Solutions of Homework #4 (ASEN5022, Spring 2004)

Problem : A beam with uncertain boundary conditions.

A group of engineers conducted vibration tests on a bridge that needs to be retrofitted with reinforced structural elements in order to improve earthquake vulnerability. After a careful modal analysis based on their measured acceleration output data sets, they discovered that the fundamental frequency is equivalent to $\beta L = 4.25$, and the peak amplitude of the corresponding mode shape occurring at the beam span $x = 0.55L$. They have concluded that there is no discernable boundary inertia effects.

4.1 Formulate the continuum equations of motion for this beam, complete with the uncertain boundary conditions.

From Lecture Notes 13, we find (There were two terms missing in equation (11) of that lecture notes, viz., k_{w1} and k_{w2} terms):

$$\begin{aligned}
&-\int_{t_1}^{t_2} \{-[EI\ w(L,t)_{xx}]_x + k_{w2}\ w(L,t) - Q_2\ \}\ \delta w(L,t)dt \\
&+\int_{t_1}^{t_2} \{-[EI\ w(0,t)_{xx}]_x - k_{w1}\ w(0,t) + Q_1\ \}\ \delta w(0,t)dt \\
&-\int_{t_1}^{t_2} \{[EI\ w(L,t)_{xx}] + k_{\theta 2}\ w(L,t)_x\ - M_2\ \}\ \delta w(L,t)_x dt \\
&+\int_{t_1}^{t_2} \{[EI\ w(0,t)_{xx}] - k_{\theta 1}\ w(0,t)_x\ + M_1\ \}\ \delta w(0,t)_x dt \\
&-\int_{t_1}^{t_2}\int_0^L \{\ m(x) w(x,t)_{tt} + [EI\ w(x,t)_{xx}]_{xx} \\
&\qquad - f(x,t)\ \}\ \delta w(x,t)\ dxdt = 0
\end{aligned} \tag{1}$$

The above Hamilton's principle yields the governing equation of motion of the form

$$m(x)\ w(x,t)_{tt} + [EI\ w(x,t)_{xx}]_{xx} - f(x,t)\ =\ 0 \tag{2}$$

and the boundary conditions with $Q_1 = Q_2 = 0,\ M_1 = M_2 = 0$:

$$\begin{aligned}
\{-[EI\ w(L,t)_{xx}]_x + k_{w2}\ w(L,t)\ \}\ \delta w(L,t) = 0 \\
\{-[EI\ w(0,t)_{xx}]_x - k_{w1}\ w(0,t)\ \}\ \delta w(0,t) = 0 \\
\{[EI\ w(L,t)_{xx}] + k_{\theta 2}\ w(L,t)_x\ \}\ \delta w(L,t)_x = 0 \\
\{[EI\ w(0,t)_{xx}] - k_{\theta 1}\ w(0,t)_x\ \}\ \delta w(0,t)_x = 0
\end{aligned} \tag{3}$$

Thus, the governing differential equation and the four natural boundary conditions are as follow.

$$
\begin{aligned}
m(x)\; w(x,t)_{tt} + [EI\; w(x,t)_{xx}]_{xx} &= \; 0 \\
\{-[EI\; w(L,t)_{xx}]_x + k_{w2}\; w(L,t)\;\} &= 0 \\
\{-[EI\; w(0,t)_{xx}]_x - k_{w1}\; w(0,t)\;\} &= 0 \\
\{[EI\; w(L,t)_{xx}] + k_{\theta 2}\; w(L,t)_x\;\} &= 0 \\
\{[EI\; w(0,t)_{xx}] - k_{\theta 1}\; w(0,t)_x\;\} &= 0
\end{aligned}
\tag{4}
$$

