# Electro-Thermal-Mechanical MEMS Devices

# Multi-Physics Problems

Presentation for ASEN 5519


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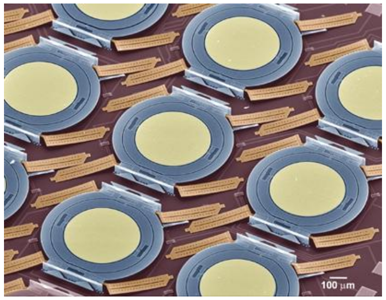
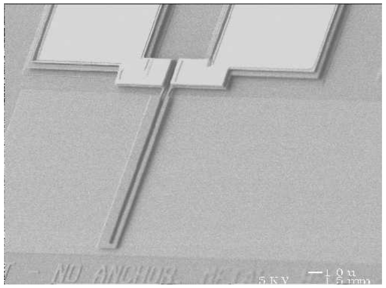
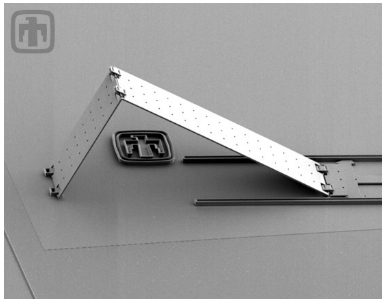
# The ETM Problems

## – Background –

- Electro-thermal-mechanical coupling
  - Joule heating
  - Thermal expansion

- Application
  - MEMS



# The ETM Problem: Electrostatic

## – Governing Physics –

Poisson Equation

$$\frac{\partial}{\partial x}\left(\sigma_{xx}\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(\sigma_{yy}\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(\sigma_{zz}\frac{\partial V}{\partial z}\right) = -z$$

Isotopic conduction ($\sigma = \sigma_{xx} = \sigma_{yy} = \sigma_{zz}$)

$$\sigma \nabla^2 V = -z$$

Electrostatic conduction finite element discretization

$$\begin{aligned} R_E = \tilde{\mathbf{K}_\mathbf{E}}\mathbf{V} - \mathbf{Z} &= \mathbf{0} \\ \mathbf{V} &= \hat{\mathbf{V}} \text{ on } \boldsymbol{\Omega}_\mathbf{V} \\ \mathbf{Z} &= \hat{\mathbf{Z}} \text{ on } \boldsymbol{\Omega}_\mathbf{Z} \end{aligned}$$

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# The ETM Problem: Coupling

## – Electrothermal –

Joule Heating

$$
\begin{aligned}
q &= \mathbf{J} \cdot \mathbf{E} \\
\mathbf{J} &= \sigma \mathbf{E} \\
\mathbf{E} &= \nabla V
\end{aligned}
$$

Element expression

$$\mathbf{Q_J} = \int_{\mathbf{\Omega}} \sigma \mathbf{E^T E d \Omega}$$

# The ETM Problem: Thermal

## – Governing Physics –

Poisson Equation

$$\frac{\partial}{\partial x}\left(k_{xx}\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_{yy}\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k_{zz}\frac{\partial T}{\partial z}\right)=-q$$

Isotopic conduction $(k = k_{xx} = k_{yy} = k_{zz})$

$$k\nabla^2 T = -q$$

Conduction finite element discretization

$$\begin{aligned} R_Q = \tilde{\mathbf{K_Q}}\mathbf{T} - \mathbf{Q}^{\mathbf{ext}} &= \mathbf{0} \\ \mathbf{T} = \hat{\mathbf{T}} & \quad \text{on} \;\; \Omega_T \\ \nabla\mathbf{T} = \hat{\mathbf{f}} & \quad \text{on} \;\; \Omega_F \end{aligned}$$

# The ETM Problem: Thermal

## – Governing Physics –

$$
\begin{aligned}
R_Q = \tilde{\mathbf{K}_{\mathbf{Q}}}\mathbf{T} - \mathbf{Q}^{\mathbf{ext}} &= \mathbf{0} \\
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\nabla \mathbf{T} = \hat{\mathbf{f}} & \quad \text{on} \quad \Omega_F
\end{aligned}
$$

