

# 13.811

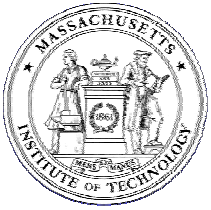
# Advanced Structural Dynamics and Acoustics

Lectures: MW 2.30-4.00, Rm. 5-233  
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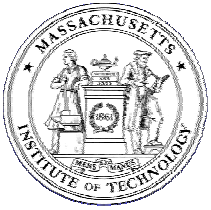
Office hours: Thursday 4-5

Textbook: Fourier Acoustics  
E.G. Williams  
Academic Press, 1999.

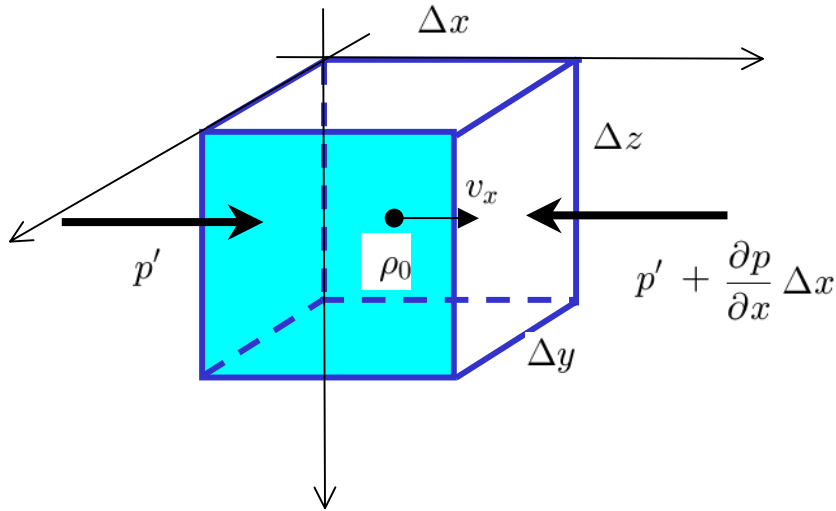


**13.811**  
**Spring 2004 Class Schedule**

Date	Lecturer	Lecture Topic	Chap.	Hwk. Due
29- Mar	Schmidt	Introduction. Overview of Ocean Acoustics	2.1-2.3	
31- Mar	Schmidt	Acoustic wave equation. Plane waves. Wavenumber representation.	2.4-2.6	
5- Apr	Schmidt	Radiation from infinite plate. 2-d wavenumber space. Spherical waves from point source.	2.7-2.8	
7- Apr	Schmidt	Directivity function. Radiation from baffled rectangular piston.	2.11.1-2.11.5	
12- Apr	Schmidt	Radiation from planar sources.	2.11.6, 2.12-2.13	
14- Apr	Schmidt	Radiation from elastic plates	2.14-2.15	#1
19- Apr	n/a	No classes		
21- Apr		Quiz		
26- Apr	Battle	Spherical waves. Spherical harmonics.	6.1-6.4	
28- Apr	Battle	Spherical multipole radiation	6.5-6.6	#2
3- May	Battle	Scattering from Spheres	6.1	
5- May	Battle	Cylindrical waves	4.1-4.3	#3
10- May	Battle	Helical waves	4.4	
12- May	Battle	Radiation from cylinders	4.5-4.7	#4
<b>Text: E.G. Williams, Fourier Acoustics. Academic Press, 1999</b>				
<b>Background reading: Section 1.</b>				



# The Acoustic Equation of Motion



## Perturbation

$$p = p_0 + p'$$

$$\rho = \rho_0 + \rho'$$

## 1D equation of motion

$$\rho_0 \Delta x \Delta y \Delta z \frac{\partial v_x}{\partial t} = - \left( \frac{\partial p}{\partial x} \Delta x \right) \Delta y \Delta z$$

$$\Rightarrow \rho_0 \frac{\partial v_x}{\partial t} = - \frac{\partial p}{\partial x}$$

## Other Coordinates

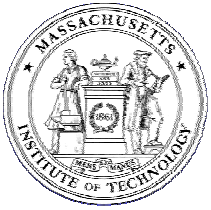
$$\rho_0 \frac{\partial v_y}{\partial t} = - \frac{\partial p}{\partial y}$$

$$\rho_0 \frac{\partial v_z}{\partial t} = - \frac{\partial p}{\partial z}$$

## Equation of Motion

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = - \nabla p'$$

$$\nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$



# The Acoustic Wave Equation

Conservation of Mass

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \rho \mathbf{v}$$

Equation of Motion

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\nabla p'$$

Constitutive Equation

$$p = p_0 + \rho' \left[ \frac{\partial p}{\partial \rho} \right]_S + \frac{1}{2} (\rho')^2 \left[ \frac{\partial^2 p}{\partial \rho^2} \right]_S + \dots$$

