Solutions of Homework #3 (ASEN5022, Spring 2004)

Problem : A cable with a flexible support at the middle.

A construction company has been designing a cable that will run between two towers in a mountain. After a preliminary analysis and a scale model test, the management came to the conclusion that a third flexible tower needs to be placed.

3.1 Formulate the equations of motion for this new cable system complete with the appropriate boundary conditions. Assume that the original two supporting towers are substantially stiffer than the new one to be placed at the middle of the original cable. The design team has concluded that they should model not only the flexibility but also the inertia effect of the middle tower.

3.2 If the new system is to have its fundamental frequency about 50% higher than the original system (which was the reason why a third tower is needed), how are you going to accomplish this requirement? Observe that the cable tension, T, of the original system could not be increased any further as that will result in an unacceptable bending on the original two towers.

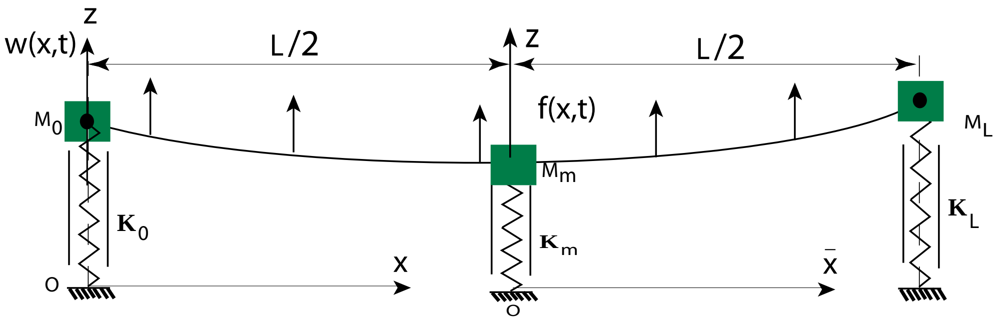


Figure 1 Cable with a middle support

1. Selection of the Coordinate System.

There are two ways of selecting the coordinate system. Once is to use one coordinate system. In the present solution, we employ two independent coordinate systems, one for the left cable and another for the right cable as shown in Figure 1 above.

2. Hamilton's principle for this problem.

Introduce $w(x,t)$ and $w(\bar{x},t)$ whose domains are defined by

$$\begin{cases} \text{For the left cable: } w(x,t), \quad 0 \le x \le \ell \\ \text{For the right cable: } w(\bar{x},t), \quad \ell \le \bar{x} \le L, \quad \ell = L/2 \end{cases} \tag{1}$$

Using these distinct displacements, we have

$$\int_{t_1}^{t_2} [\delta L + \delta \bar{W}_{noncons}] dt = 0, \quad L = T - V$$

$$\begin{aligned} T = & \int_0^{\ell} \tfrac{1}{2} \rho(x) \dot{w}^2(x,t)\, dx + \int_{\ell}^{L} \tfrac{1}{2} \rho(\bar{x}) \dot{w}^2(\bar{x},t)\, d\bar{x} \\ & + \tfrac{1}{2} M_0\, \dot{w}^2(0,t) + \tfrac{1}{2} M_L\, \dot{w}^2(L,t) + \tfrac{1}{2} M_m\, \dot{w}^2(\ell,t) \end{aligned} \tag{2}$$

$$\begin{aligned} V = & \int_0^{\ell} \tfrac{1}{2} T(x)\, w_x^2(x,t)\, dx + \int_{\ell}^{L} \tfrac{1}{2} T(\bar{x})\, w_x^2(\bar{x},t)\, d\bar{x} \\ & + \tfrac{1}{2} K_0\, w^2(0,t) + \tfrac{1}{2} K_L\, w^2(L,t) + + \tfrac{1}{2} K_m\, w^2(\ell,t) \end{aligned}$$