- $\mathbf{Q}^{\mathbf{ext}}$ contributions

  1. Joule heating
  2. Convection
  3. Radiation

# The ETM Problem: Thermal

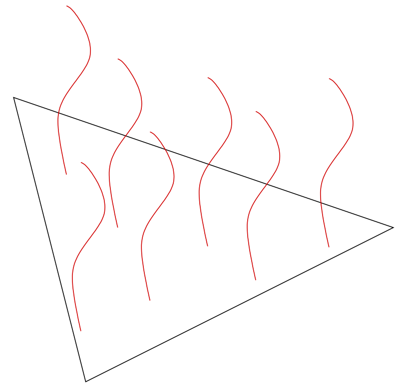
## – Convection –

Convective bc's proportional to temperature

$$\mathbf{Q_c} \propto \mathbf{h}(\mathbf{T}_\infty - \mathbf{T})$$

For shell elements convecting through one lateral side

$$\mathbf{Q_c} = \tilde{\mathbf{K}}_\mathbf{c}(\mathbf{T}_\infty - \mathbf{T})$$



# The ETM Problem: Thermal

## – Radiation –

Blackbody radiation proportional to temperature to the fourth

$$\mathbf{Q_r} \propto \mathbf{h_r}(\mathbf{T_r^4} - \mathbf{T^4})$$

Element radiation from one end will produce load vector

$$\mathbf{Q_r} = \int_{\mathbf{\Omega}} \mathbf{N^{(e)}} \mathbf{h_r}(\mathbf{T_r^4} - \mathbf{T^4}) \mathbf{d\Omega}$$

Linearization of residual

$$\begin{aligned} \left( \tilde{\mathbf{K_Q}} + \tilde{\mathbf{K_c}} - \frac{\partial \mathbf{Q_r}}{\partial \mathbf{T}} \right) \mathbf{\Delta T} \quad &+ \quad R_Q(\mathbf{T}*) = \mathbf{0} \\ \mathbf{K_Q} \mathbf{\Delta T} \quad &+ \quad R_Q(\mathbf{T}*) = \mathbf{0} \end{aligned}$$

# The ETM Problem: Coupling

## – Structural-Thermal –

Thermal strain (in plane)

$$e_{ij}^t = \begin{cases} \alpha_{ij}(T - T_0) & \text{if } i = j = 1,2 \\ 0 & \text{else} \end{cases}$$

Elemental external force

$$\mathbf{f}^{ext} = \int_\Omega C_{ijkl} e_{kl}^t d\Omega$$

- Element orientation independent force in local coord

# The ETM Problem: Structure

## – Governing Physics –

Continuum mechanics

$$
\begin{aligned}
\nabla\sigma_{ij} + p_i &= 0 \\
u_i = \bar{u}_i & \qquad \text{on} \;\; \Omega_u \\
\sigma_{ij} n_j = \hat{t}_i & \qquad \text{on} \;\; \Omega_t
\end{aligned}
$$

TPE functional

$$
\begin{aligned}
\Pi[\mathbf{u}] &= U - W = \int_\Omega \sigma_{ij} e_{ij} d\Omega - \int_\Omega p_i u_i d\Omega - \int_{\Omega_t} \hat{t}_i n_i d\Omega_t \\
& \sigma_{ij} = C_{ijkl}(e_{kl} - e^t_{kl})
\end{aligned}
$$

# The ETM Problem

## – Governing Equations –

Residual finite element equations in each domain

Structural: $R_S = |\mathbf{f}^{\mathbf{int}}(\mathbf{u}) - \mathbf{f}^{\mathbf{ext}}(\mathbf{u}, \mathbf{T})| = \mathbf{0}$

Thermal: $R_Q = |\mathbf{Q}^{\mathbf{int}}(\mathbf{u}, \mathbf{T}) - \mathbf{Q}^{\mathbf{ext}}(\mathbf{u}, \mathbf{V})| = \mathbf{0}$