In order to formulate the beam vibration problem, first, we assume $w(x,t)$ in the form

$$
w(x,t) = W(x)e^{j\omega t} \tag{5}
$$

which, when substituted into (4), yields

$$
\boxed{
\begin{aligned}
-\frac{\omega^2 m}{EI}\; W(x) + W(x)_{xxxx} = \; 0, \quad 0 \le x \le L \\
-W(0)_{xxx} - \frac{k_{w1}}{EI}\; W(0) \;&= 0 \\
w(0)_{xx} - \frac{k_{\theta 1}}{EI}\; W(0)_x \;&= 0 \\
-W(L)_{xxx} + \frac{k_{w2}}{EI}\; W(L) \;&= 0 \\
w(L)_{xx} + \frac{k_{\theta 2}}{EI}\; W(L)_x \;&= 0
\end{aligned}
}
\tag{6}
$$

which gives the following characteristic equation:

$$
\det \begin{bmatrix}
0 & -1 & 0 & -1 \\
-\bar{k}_{\theta 1} & -\bar{\beta} & -\bar{k}_{\theta 1} & \bar{\beta} \\
-\sin\bar{\beta} & -\cos\bar{\beta} & -\sinh\bar{\beta} & -\cosh\bar{\beta} \\
\begin{matrix}\bar{\beta}\sin\bar{\beta} \\ -\bar{k}_{\theta 2}\cos\bar{\beta}\end{matrix} & \begin{matrix}\bar{\beta}\;\cos\bar{\beta} \\ +\bar{k}_{\theta 2}\sin\bar{\beta}\end{matrix} & \begin{matrix}-\bar{\beta}\;\sinh\bar{\beta} \\ -\bar{k}_{\theta 2}\cosh\bar{\beta}\end{matrix} & \begin{matrix}-\bar{\beta}\;\cosh\bar{\beta} \\ -\bar{k}_{\theta 2}\sinh\bar{\beta}\end{matrix}
\end{bmatrix}
\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix} = 0
\tag{7}
$$

$$\Downarrow$$

$$
\begin{aligned}
\bar{\beta}^2 \sin\bar{\beta}\sinh\bar{\beta} + \bar{k}_{\theta 1}\;\bar{\beta}\;(\sin\bar{\beta}\cosh\bar{\beta} - \cos\bar{\beta}\sinh\bar{\beta}) \\
- \bar{k}_{\theta 2}\;\bar{\beta}\cos\bar{\beta}\sinh\bar{\beta} + \bar{k}_{\theta 1}\;\bar{k}_{\theta 2}\;(1 - \cos\bar{\beta}\cosh\bar{\beta}) = 0
\end{aligned}
$$

$$
\bar{\beta} = \beta L, \quad \bar{k}_{\theta i} = k_{\theta i}/(EI/L)
$$

After computing the first mode of the 5 ideal boundary conditions using *BeamModeShapeFinder.m*, the mode and mode shape of the present problem are found to fall between the *fixed-fixed* and the *simply supported-simply supported* beams as shown in Figure 1 below. Notice the frequency parameter βL of the problem is $\beta L = 4.25$ whereas the fixed-simple support and the fixed-fixed beams are $\beta L = 3.9266$ and $\beta L = 4.73$, respectively. In addition, the maximum mode shape amplitude of the problem occurs at $x/L = 0.55$ which is between that of the fixed-simply supported beam($x/L = 0.60$) and of the fixed-fixed beam ($x/L = 0.5$).

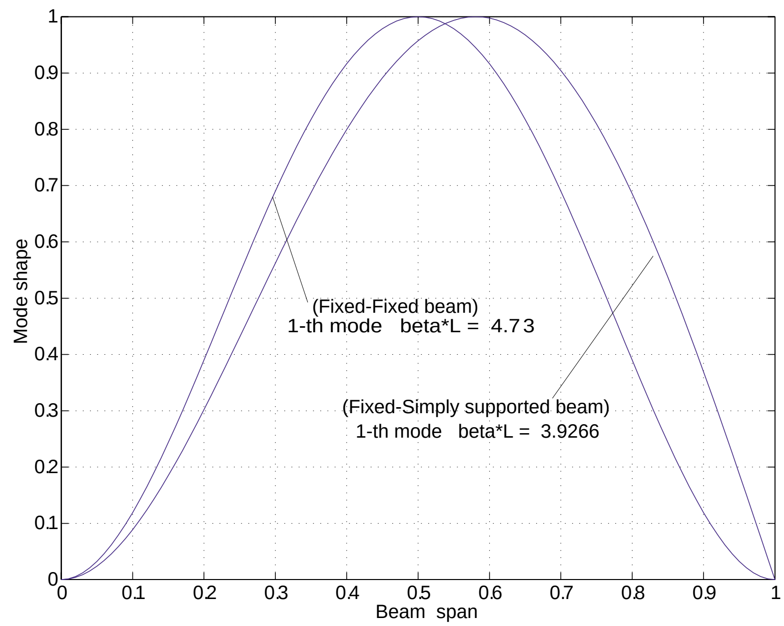

Fig. 1 Mode and mode shapes of fixed-fixed and fixed-simple support beams

Hence, the stage is set for finding the rotational spring parameters $k_{\theta 1}$ and $k_{\theta 2}$. A possible beam model is shwon in Figure 2 below.