Evaluation of $\int_{t_1}^{t_2} \delta T\, dt$

$$\boxed{\begin{aligned} \int_{t_1}^{t_2} \delta T dt = & -\int_{t_1}^{t_2} \{ [\int_0^{\ell} \rho(x) \ddot{w}(x,t)\, \delta w(x,t) dx\ ] + [\int_{\ell}^{L} \rho(\bar{x}) \ddot{w}(\bar{x},t)\, \delta w(\bar{x},t) d\bar{x}\ ] \\ & + M_0 \ddot{w}(0,t)\, \delta w(0,t) + M_L \ddot{w}(L,t)\, \delta w(L,t) + M_m \ddot{w}(\ell,t)\, \delta w(\ell,t) \}\, dt \end{aligned}} \tag{3}$$

Evaluation of δV

$$\begin{aligned} \delta V = & -\int_0^{\ell} T(x)\, w_{xx}(x,t) \delta w(x,t)\, dx - \int_{\ell}^{L} T(\bar{x})\, w_{\bar{x}\bar{x}}(\bar{x},t) \delta w(\bar{x},t)\, d\bar{x} \\ & + \{\ [T(x)\, w_x(x,t) \delta w(x,t)]_{x=\ell} - [T(x)\, w_x(x,t) \delta w(x,t)]_{x=0}\ \} \\ & + \{\ [T(\bar{x})\, w_{\bar{x}}(\bar{x},t) \delta w(\bar{x},t)]_{\bar{x}=L} - [T(\bar{x})\, w_{\bar{x}}(\bar{x},t) \delta w(\bar{x},t)]_{\bar{x}=\ell}\ \} \\ & + [K_0\, w(x,t) \delta w(x,t)]_{x=0} + [K_L\, w(\bar{x},t) \delta w(\bar{x},t)]_{\bar{x}=L} + [K_m\, w(x,t) \delta w(x,t)]_{x=\ell} \end{aligned} \tag{4}$$

Virtual work due to nonconservative force $f(x,t)$

$$\delta \bar{W}_{noncons} = 0 \tag{5}$$

due to free vibration problem of interest.

Substituting (5), (4) and (3) into (1), one obtains Hamilton's principle:

$$
\begin{aligned}
&\int_{t_1}^{t_2} \langle \int_0^{\ell} \{[-\rho(x)\ddot{w}(x,t) + T(x)\ w_{xx}(x,t)]\ \delta w(x,t)\} dx \\
&\qquad \int_{\ell}^{L} \{[-\rho(\bar{x})\ddot{w}(\bar{x},t)\ + T(\bar{x})\ w_{\bar{x}\bar{x}}(\bar{x},t)]\ \delta w(\bar{x},t)\} d\bar{x} \\
&\qquad - \{[-T(x)\ w_x(x,t) + K_0 w(x,t) + M_0 \ddot{w}(x,t)]\delta w(x,t)\}_{x=0} \\
&\qquad - \{[(T(x)\ w_x(x,t) + K_m w(x,t) + M_m \ddot{w}(x,t)]\delta w(x,t)\}_{x=\ell} \\
&\qquad - \{[-T(\bar{x})\ w_{\bar{x}}(\bar{x},t)]\delta w(\bar{x},t)\}_{x=\ell} \\
&\qquad - \{[(T(\bar{x})\ w_{\bar{x}}(\bar{x},t) + K_L w(\bar{x},t) + M_L \ddot{w}(\bar{x},t)]\delta w(x,t)\}_{x=L} \\
&\rangle\ dt = 0
\end{aligned}
\tag{6}
$$

Since $\delta w(x,t)|_{x=\ell}$ and $\delta w(\bar{x},t)|_{\bar{x}=\ell}$ are the same, viz.,

$$
w(x,t)|_{x=\ell} - w(\bar{x},t)|_{\bar{x}=\ell} = 0 \tag{7}
$$

the fourth and fifth rows in (6) must be concolidated to read:

$$
\begin{aligned}
&- \{[(T(x)\ w_x(x,t) + K_m w(x,t) + M_m \ddot{w}(x,t)]\delta w(x,t)\}_{x=\ell} \\
&- \{[-T(\bar{x})\ w_{\bar{x}}(\bar{x},t)]\delta w(\bar{x},t)\}_{x=\ell} \\
&= -\{[(T(x)\ w_x(x,t) + K_m w(x,t) + M_m \ddot{w}(x,t) - T(\bar{x})\ w_{\bar{x}}(\bar{x},t)]\delta w(x,t)\}_{x=\ell} \\
&= -\{[(T(x)\ w_x(x,t) - T(\bar{x})\ w_{\bar{x}}(\bar{x},t) + K_m w(x,t) + M_m \ddot{w}(x,t)]\delta w(x,t)\}_{x=\ell}
\end{aligned}
\tag{8}
$$