Electrostatic: $R_E = |\mathbf{Z}^{\mathbf{int}}(\mathbf{u}, \mathbf{V}) - \mathbf{Z}^{\mathbf{ext}}| = \mathbf{0}$

$$\begin{bmatrix} \mathbf{K_T} & -\frac{\partial \mathbf{f}^{\mathbf{ext}}}{\partial \mathbf{T}} & \mathbf{0} \\ \frac{\partial R_Q}{\partial \mathbf{u}} & \mathbf{K_Q} & -\frac{\partial \mathbf{Q}^{\mathbf{ext}}}{\partial \mathbf{V}} \\ \frac{\partial R_E}{\partial \mathbf{u}} & \mathbf{0} & \mathbf{K_E} \end{bmatrix} \begin{bmatrix} \Delta \mathbf{u}^{(\mathbf{n})} \\ \Delta \mathbf{T}^{(\mathbf{n})} \\ \Delta \mathbf{V}^{(\mathbf{n})} \end{bmatrix} = \begin{bmatrix} -R_S^{(n)} \\ -R_Q^{(n)} \\ -R_E^{(n)} \end{bmatrix}$$

$$\mathbf{u}^{(\mathbf{n+1})} = \mathbf{u}^{(\mathbf{n})} + \boldsymbol{\Delta}\mathbf{u}^{(\mathbf{n})}$$

$$\mathbf{T}^{(\mathbf{n+1})} = \mathbf{T}^{(\mathbf{n})} + \boldsymbol{\Delta}\mathbf{T}^{(\mathbf{n})}$$

$$\mathbf{V}^{(\mathbf{n+1})} = \mathbf{V}^{(\mathbf{n})} + \boldsymbol{\Delta}\mathbf{V}^{(\mathbf{n})}$$

# The ETM Problem

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Electrostatic: $R_E = |\mathbf{Z}^{\mathbf{int}}(\mathbf{u}, \mathbf{V}) - \mathbf{Z}^{\mathbf{ext}}| = \mathbf{0}$

$$\begin{bmatrix} \mathbf{K_T} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{K_Q} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{K_E} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{u}^{(\mathbf{n})} \\ \Delta\mathbf{T}^{(\mathbf{n})} \\ \Delta\mathbf{V}^{(\mathbf{n})} \end{bmatrix} = \begin{bmatrix} -R_S(\mathbf{u}^{(\mathbf{n})}, \mathbf{T}^{(\mathbf{n})}) \\ -R_Q(\mathbf{u}^{(\mathbf{n})}, \mathbf{T}^{(\mathbf{n})}, \mathbf{V}^{(\mathbf{n})}) \\ -R_E(\mathbf{u}^{(\mathbf{n})}, \mathbf{V}^{(\mathbf{n})}) \end{bmatrix}$$

$$\mathbf{u}^{(\mathbf{n+1})} = \mathbf{u}^{(\mathbf{n})} + \boldsymbol{\Delta}\mathbf{u}^{(\mathbf{n})}$$

$$\mathbf{T}^{(\mathbf{n+1})} = \mathbf{T}^{(\mathbf{n})} + \boldsymbol{\Delta}\mathbf{T}^{(\mathbf{n})}$$

$$\mathbf{V}^{(\mathbf{n+1})} = \mathbf{V}^{(\mathbf{n})} + \boldsymbol{\Delta}\mathbf{V}^{(\mathbf{n})}$$

# The ETM Problem: Coupling

## – One Way Coupling –

The Linearized operator

$$\Phi = \begin{bmatrix} \mathbf{K_T} & -\frac{\partial \mathbf{f}^{\mathbf{ext}}}{\partial \mathbf{T}} & 0 \\ \frac{\partial R_Q}{\partial \mathbf{u}} & \mathbf{K_Q} & -\frac{\partial \mathbf{Q}^{\mathbf{ext}}}{\partial \mathbf{V}} \\ \frac{\partial R_E}{\partial \mathbf{u}} & 0 & \mathbf{K_E} \end{bmatrix}$$