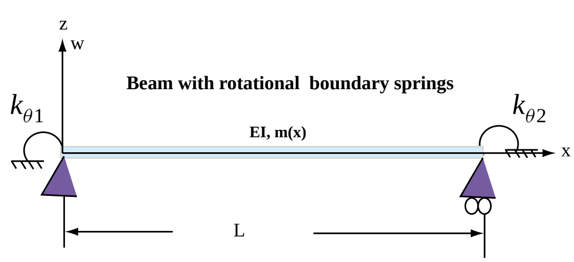

Fig. 2 A beam with two rotational support springs

4.2 Model this beam in terms of discrete springs and masses employing the discrete modeling approaches discussed in the class. Can you have a rough estimate of the boundary springs from your discrete model? Show your rationale as to how well your proposed discrete model can guide you to a reasonable set of model parameters.

We will examine this in three ways.

4.2.1 A crude one-element beam with two support springs:

If one models the entire beam with one element and eliminates the vertical degrees of freedom, $w_1 = w + 2 = 0$, from Lecture 14 on the finite element modeling of beams as shown in Figure 2, we obtain the following equation when weconsider only the two end rotations:

$$\mathbf{m}\ddot{\mathbf{x}} + \mathbf{k}\mathbf{x} = 0, \quad \mathbf{x} = [\theta_1,\ \theta_2]^T$$

$$\mathbf{m} = \frac{m\ \ell^3}{420}\begin{bmatrix} 4 & -3 \\ -3 & 4 \end{bmatrix}, \quad m = \rho A \tag{8}$$

$$\mathbf{k} = \frac{EI}{\ell}\begin{bmatrix} 4+\kappa_{\theta 1} & 2 \\ 2 & 4+\kappa_{\theta 2} \end{bmatrix}, \quad \kappa_\theta = \frac{k_\theta \ell}{EI}$$

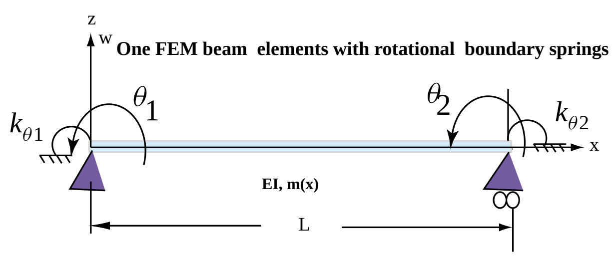

Fig. 2 One element model with two rotational support springs

The preceding equation leads to the following eigenvalue problem:

$$[\mathbf{k} - \omega^2\mathbf{m}]\mathbf{x} = \mathbf{0}$$
$$\Downarrow$$
$$\det\left|\begin{bmatrix} 4+\kappa_{\theta 1} & 2 \\ 2 & 4+\kappa_{\theta 2} \end{bmatrix} - \frac{(\beta L)^4}{420}\begin{bmatrix} 4 & -3 \\ -3 & 4 \end{bmatrix}\right| = \mathbf{0}, \quad \beta^4 = \frac{\omega^2 m}{EI} \tag{9}$$