Speed of Sound

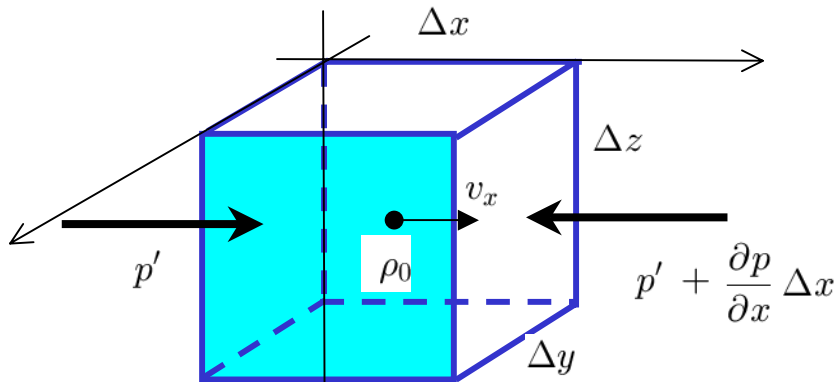
$$c^2 \equiv \left[ \frac{\partial p}{\partial \rho} \right]_S \quad (\text{Sound speed})$$

Linearly Elastic Fluid

$$p = p_0 + p'$$

$$\rho = \rho_0 + \rho'$$

$$p' = \rho' c^2.$$

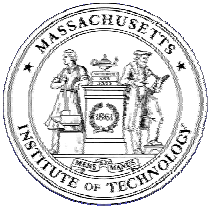


Pressure Wave Equation

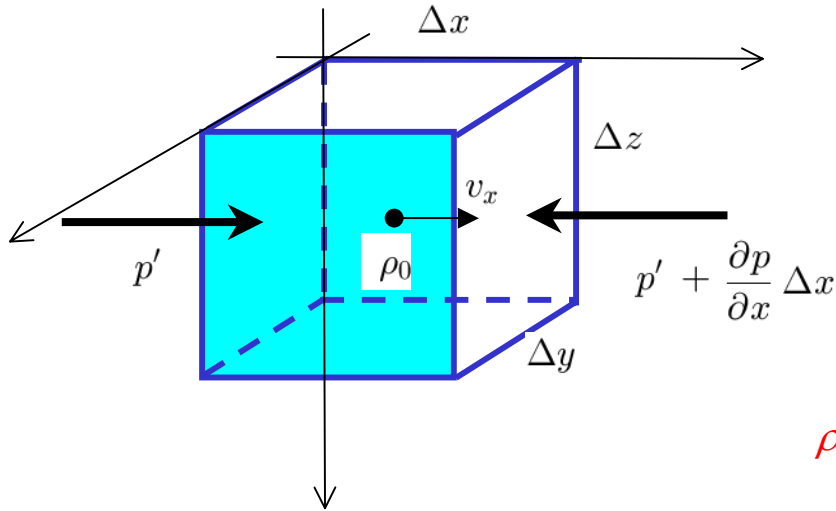
$$\rho \nabla \cdot \left( \frac{1}{\rho} \nabla p \right) - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = 0,$$

Particle Velocity Wave Equation

$$\frac{1}{\rho} \nabla (\rho c^2 \nabla \cdot \mathbf{v}) - \frac{\partial^2 \mathbf{v}}{\partial t^2} = \mathbf{0}.$$



# Potential Wave Equations



$\rho$  constant

## Pressure Wave Equation

$$\rho \nabla \cdot \left( \frac{1}{\rho} \nabla p \right) - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = 0,$$

## Particle Velocity Wave Equation

$$\frac{1}{\rho} \nabla (\rho c^2 \nabla \cdot \mathbf{v}) - \frac{\partial^2 \mathbf{v}}{\partial t^2} = \mathbf{0}.$$

## Wavefield Potentials

### Velocity Potential

$$\mathbf{v} = \nabla \phi,$$

$$\nabla \left( c^2 \nabla^2 \phi - \frac{\partial^2 \phi}{\partial t^2} \right) = \mathbf{0}.$$

$$\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = 0,$$

### Displacement Potential

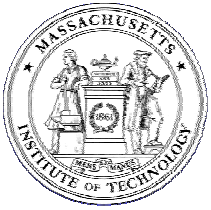
$$\mathbf{u} = \nabla \psi,$$

$$\nabla^2 \psi - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = 0.$$

$$p = -K \nabla^2 \psi,$$

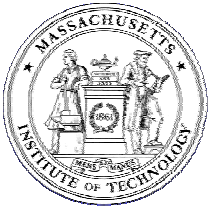
$$K = \rho c^2.$$

$$p = -\rho \frac{\partial^2 \psi}{\partial t^2}.$$



# Solution of the Wave Equations

- 4-D Partial Differential Equation
- Analytical solutions only for few canonical problems
- Direct Numerical Solution (FDM, FEM)
  - Computationally intensive ( $\Delta x \ll \lambda$ ,  $\Delta t \ll T$ ).
- Dimension Reduction for PDE
  - Geometrical symmetries (Plane or axisymmetric problems)
  - Integral transforms
  - Analytical or numerical solution of ODE or low dimensional PDE.
  - Evaluation of inverse transforms (analytical or numerical)