Assume thermal and electrostatic independent of structural response

$$\frac{\partial R_Q}{\partial \mathbf{u}} = \mathbf{0}$$

$$\frac{\partial R_E}{\partial \mathbf{u}} = \mathbf{0}$$

# The ETM Problem: Coupling

## – One Way Coupling –

The "one way" linearized operator

$$\Phi = \begin{bmatrix} \mathbf{K_T} & -\frac{\partial \mathbf{f^{ext}}}{\partial \mathbf{T}} & 0 \\ 0 & \mathbf{K_Q} & -\frac{\partial \mathbf{Q^{ext}}}{\partial \mathbf{V}} \\ 0 & 0 & \mathbf{K_E} \end{bmatrix}$$

- Eliminate costly computations
- Upper diagonal $\rightarrow$ one solve per domain
- Effect on solution/accuracy?

# The ETM Problem: Verification

## – Energy Balance –

- Check that power balance is correct

- Electrical Power in equals thermal dissipation

- Electrical Power: $P = IV = I^2R = V/R^2$

  $I = \int \mathbf{J} \cdot \hat{\mathbf{n}} dA$

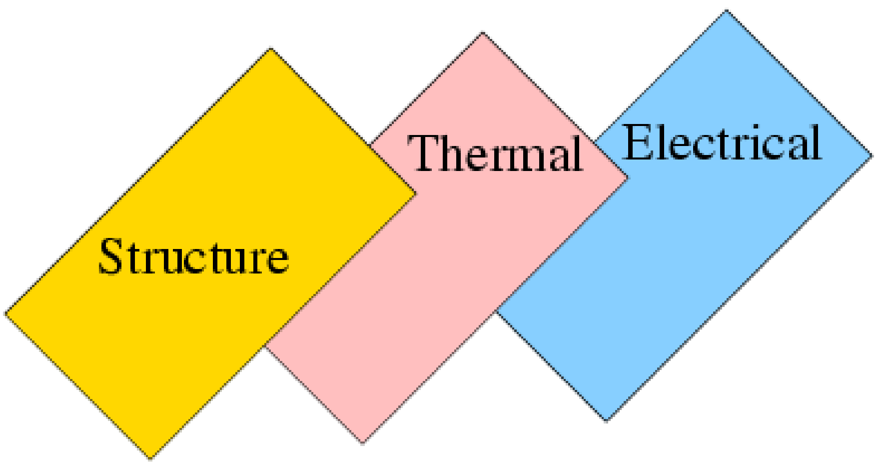
- Thermal Power: $P = \int \hat{\mathbf{f}} \cdot \hat{\mathbf{n}} dA$

  $\hat{\mathbf{f}} = k \nabla T$

# The ETM Problem: Solution

## – Solution Procedure –

- How does one run these problems
  - Modularity
  - Parallel computation
- One-to-one interface
- Body is the interface



# The ETM Problem: Solution

## – Modularity –

- Which values should be sent from code to code?

- Codes should be kept "dumb" to each other
  - Thermal codes at the middle of coupling
  - Master-slave relationship

# The ETM Problem: Solution

## – Parallel computation –

- Modular requirements

- Split problem into manageable sizes

$$\Phi \quad = \quad \begin{bmatrix} \mathbf{K_T} & -\frac{\partial \mathbf{f^{ext}}}{\partial \mathbf{T}} & 0 \\ 0 & \mathbf{K_Q} & -\frac{\partial \mathbf{Q^{ext}}}{\partial \mathbf{V}} \\ 0 & 0 & \mathbf{K_E} \end{bmatrix}$$

If domains have 10k nodes, full system is larger than 30k by 30k!

# The ETM Problem: Solution

## – Parallel computation –

- Handling communications between domains (e.g. MPI)
- Nontrivial issue
- Pass minimum amount of information (vectors vs matrix)
- Computations carried out within each domain
- Resultants sent back