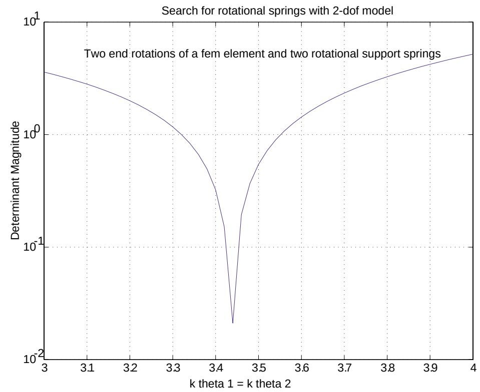

Fig. 3 Search for $\kappa_{\theta 1} = \kappa_{\theta 2}$ by one element model

In order to get an idea of an order of magnitude estimate, we set $\kappa = \kappa_{\theta 1} = \kappa_{theta2}$ and searched for its value with the desired $\beta L = 4.25$. The result is shown in Figure 3, which shows that for the case of $\kappa_{\theta 1} = \kappa_{theta2}$ it hovers around

$$(\kappa_1)_{(2-dof\ \ fem)} \approx \ 3.45 \tag{10}$$

. Note, however, that this equal rotational springs will result in a symmetric mode shape, meaning that the maximum mode shape amplitude will occur at $x/L = 0.5$.

4.2.2 A two-element beam with two support springs:

If one models the entire beam with two elements and eliminates the vertical degrees of freedom at the supports as shown in Figure 4, we obtain the following equation when we consider only the two end rotations:

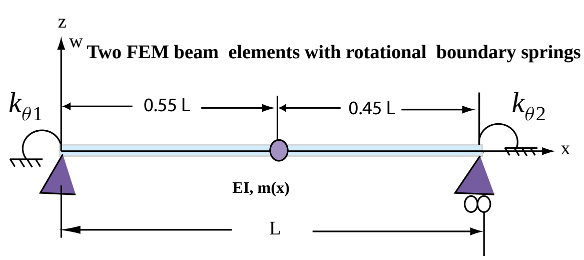

Fig. 4 Two beam elements with two rotational support springs

$$\mathbf{m}\ddot{\mathbf{x}} + \mathbf{k}\mathbf{x} = \mathbf{0}, \quad \mathbf{x} = [\theta_1, w_m, \theta_m, \theta_2]^T$$

$$\mathbf{m} = \begin{bmatrix} \frac{m_1\ell_1^2}{105} & \frac{13m_1\ell_1}{420} & -\frac{m_1\ell_1^2}{140} & 0 \\ \frac{13m_1\ell_1}{420} & \frac{13(m_1+m_2)}{35} & \frac{11(m_2\ell_2-m_1\ell_1)}{210} & -\frac{13m_2\ell_2}{420} \\ -\frac{m_1\ell_1^2}{140} & \frac{11(m_2\ell_2-m_1\ell_1)}{210} & \frac{(m_2\ell_2^2+m_1\ell_1^2)}{105} & -\frac{m_2\ell_2^2}{140} \\ 0 & -\frac{13m_2\ell_2}{420} & -\frac{m_2\ell_2^2}{140} & \frac{m_2\ell_2^2}{105} \end{bmatrix}$$

$$\mathbf{k} = \begin{bmatrix} 4k_1\ell_1^2 + k_{\theta 1} & -6k_1\ell_1 & 2k_1\ell_1^2 & 0 \\ -6k_1\ell_1 & 12(k_1+k_2) & -6(k_1\ell_1 - k_2\ell_2) & 6k_2\ell_2 \\ 2k_1\ell_1^2 & -6(k_1\ell_1 - k_2\ell_2) & 4(k_1\ell_1^2 + k_2\ell_2^2) & 2k_2\ell_2^2 \\ 0 & 6k_2\ell_2 & 2k_2\ell_2^2 & 4k_2\ell_2^2 + k_{\theta 2} \end{bmatrix} \qquad (11)$$

$$m_1 = \rho A \ell_1 = m\ell_1, \quad m_2 = \rho A \ell_2 = m\ell_2, \quad k_1 = \frac{EI}{\ell_1^3}, \quad k_2 = \frac{EI}{\ell_2^3}$$