# Helmholtz Equation

## Frequency-Time Fourier Transform

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\omega) e^{-i\omega t} d\omega ,$$

$$f(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt ,$$

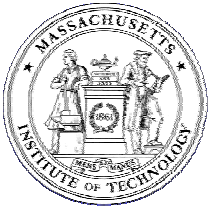
*Helmholtz Equation*

$$[\nabla^2 + k^2(\mathbf{r})] \psi(\mathbf{r}, \omega) = 0 ,$$

$$k(\mathbf{r}) = \frac{\omega}{c(\mathbf{r})} .$$

## Solution of Helmholtz Equation

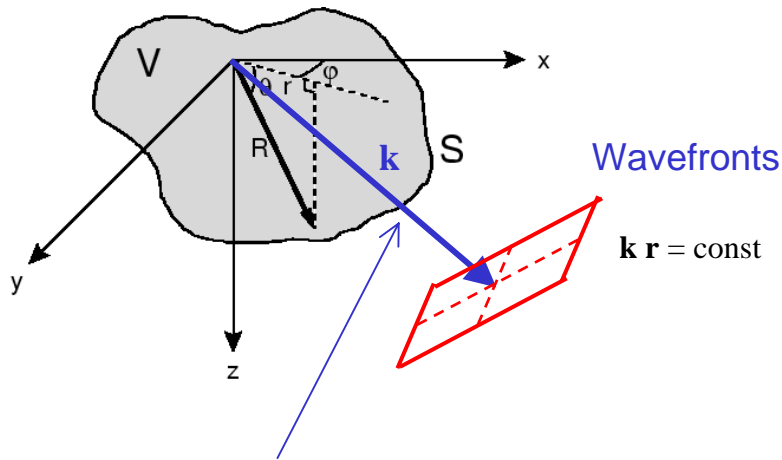
- Dimensionality of the problem.
- Medium wavenumber variation  $k(\mathbf{r})$ , i.e., the sound speed variation  $c(\mathbf{r})$ .
- Boundary conditions.
- Source–receiver geometry.
- Frequency and bandwidth.



# Helmholtz Equation Homogeneous Media

Helmholts Equation

$$[\nabla^2 + k^2] \psi(\mathbf{r}, \omega) = 0,$$



Wavevector  $\mathbf{k}$  determines propagation direction of plane waves

Laplacian in Cartesian Coordinates

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},$$

Plane Waves

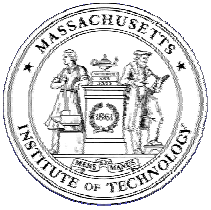
$$\psi(x, y, z) = \begin{cases} A e^{i\mathbf{k}\cdot\mathbf{r}} \\ B e^{-i\mathbf{k}\cdot\mathbf{r}} \end{cases},$$

Wavevector

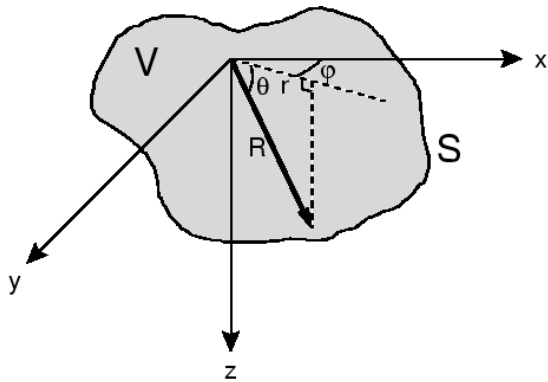
$$\mathbf{k} = (k_x, k_y, k_z) \\ |\mathbf{k}| = k = \omega/c$$

1-D propagation:  $k_y, k_z = 0$ :

$$\psi(x) = \begin{cases} A e^{ikx} & \text{Forward propagating} \\ B e^{-ikx} & \text{Backward propagating} \end{cases}$$



# Helmholtz Equation Homogeneous Media



## Cylindrical Coordinates

$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} .$$

*Axial Symmetry*

$$\left[ \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + k^2 \right] \psi(r) = 0 ,$$