# The ETM Problem: Parallel Computation

## – Optimization –

1. Problem definition

$$\min_{s} z(\mathbf{s})$$

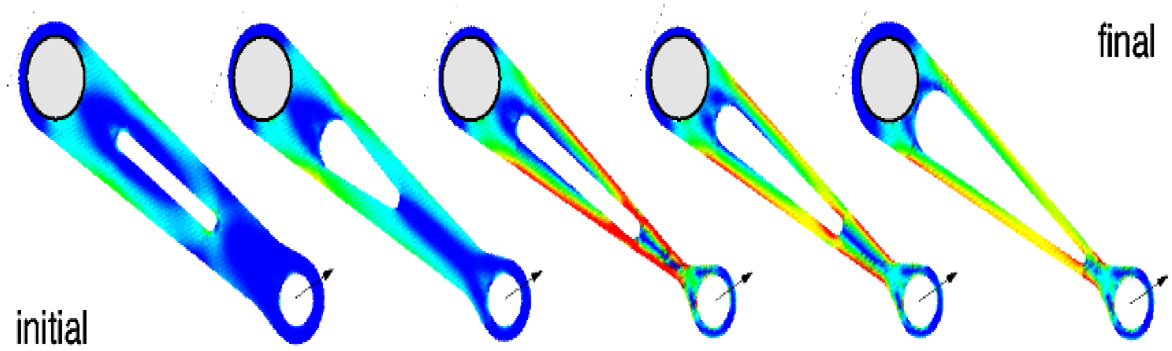
$$\begin{aligned} g_i(\mathbf{s}) &\geq 0 \\ h_i(\mathbf{s}) &= 0 \\ \mathbf{s_L} &\leq \mathbf{s} \leq \mathbf{s_U} \end{aligned}$$

2. Analysis of physical fields (FEM)
   - Optimization criteria $q = q(\mathbf{s}, \mathbf{u}(\mathbf{s}), \mathbf{T}(\mathbf{s}), \mathbf{V}(\mathbf{s}))$
   - State variables $\mathbf{u} = \mathbf{u}(\mathbf{s})$, $\mathbf{T} = \mathbf{T}(\mathbf{s})$, $\mathbf{V} = \mathbf{V}(\mathbf{s})$

# ETM Optimization

## – Background –

- Linking optimization variables to physical problem
  - Size optimization
  - Shape optimization



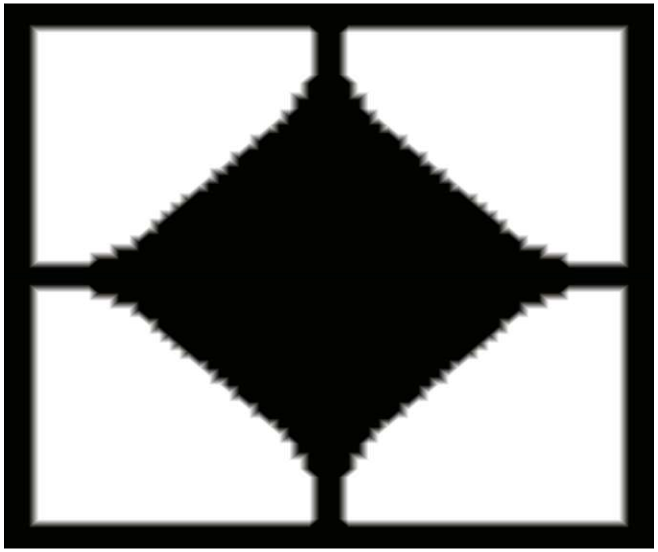
# ETM Optimization

## – Background –

- Topology optimization is a material distribution problem
  - Indicator function $\chi$

$$\chi(r) = \begin{cases} 1 & r \in \Omega_1 \\ 0 & r \in \Omega_0 \end{cases}$$

$$\Omega = \Omega_0 \cup \Omega_1$$



# ETM Optimization

## – Sensitivity Analysis –

Require criteria gradients

$$\frac{dq_j}{ds_i} = \frac{\partial q_j}{\partial s_i} + \frac{\partial q_j}{\partial \mathbf{u}}\frac{d\mathbf{u}}{ds_i} + \frac{\partial q_j}{\partial \mathbf{T}}\frac{d\mathbf{T}}{ds_i} + \frac{\partial q_j}{\partial \mathbf{V}}\frac{d\mathbf{V}}{ds_i}$$