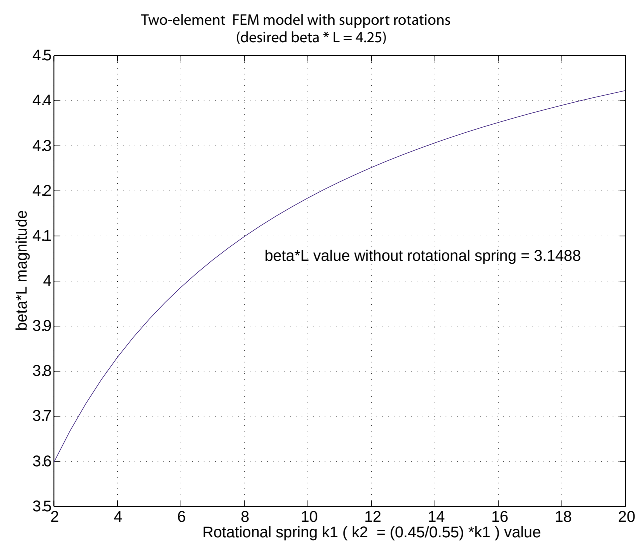

Fig. 5 Two beam elements with two rotational support springs

After solving for various $[\kappa_1 = \frac{k_{\theta 1}L}{EI}, \kappa_2 = (0.45/0.55)\kappa_1]$, the result is illustraated in Figure 5. Hence, one can estimate the support spring to be:

$$(\kappa_1)_{(4-dof\ \ fem)} \approx 12.0$$

.

4.2.3 Classical approach with two end rotational degrees of freedom: A classical approach (i.e., assumed mode approximation) of the following form may be utilized for the present purposes:

$$W(x,t) = c_1(t)\sin(\pi x/L) + c_2(t)\sin(2\pi/L) \tag{12}$$

This approximation satisfies the two end conditions:

$$w(0,t) = w(L,t) = 0 \tag{13}$$

and setting $\theta(x,t) = w(x,t)_x$, $[c_1(t), c_2(t)]$ can be expressed by the two end rotations, $[\ \theta_1(t) = \theta(0,t),\ \theta_2(t) = \theta(L,t)\ ]$. Therefore, the total kinetic and potential energy can be obtained by

$$\begin{aligned} T &= \tfrac{1}{2}\int_0^L m[\dot{w}(x,t)]^2\,dx,\ m = \rho A \\ V &= \tfrac{1}{2}\int_0^L EI[w(x,t)_{xx}]^2\,dx + \tfrac{1}{2}k_{\theta_1}\theta_1^2 + \tfrac{1}{2}k_{\theta 2}\theta_2^2 \end{aligned} \tag{14}$$

which is a function of $[\dot{\theta}_1, \dot{\theta}_2, \theta_1, \theta_2]$.

Carrying out the necessary variational process, one finds the following homogeneous equation:

$$\begin{aligned} &\mathbf{m}\ddot{\mathbf{x}} + \mathbf{k}\mathbf{x} = 0, \quad \mathbf{x} = [\theta_1, \theta_2]^T \\ &\mathbf{m} = m\begin{bmatrix} 5/32 & -3/32 \\ -3/32 & 5/32 \end{bmatrix} \\ &\mathbf{k} = \begin{bmatrix} 5k/8 + k_{\theta 1} & 3k/8 \\ 3k/8 & 5k/8 + k_{\theta 1} \end{bmatrix} \\ &m = mL^3/\pi^2, \quad k = \pi^2 EI/L \end{aligned} \tag{15}$$

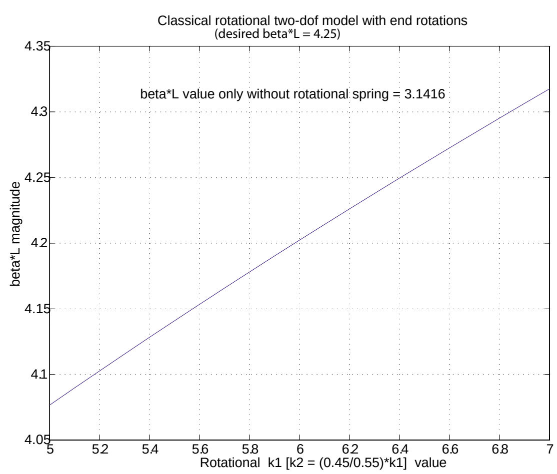

Fig. 5 Classical assumed mode model with two rotational support springs

Figure 5 shows the βL vs. κ_1, which estimates the desired κ_1 to be around:

$$(\kappa_1)_{(2-dof\ classical)} \approx 6.4 \tag{16}$$

4.3 Utilizing the technical insight you gained from the above two tasks, develop a strategy of how you can employ the continuum model derived in problem (4.1) to arrive at the uncertain boundary condition parameters.