Substituting this into (6) leads to the following variation equation:

$$
\begin{aligned}
&\int_{t_1}^{t_2} \langle \int_0^{\ell} \{[-\rho(x)\ddot{w}(x,t) + T(x)\ w_{xx}(x,t)]\ \delta w(x,t)\} dx \\
&\qquad \int_{\ell}^{L} \{[-\rho(\bar{x})\ddot{w}(\bar{x},t)\ + T(\bar{x})\ w_{\bar{x}\bar{x}}(\bar{x},t)]\ \delta w(\bar{x},t)\} d\bar{x} \\
&\qquad - \{[-T(x)\ w_x(x,t) + K_0 w(x,t) + M_0 \ddot{w}(x,t)]\delta w(x,t)\}_{x=0} \\
&\qquad - \{[(T(x)\ w_x(x,t) - T(\bar{x})\ w_{\bar{x}}(\bar{x},t) + K_m w(x,t) + M_m \ddot{w}(x,t)]\delta w(x,t)\}_{x=\ell} \\
&\qquad - \{[(T(\bar{x})\ w_{\bar{x}}(\bar{x},t) + K_L w(\bar{x},t) + M_L \ddot{w}(\bar{x},t)]\delta w(x,t)\}_{x=L} \\
&\rangle\ dt = 0
\end{aligned}
\tag{9}
$$

The preceding Hamilton’s equation yields two governing differential equations:

The governing equation of motion:

$$\boxed{\begin{aligned}&\rho(x)\ddot{w}(x,t) = T(x)\ w_{xx}(x,t),\ \ 0 \le x \le \ell,\ \ \ell = L/2\\&\rho(\bar{x})\ddot{w}(\bar{x},t) = T(\bar{x})\ w_{\bar{x}\bar{x}}(\bar{x},t),\ \ \ell \le \bar{x} \le L\end{aligned}} \tag{10}$$

The three natural boundary conditions:

$$\boxed{\{[-T(x)\ w_x(x,t) + K_0 w(x,t) + M_0\ddot{w}(x,t)]\delta w(x,t)\}_{x=0} = 0} \tag{11}$$

$$\boxed{\{[(T(x)\ w_x(x,t) - T(\bar{x})\ w_{\bar{x}}(\bar{x},t) + K_m w(x,t) + M_m\ddot{w}(x,t)]\delta w(x,t)\}_{x=\ell} = 0} \tag{12}$$

$$\boxed{\{[T(\bar{x})\ w_{\bar{x}}(\bar{x},t) + K_L w(\bar{x},t) + M_L\ddot{w}(\bar{x},t)]\delta w(\bar{x},t)\}_{x=L} = 0} \tag{13}$$

Together with the above three natural boundary conditions, one must augment them with the continuity condition at the middle support given by (7), which provide the critical fourth constraint condition.

The three natural boundary conditions plus the continuity condition at the middle support:

$$\boxed{\begin{aligned}&[-T\ w_x(x,t) + K_0 w(x,t) + M_0\ddot{w}(x,t)]_{x=0} = 0\\[2ex]&[T\ w_x(x,t) - T\ w_{\bar{x}}(\bar{x},t) + K_m w(x,t) + M_m\ddot{w}(x,t)]_{x=\ell} = 0,\ \ \ell = L/2\\[2ex]&[T\ w_{\bar{x}}(\bar{x},t) + K_L w(\bar{x},t) + M_L\ddot{w}(\bar{x},t)]_{x=L} = 0\\[2ex]&[w(x,t)|_{x=\ell} - w(\bar{x},t)]_{\bar{x}=\ell} = 0\\[2ex]&\text{with }\ T = T(x) = T(\bar{x})\end{aligned}} \tag{14}$$

