\,\mu_0, \qquad \mu_0 = \frac{M_0}{\rho L} \\ &\omega^2 M_L L/EA = (\beta^2\frac{EA}{\rho})(M_L L/EA) = (\beta L)^2\frac{M_L}{\rho L} = (\beta L)^2\,\mu_L, \qquad \mu_L = \frac{M_L}{\rho L} \end{aligned} \tag{23}$$

Utilizing these nondimensionalization, (22) becomes:

$$\begin{bmatrix} (\kappa_{01}+\kappa_{02}-(\beta L)^2\mu_0) & 0 & 0 & -\kappa_{02} \\ 0 & (\kappa_{L1}+\kappa_{L2}-(\beta L)^2\mu_L) & -\kappa_{L1}\sin\beta L & -\kappa_{L1}\cos\beta L \\ 0 & -\kappa_{L1} & (\kappa_{L1}\sin\beta L+(\beta L)\cos\beta L) & (\kappa_{L1}\cos\beta L-(\beta L)\sin\beta L) \\ -\kappa_{02} & 0 & -(\beta L) & \kappa_{02} \end{bmatrix} \begin{Bmatrix} u_0(t) \\ u_L(t) \\ C_1(t) \\ C_2(t) \end{Bmatrix} = \mathbf{0} \tag{24}$$

The characteristic equation of the above equation (after relying on Mathematica!) is obtained by setting the determinant of its solution matrix to be zero:

$$\begin{aligned} \triangle = & ((-k_{L1}^2)\ ((x\ ((k_{01}+k_{02}-\mu_0\ x^2))\ \cos[x]+k_{02}\ ((k_{01}-\mu_0\ x^2))\ \sin[x])) \\ & +((k_{L1}+k_{L2}-mu_L\ x^2))\ ((k_{02}\ ((k_{01}-\mu_0\ x^2))\ ((x\ \cos[x]+k_{L1}\ \sin[x])) \\ & -x\ ((k_{01}+k_{02}-\mu_0\ x^2))\ (((-k_{L1})\ Cos[x]+x\ \sin[x]))))) \end{aligned} \tag{25}$$

Alternatively, one may eliminate $(u_0(t), u_L(t))$ (17) and (18) to obtain

$$\begin{aligned} u_0(t) &= (K_{01}+K_{02}-\omega^2 M_0)^{-1}\ K_{02}\ u(0,t) \\ u_L(t) &= (K_{L1}+K_{L2}-\omega^2 M_L)^{-1}\ K_{L1}\ u(L,t) \end{aligned} \tag{26}$$

Eliminating the discrete displacements, $(u_0(t), u_L(t))$, via the preceding relation in the two boundary conditions (19), we obtain:

$$\begin{aligned} EA(L)\ u_x(L,t)+K_{L1}\ [\ 1-(K_{L1}+K_{L2}-\omega^2 M_L)^{-1}\ K_{L1}]\ u(L,t)\ &=0 \\ -\ EA(0)\ u_x(0,t)+K_{02}\ [1-(K_{01}+K_{02}-\omega^2 M_0)^{-1}\ K_{02}\ ]u(0,t) &= 0 \end{aligned} \tag{27}$$

Rearranging the the above equation by multiplying $(K_{L1}+K_{L2}-\omega^2 M_L)$ and $(K_{01}+K_{02}-\omega^2 M_0)$ yields:

$$\begin{aligned} (K_{L1}+K_{L2}-\omega^2 M_L)\ EA(L)\ u_x(L,t)+K_{L1}\ [\ (K_{L1}+K_{L2}-\omega^2 M_L)-\ K_{L1}\}\ u(L,t)\ &=0 \\ -\ (K_{01}+K_{02}-\omega^2 M_0)\ EA(0)\ u_x(0,t)\ +K_{02}\ [(K_{01}+K_{02}-\omega^2 M_0)-K_{02}\ ]\ u(0,t) &= 0 \end{aligned} \tag{28}$$

Classical Boundary Conditions

(a) *Free-free bar:* This case is realized from the above equation by setting

$$M_0 = M_L = 0, \quad K_{01} = K_{02} = K_{L1} = K_{L2} = \epsilon << 1 \tag{29}$$

so that (29) reads:

$$\begin{aligned} 2\epsilon\ EA(L)\, u_x(L,t) + \epsilon^2\, u(L,t)\ &= 0 \\ -\,2\epsilon\ EA(0)\, u_x(0,t)\ + \epsilon^2\, u(0,t) &= 0 \end{aligned} \tag{30}$$

Dividing by 2ϵ yields:

$$\begin{aligned} EA(L)\, u_x(L,t) + \tfrac{1}{2}\epsilon\, u(L,t) \approx EA(L)\, u_x(L,t) &=\ 0 \\ -\ EA(0)\, u_x(0,t)\ + \tfrac{1}{2}\epsilon\, u(0,t) \approx -\, EA(0)\, u_x(0,t)\ &= 0 \end{aligned} \tag{31}$$

which agrees with the previous derivation.