Differentiate Finite Element Residual Equations with respect to $s_i$

$$\begin{bmatrix} \mathbf{K_T} & -\frac{\partial \mathbf{f}^{\mathbf{ext}}}{\partial \mathbf{T}} & 0 \\ \frac{\partial R_Q}{\partial \mathbf{u}} & \mathbf{K_Q} & -\frac{\partial \mathbf{Q}^{\mathbf{ext}}}{\partial \mathbf{V}} \\ \frac{\partial R_E}{\partial \mathbf{u}} & 0 & \mathbf{K_E} \end{bmatrix} \begin{bmatrix} \frac{d\mathbf{u}}{ds_i} \\ \frac{d\mathbf{T}}{ds_i} \\ \frac{d\mathbf{V}}{ds_i} \end{bmatrix} = \Phi \begin{bmatrix} \frac{d\mathbf{u}}{ds_i} \\ \frac{d\mathbf{T}}{ds_i} \\ \frac{d\mathbf{V}}{ds_i} \end{bmatrix} = \begin{bmatrix} -\frac{\partial R_S}{\partial s_i} \\ -\frac{\partial R_Q}{\partial s_i} \\ -\frac{\partial R_E}{\partial s_i} \end{bmatrix}$$

# ETM Optimization

## – Sensitivity Analysis –

$\Rightarrow$ Rewrite criteria sensitivity expression

$$\frac{dq_j}{ds_i} = \frac{\partial q_j}{\partial s_i} - \left[ \begin{array}{ccc} \frac{\partial q_j}{\partial \mathbf{u}} & \frac{\partial q_j}{\partial \mathbf{T}} & \frac{\partial q_j}{\partial \mathbf{V}} \end{array} \right] \mathbf{\Phi}^{-1} \left[ \begin{array}{c} \frac{\partial R_S}{\partial s_i} \\ \frac{\partial R_Q}{\partial s_i} \\ \frac{\partial R_E}{\partial s_i} \end{array} \right]$$

- Matrix-vector solves required
- Minimize this expensive procedure
  - Direct vs Adjoint

# ETM Optimization

## – Staggered Solution –

$$\begin{bmatrix} \mathbf{K_T} & -\frac{\partial \mathbf{f}^{\text{ext}}}{\partial \mathbf{T}} & 0 \\ \frac{\partial R_Q}{\partial \mathbf{u}} & \mathbf{K_Q} & -\frac{\partial \mathbf{Q}^{\text{ext}}}{\partial \mathbf{V}} \\ \frac{\partial R_E}{\partial \mathbf{u}} & 0 & \mathbf{K_E} \end{bmatrix} \begin{bmatrix} \frac{d\mathbf{u}}{ds_i} \\ \frac{d\mathbf{T}}{ds_i} \\ \frac{d\mathbf{V}}{ds_i} \end{bmatrix} = \begin{bmatrix} -\frac{\partial R_S}{\partial s_i} \\ -\frac{\partial R_Q}{\partial s_i} \\ -\frac{\partial R_E}{\partial s_i} \end{bmatrix}$$

Gauss-Seidel solve on submatricies

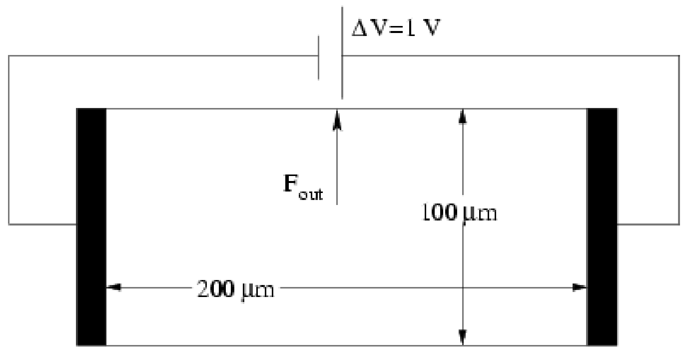
$$\begin{bmatrix} \mathbf{K_T} & 0 & 0 \\ 0 & \mathbf{K_Q} & 0 \\ 0 & 0 & \mathbf{K_E} \end{bmatrix} \begin{bmatrix} \frac{d\mathbf{u}}{ds_i} \\ \frac{d\mathbf{T}}{ds_i} \\ \frac{d\mathbf{V}}{ds_i} \end{bmatrix} = \begin{bmatrix} -\frac{\partial R_S}{\partial s_i} + \frac{\partial \mathbf{f}^{\text{ext}}}{\partial \mathbf{T}} \frac{d\mathbf{T}}{ds_i} \\ -\frac{\partial R_Q}{\partial s_i} - \frac{\partial R_Q}{\partial \mathbf{u}} \frac{d\mathbf{u}}{ds_i} + \frac{\partial \mathbf{Q}^{\text{ext}}}{\partial \mathbf{V}} \frac{d\mathbf{V}}{ds_i} \\ -\frac{\partial R_E}{\partial s_i} - \frac{\partial R_E}{\partial \mathbf{u}} \frac{d\mathbf{u}}{ds_i} \end{bmatrix}$$