The general form of solution from the governing two equations of motion (10):

$$\begin{aligned}w(x,t) &= F(t)\ W(x),\ \ W(x) = [C_1\sin\beta x + C_2\cos\beta x],\ \ 0 \le x \le \ell\\w(\bar{x},t) &= F(t)\ W(\bar{x}),\ \ W(\bar{x}) = [C_3\sin\beta\bar{x} + C_4\cos\beta\bar{x}],\ \ \ell \le \bar{x} \le L\\F(t) &= \hat{f}e^{j\omega t}\end{aligned} \tag{15}$$

When ($K_0 \to \infty,\ K_L \to \infty$), by substituting this condition and the general solution form (15), we find:

$$\boxed{\begin{aligned} & W(0) = 0 \\ & [T\ W(x)_x - T\ W(\bar{x})_{\bar{x}} + K_m W(x) - \omega^2 M_m W(x)]_{x=\ell} = 0, \quad \ell = L/2 \\ & W(L) = W(2\ell) = 0 \\ & [W(x)|_{x=\ell} - W(\bar{x})]_{\bar{x}=\ell} = 0 \end{aligned}} \tag{16}$$

Let's focus on the second equation of the above equation set.

First, dividing by the tension T and multiplying by ℓ, we have

$$[\ \ell\ W(x)_x -\ \ell\ W(\bar{x})_{\bar{x}} + \frac{K_m \ell}{T} W(x) - \frac{\omega^2 M_m \ell}{T} W(x)]_{x=\ell} = 0 \tag{17}$$

Second, we note that with $\beta^2 = \frac{\omega^2 \rho}{T}$

$$\begin{aligned} \frac{\omega^2 M_m \ell}{T} &= M_m \ell\ \frac{\omega^2}{T} = \frac{M_m \ell^2}{\rho \ell}\ \frac{\omega^2 \rho}{T} = \frac{M_m \ell^2}{\rho \ell}\ \beta^2 = \frac{M_m}{\rho \ell}\ (\beta \ell)^2 \\ &= \mu_m\ \bar{\beta}^2 \\ \mu_m &= \frac{M_m}{\rho \ell}, \quad \bar{\beta} = \beta \ell, \ \ \ell = L/2 \end{aligned} \tag{18}$$

Third, introducing

$$\kappa_m = \frac{K_m \ell}{T} \tag{19}$$

equation (17) can be written as

$$[\ \ell\ W(x)_x -\ \ell\ W(\bar{x})_{\bar{x}} + \kappa_m\ W(x) - \mu_m \bar{\beta}^2 W(x)]_{x=\ell} = 0 \tag{20}$$

Therefore, (16) simplifies to

$$\boxed{\begin{aligned} & W(0) = 0 \\ & [\ \ell\ W(x)_x -\ \ell\ W(\bar{x})_{\bar{x}} + \kappa_m\ W(x) - \mu_m \bar{\beta}^2 W(x)]_{x=\ell} = 0 \\ & W(L) = W(2\ell) = 0 \\ & [W(x)|_{x=\ell} - W(\bar{x})]_{\bar{x}=\ell} = 0 \end{aligned}} \tag{21}$$

Finally, using

$$\begin{aligned} W(x) &= C_1 \sin(\beta x) + C_2 \cos(\beta x), 0 \leq x \leq \ell \\ W(\bar{x}) &= C_3 \sin(\beta x) + C_4 \cos(\beta x), \ell \leq \bar{x} \leq L \end{aligned} \tag{22}$$

the preceding boundary and constraint set (21) leads to the following characteristic equation:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ [\bar{\beta}\ \cos(\bar{\beta}) + (\kappa_m - \mu_m \bar{\beta}^2)\ \sin(\bar{\beta})] & [-\bar{\beta}\ \sin(\bar{\beta}) + (\kappa_m - \mu_m \bar{\beta}^2)\ \cos(\bar{\beta})] & -\bar{\beta}\ \cos(\bar{\beta}) & \bar{\beta}\ \sin(\bar{\beta}) \\ 0 & 0 & \sin(2\bar{\beta}) & \cos(2\bar{\beta}) \\ \sin(\bar{\beta}) & \cos(\bar{\beta}) & -\sin(\bar{\beta}) & -\cos(\beta) \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{23}$$