An analysis for the estimation of the support rotational springs from Problem 4.2 has provided the following:

$$\begin{aligned} \text{One-element with two end rotational DOFs:} \quad & \kappa_1 \approx 3.45 \\ \text{Two-element with one interior translation and two end rotational DOFs:} \quad & \kappa_1 \approx 12.0 \\ \text{Classical approach with two end rotational DOFs:} \quad & \kappa_1 \approx 6.4 \end{aligned} \tag{17}$$

It should be noted that the above estimates are based on the assumption that the peak amplitude of the mode shape occurs at the beam mid-span.

Armed with these estimates, an iterative search for κ_1 and κ_2 was launched by utilizing the theoretical (4x4)-characteristic matrix routine, `CmatrixBeamGeneral.m`, while varying κ_1 in the outer loop and κ_2 in the inner loop. A limiting solution has been found if $\kappa_{\theta 1} = k_{\theta 1} L/EI$ is chosen to be excessively large, e.g.,

$$\boxed{\begin{aligned} \kappa_{\theta 1} &= 10^6 \\ \kappa_{\theta 2} &= \frac{k_{\theta 2} L}{EI} \approx 4 \end{aligned}} \tag{18}$$

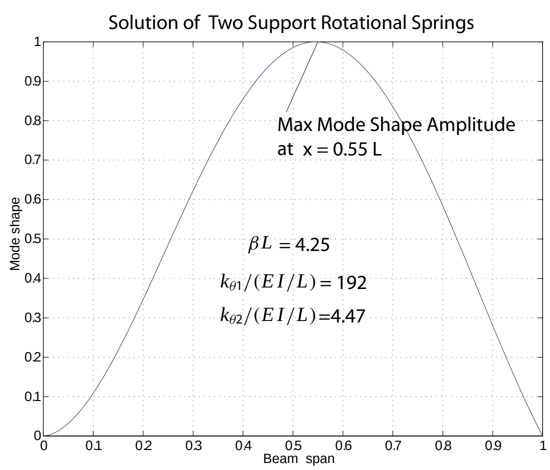

Fig. 6 Classical assumed mode model with two rotational support springs

A further iterative search discovered that a solution (not unique!) has been discovered as

$$\boxed{\begin{aligned} \kappa_{\theta 1} &= \frac{k_{\theta 1} L}{EI} \approx 192 \\ \kappa_{\theta 2} &= \frac{k_{\theta 2} L}{EI} \approx 4.35 \end{aligned}} \tag{19}$$

The mode shape of this choice is plotted in Figure 6.

$\{Remark : What\ if\ \beta L = 4.0\,?$

For this case one can find the following parameters:

$$\boxed{\begin{array}{l} \dfrac{k_{\theta 1} L}{EI} \approx 26.5 \\ \dfrac{k_{\theta 2} L}{EI} \approx 1.95 \end{array}} \tag{20}$$

which shows that, for a drop of frequency 11 percent (note that the frequency is proportional to $(\beta L)^2$), an appreciable reduction of the support springs will result. In practice, the frequency drop is an indication of how much the bridge has deteriorated, a damage indicator. It is for this reason the support spring models play a crucial role in assessing the health of the bridge via dynamic testing. }

4.3 Discuss complementary features of both the rigorous continuum modeling approach and ruthless discrete modeling approaches, if any. What have you learned?

The estimate of $\kappa_2 = \frac{k_{\theta 2} L}{EI}$ by the two to four-DOF models has been proved to be fairly useful. Judging from that fact that the peak amplitude of the mode shape occurs at $x?L = 0.55$, we see that the left-end rotational support spring ($_{\theta 1}$) must be stiffer than the right-end support spring ($_{\theta 2}$). In using the exact continuum formula (7), we have found that in fact one could treat the left-hand boundary condition is very close to an ideal clamped or fixed support. Therefore, if one is charged to inspect or repair the bridge, one should carefully look into any loosened connections on the right-side of the bridge and the right-end end condition.

Finally, with the preliminary estimate, a refined finite element model can be constructed, which can be used to study not only the end conditions but also the many connectors in the bridge structures. For this to be meaningful, there should be more measured data along the beam span. This subject is called *structural health monitoring* which has been a growing activity around the world.