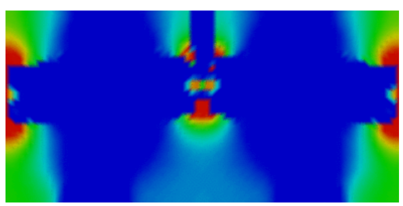
$\Rightarrow$ Effect of one way coupling? $\Leftarrow$

# ETM Optimization

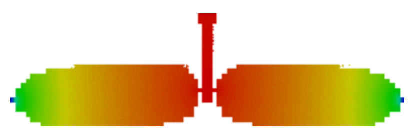
## – Examples –

- Maximize output force
- Uses less than 30% of available mass



(d) Current Density



(e) Temperature

(f) Deformation

# ETM Optimization

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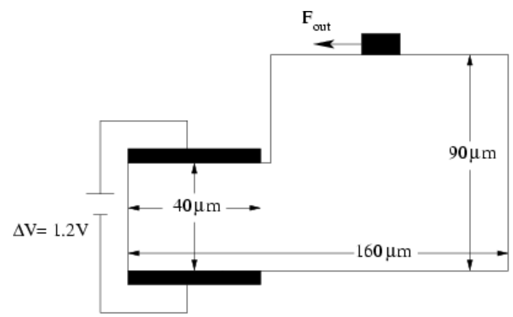


(a) Current Density

(b) Temperature

(c) Deformation

# ETM Optimization

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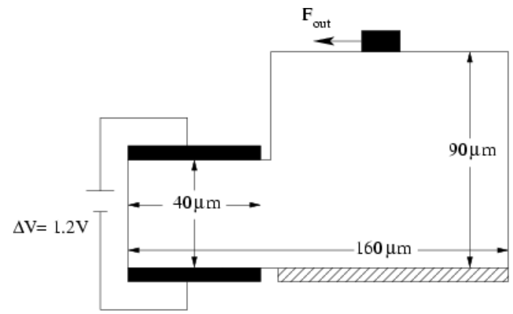
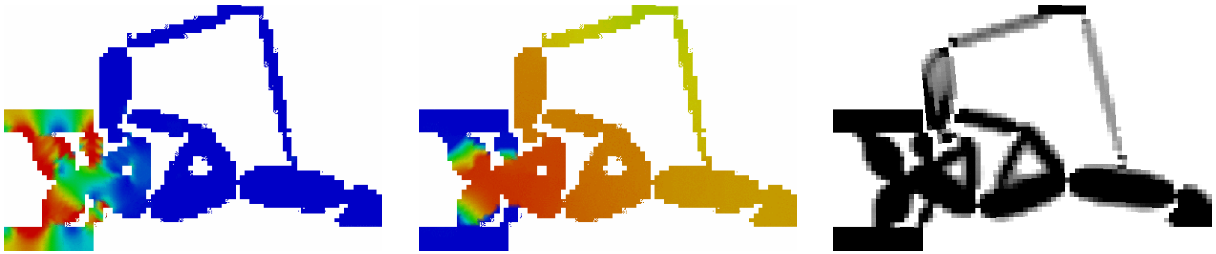
- Uses less than 30% of available mass



(a) Current Density (b) Temperature (c) Deformation