By successive manipulations of the third and fourth columns, it can be shown that the above equation simplifies to

$$\begin{aligned} &\begin{bmatrix} 0 & 1 & 0 & 0 \\ [\bar{\beta}\ \cos(\bar{\beta}) + (\kappa_m - \mu_m \bar{\beta}^2)\ \sin(\bar{\beta})] & [-\bar{\beta}\ \sin(\bar{\beta}) + (\kappa_m - \mu_m \bar{\beta}^2)\ \cos(\bar{\beta})] & -\bar{\beta} & 0 \\ 0 & 0 & \sin(\bar{\beta}) & \cos(\bar{\beta}) \\ \sin(\bar{\beta}) & \cos(\bar{\beta}) & 0 & -1 \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \\ &\Downarrow \\ &\mathbf{AC} = \mathbf{0} \end{aligned} \tag{24}$$

This simplification is equivalent to taking the left and right cable coordinates independently for both cases as $(0 \leq x \leq \ell,\ \ 0 \leq \bar{x} \leq \ell)$.

Setting $\det(\mathbf{A}) = 0$, one finds with $\mu_m = \frac{M_m}{\rho \ell}$:

$$\kappa_m = \frac{K_m \ell}{T} = (\beta \ell)^2 - 2(\beta \ell)\ \frac{\cos(\beta \ell)}{\sin(\beta \ell)} \tag{25}$$

Since the desired frequency is

$$\beta L = 1.5\pi \quad \Rightarrow \quad \beta \ell = 1.5\pi/2 \tag{26}$$

we find the spring constant to be

$$\kappa_m = \frac{K_m \ell}{T} = 10.26404145599745 \tag{27}$$

The fundamental mode shape is plotted in Figure 2 using the computation routines listed below.