(b) *Fixed-fixed bar:* This case is realized from equation by setting

$$M_0 = M_L = 0, \quad K_{01} = K_{02} = K_{L1} = K_{L2} = K_m >> 1 \tag{32}$$

so that (32) reads:

$$\begin{aligned} 2K_m\ EA(L)\, u_x(L,t) + K_m^2\, u(L,t)\ &= 0 \\ -\,2K_m\ EA(0)\, u_x(0,t)\ + K_m^2\, u(0,t) &= 0 \end{aligned} \tag{33}$$

which, when divided by $K_m >> 1$ becomes

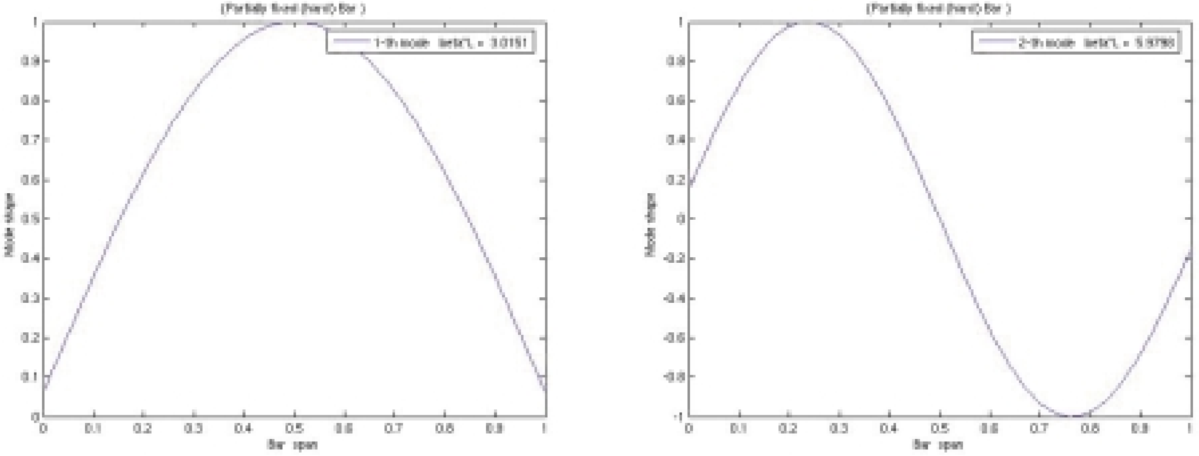
$$\begin{aligned} \frac{2}{K_m}\ EA(L)\, u_x(L,t) +\ u(L,t) \approx u(L,t) &= 0 \\ -\,\frac{2}{K_m}\ EA(0)\, u_x(0,t)\ +\ u(0,t) \approx u(0,t) &= 0 \end{aligned} \tag{34}$$

This also agrees with the classical fixed-fixed boundary conditions. For a mixed free-fixed boundary conditions, one may combineone of (33) and one of (34).

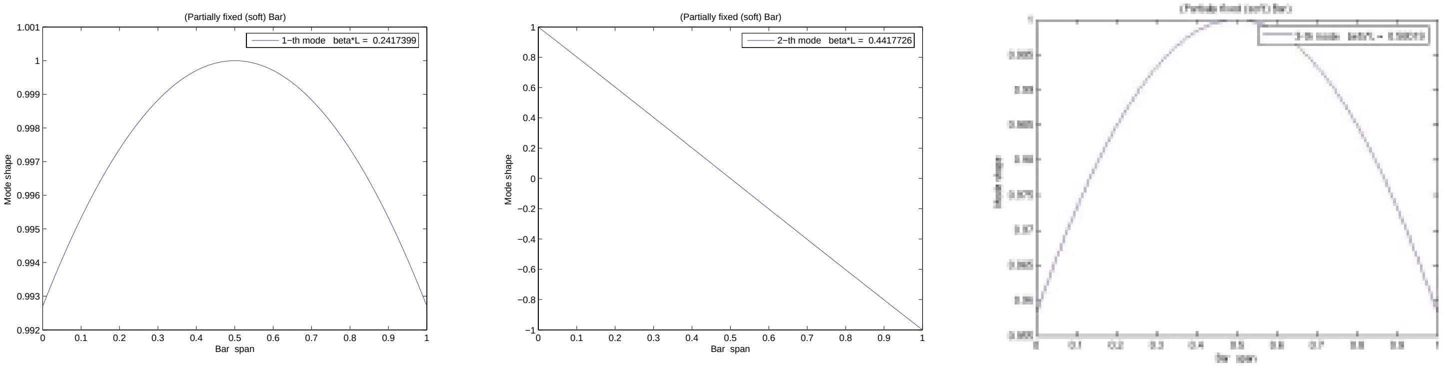
Your instructor has used a Malab-based code to solve the analytical eigenvalue problem with the following parameters:

```
Parameters used:   k=10^8; eps = 1.e-6; kh = 100;  ks = 0.1;
cases:      free-free  fixed-free  fix-fix  fixed-hard   fixed-soft
k01     =  [ eps            k          k         kh            ks ];
k02     =  [ eps            k          k         kh            ks ];
kL1     =  [ eps           eps         k         kh            ks ];
kL2     =  [ eps           eps         k         kh            ks ];
mu0     =  [ 0              0          0         1             1 ];
muL     =  [ 0              0          0         1             1 ];
```

The three cases of classical boundary conditions were first evaluated in order to check the validity of the general approach. Once assured of my code's vality, the above two cases, *hard and soft*, were analytically computed. The first two modes and mode shapes are plotted in the figures below.



Mode Shapes for Stiffly Constrained Ends



Mode Shapes for Softly Constrained Ends

Experiments with the FEM code (the instructor’s Matlab version) indicate that one needs about 80 elements for the FEM solution of the first two modes with the accuracy of 4 significant digits.