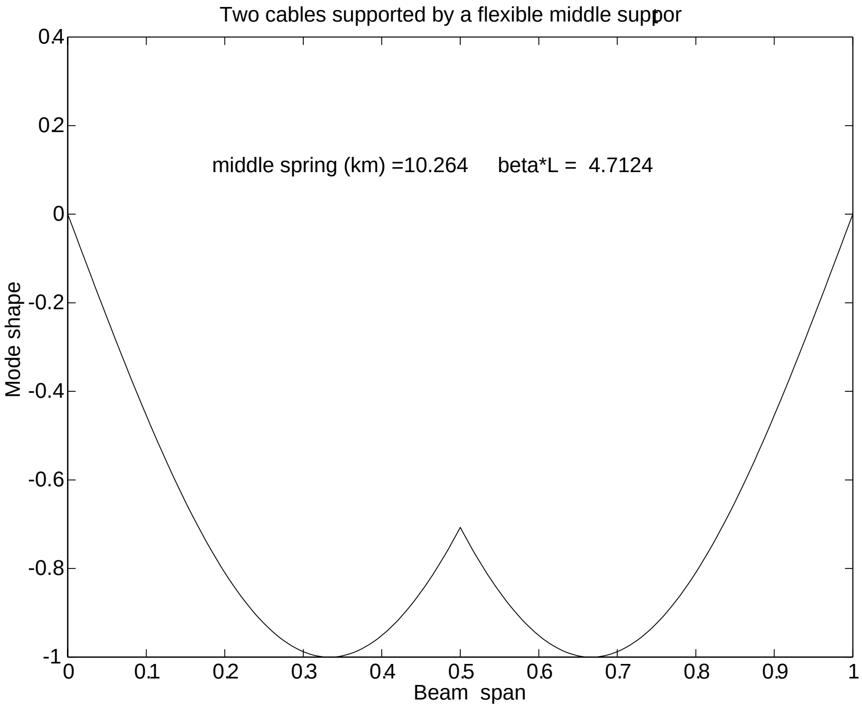


Figure 2 Fundamental mode of a cable with a middle support

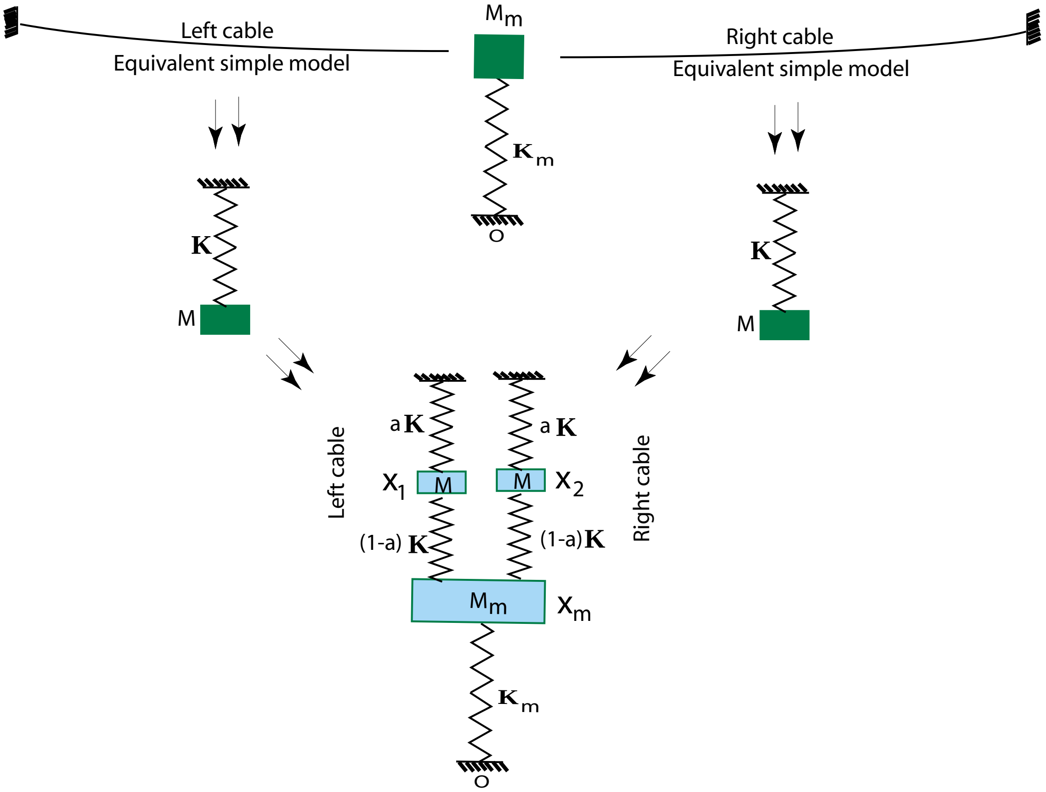
```
%  Computations of two cable vibrations
%  using a general characteristic  equation
%
clear all;
%
% Km determined from the det(C_cable)
%
km=10.26404145599745;
%
% compute mode shape  for this frequency
%
beta_ell  = 1.5*pi/2; % target frequency
% compute C_cable (4x4) matrix
Ccable = CmatrixTwoCable(km, beta_ell);
% it is assumed that the minor Cbeam(3x3) matrix is non-singular
Cminor = Ccable(2:4, 2:4);
b=-Ccable(2:4,1);
% mode shape coefficients assuming c_4 =1
coef = Cminor\b;

xL_coord=zeros(1,0);
yL_coord =zeros(1,0);
xR_coord=zeros(1,0);
yR_coord =zeros(1,0);
```

```
for i=0:100,
span = i/100;
arg =beta_ell*span;
sx = sin(arg); cx = cos(arg);
WL =  sx + coef(1)*cx;
WR =coef(2)*sx +  coef(3)*cx;
xL_coord = [xL_coord span];
yL_coord =[yL_coord WL];
xR_coord = [xR_coord span];
yR_coord =[yR_coord WR];
end;

x_coord=0.0:0.005:1.00;
y_coord=[yL_coord(:,1:100)   yR_coord ];

%normalize y_coord
y_coord = -y_coord/(max(abs(y_coord)));
figure(1);
plot(x_coord, y_coord);
xlabel('Beam  span');
ylabel('Mode shape ');
legend([ 'middle spring (km) =', num2str(km),   '  beta*L =  ', num2str(2*beta_ell)]);
title('Two cables supported by a flexible middle support');
% end of the program
%
%  compute the determinant of two-cable with middle support
%
function [C_cable] = CmatrixTwoCable(km, beta_bar);
%  characteristic matrix for two cable with middle flexible support
%vibration problem;
%  the two end supports are fixed
%input beta = beta*L/2;  km = middle spring,  m is assumed
%to be half of the cable weight.
%
%output : C_cable (4x4)
%%%%%%%%%%%%%%%%%%%
%  data preparation
%%%%%%%%%%%%%%%%%%%
c  = cos(beta_bar);
s  = sin(beta_bar);
%
C_cable =[
0                                             1                 0      0;
beta_bar*c+km*s-beta_bar^2*s  -beta_bar*s+km*c-beta_bar^2*c  -beta_bar 0;
0                                             0                  s     c;
s                                             c                  0    -1];
return;
```

A countryside engineer’s model of a cable with a flexible middle support

Suppose that you are in the countryside without access to advanced modeling tools and you are asked to design the same task. If that individual happens to be this instructor, here is an approach.

Step 1: Model each half of the cable by a lumped mass-spring system (M, K) such that the fundamental frequency is $\omega_1 = (2\pi) = \sqrt{K/M}$. This is illustrated in Figure 3.



The spring distribution factor, a, is chosen such that when Mm = Km = 0, the fundamental frequency is half of the individual frequency, viz., from 2pi to pi. Thus, a is found to be a = 0.75.

Figure 3 A simple equivalent model of a cable with a middle support

Step 2: Form the left and right cables to be in parallel and attach them to the support system characterized by (M_m, K_m).

Step 3: In doing so, distribute the cable springs such that if the middle support is removed, the fundamental frequency of the system would be half of the individual frequency. This is because for cables the fundamental frequency is given by

$$\omega_1 = \frac{n\pi}{L}\sqrt{\frac{T}{\rho}} \tag{28}$$

which states that if the cable length is doubled, the fundamental frequency $(n = 1)$ will be halved. A simple weighted distribution of the cable springs indicates that the distribution factor should be

$$a = 3/4 \tag{29}$$

Step 4: Form the model equation set given by:

$$\begin{aligned} M\ddot{x}_1 + Kx_1 - \frac{K}{4}x_m &= f_1 \\ M\ddot{x}_2 + Kx_2 - \frac{K}{4}x_m &= f_2 \\ M_m\ddot{x}_m + (K_m + \frac{K}{2})x_m - \frac{K}{4}(x_1 + x_2) &= f_m \end{aligned} \tag{30}$$

whose characteristic equation is given by

$$\det \begin{bmatrix} (K - M\,\omega^2) & 0 & -\frac{K}{4} \\ 0 & (K - M\,\omega^2) & -\frac{K}{4} \\ -\frac{K}{4} & -\frac{K}{4} & (K_m + \frac{K}{2} - M_m\,\omega^2) \end{bmatrix} = 0 \tag{31}$$

For computational expediency, divide each of the three rows of the above characteristic matrix by M and using $\omega_n = \sqrt{K/M}$, we have

$$\det \begin{bmatrix} (\omega_n^2 - \omega^2) & 0 & -\frac{\omega_n^2}{4} \\ 0 & (\omega_n^2 - \omega^2) & -\frac{\omega_n^2}{4} \\ -\frac{\omega_n^2}{4} & -\frac{\omega_n^2}{4} & (K_m/M + \frac{\omega_n^2}{2} - M_m/M\,\omega^2) \end{bmatrix} = 0 \tag{32}$$

Introducing the following non-dimensional parameters,

$$\frac{K_m}{M} = \frac{K_m}{K}\,\frac{K}{M} = \kappa_m\omega_n^2, \qquad \mu_m = \frac{M_m}{M} \tag{33}$$

equation(32) becomes

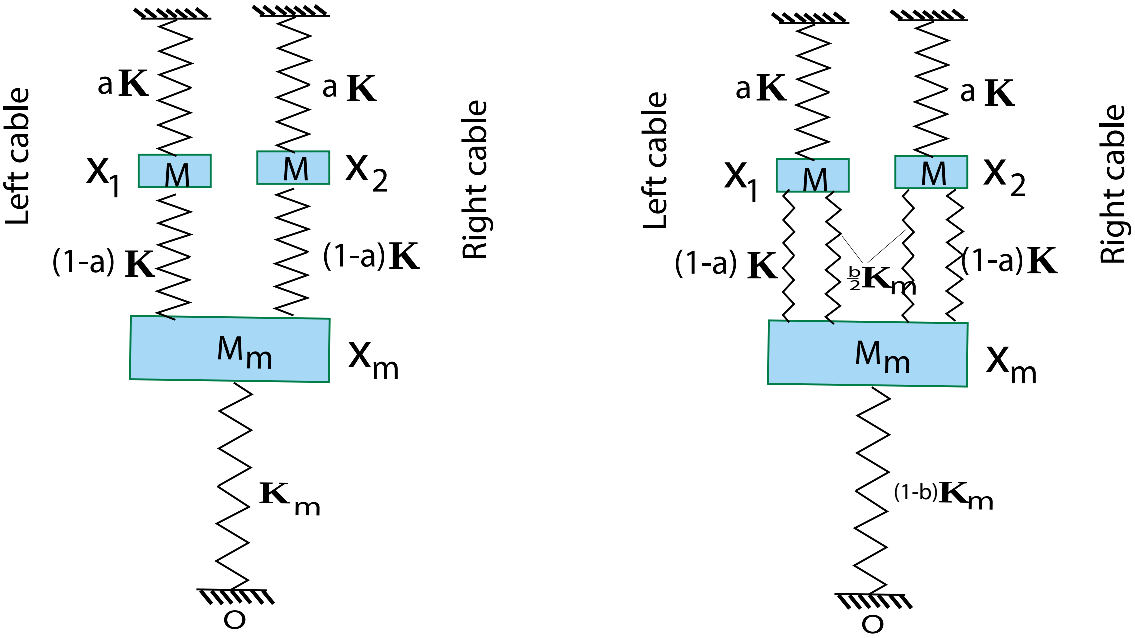
$$\det \begin{bmatrix} (\omega_n^2 - \omega^2) & 0 & -\frac{\omega_n^2}{4} \\ 0 & (\omega_n^2 - \omega^2) & -\frac{\omega_n^2}{4} \\ -\frac{\omega_n^2}{4} & -\frac{\omega_n^2}{4} & (\kappa_m\omega_n^2 + \frac{\omega_n^2}{2} - \mu_m\,\omega^2) \end{bmatrix} = 0 \tag{34}$$

For this characteristic equation, one finds κ_m for the case of $\omega = 1.5\pi$, with $\mu_m = 1$, to be

$$\boxed{\kappa_m = 13.747} \tag{35}$$

which, when compared with the exact solution given by (27), yields about 34% off.

One possible improvement for estimating the middle support spring is to distribute the spring constant K_m as shown in Figure 4.



Possible Improvement of estimating the middle support spring Km by distributing Km into three ways as shown on the right in the above figure.

Figure 4 Possible Improvement in modeling the support spring

Whether this conjecture will be valid or not is left as an exercise to curious minds.

Better yet, one may employ each half of the cable by a two-DOF model, instead of the one-DOF model I used herein. In any case, you now have a glimpse of ”old-fashioned” modeling approaches. Let me know if you would like to learn more about old-fashioned modeling practices.