*bessel Functions*

$$\psi(r) = \begin{cases} A J_0(kr) \\ B Y_0(kr) , \end{cases}$$

*Hankel Functions*

$$\psi(r) = \begin{cases} CH_0^{(1)}(kr) = C [J_0(kr) + iY_0(kr)] \\ DH_0^{(2)}(kr) = D [J_0(kr) - iY_0(kr)] . \end{cases}$$

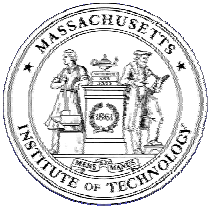
## Spherical Coordinates

$$\left[ \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + k^2 \right] \psi(r) = 0 ,$$

$$\psi(r) = \begin{cases} (A/r) e^{ikr} & \text{Diverging waves} \\ (B/r) e^{-ikr} & \text{Converging waves} \end{cases}$$

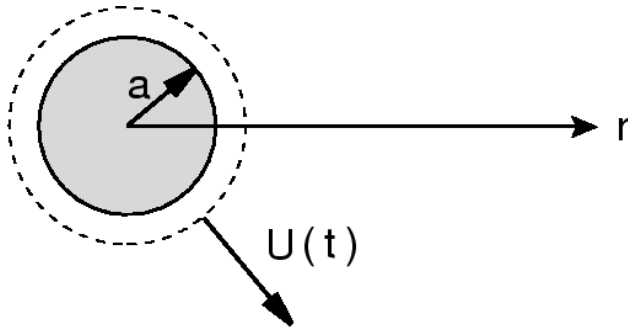
$$H_0^{(1)}(kr) \simeq \sqrt{\frac{2}{\pi kr}} e^{i(kr - \pi/4)} \text{ Diverging waves}$$

$$H_0^{(2)}(kr) \simeq \sqrt{\frac{2}{\pi kr}} e^{-i(kr - \pi/4)} \text{ Converging waves}$$



# Radiation of Sound

## The Point Source



Unbounded Homogeneous Medium  
Frequency Domain

$$u_r(a) = U(\omega) .$$

*Spherical geometry solution*

$$\psi(r) = A \frac{e^{ikr}}{r} ,$$

$$u_r(r) = \frac{\partial \psi(r)}{\partial r} = A e^{ikr} \left( \frac{ik}{r} - \frac{1}{r^2} \right) .$$

Simple Point Source

$$ka \ll 1$$

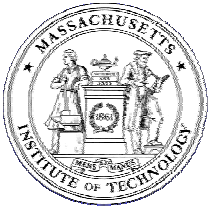
$$u_r(a) = A e^{ika} \frac{ika - 1}{a^2} \simeq -\frac{A}{a^2} ,$$

$$A = -a^2 U(\omega) .$$

$\Rightarrow$

$$\psi(r) = -S_\omega \frac{e^{ikr}}{4\pi r} .$$

$$S_\omega = 4\pi a^2 U(\omega)$$



# Green's Function

$$g_{\omega}(r, 0) = \frac{e^{ikr}}{4\pi r},$$

Source at  $r_0$

$$g_{\omega}(\mathbf{r}, \mathbf{r}_0) = \frac{e^{ikR}}{4\pi R}, \quad R = |\mathbf{r} - \mathbf{r}_0|.$$

## Helmholtz Equation for Green's function

$$[\nabla^2 + k^2] g_{\omega}(\mathbf{r}, \mathbf{r}_0) = -\delta(\mathbf{r} - \mathbf{r}_0),$$

Integrate over spherical volume  $V$  of radius  $\epsilon \rightarrow 0$ :

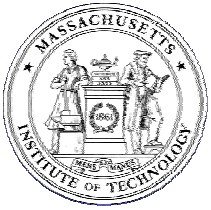
$$\int_V -\delta(\mathbf{r} - \mathbf{r}_0) dV = -1$$

$$\int_V k^2 g_{\omega}(\mathbf{r}, \mathbf{r}_0) dV \rightarrow_{\epsilon \rightarrow 0} 0$$

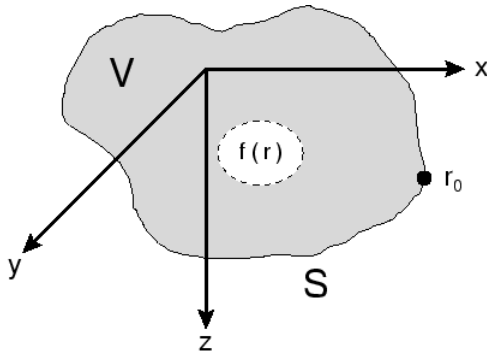
$$\begin{aligned} \int_V \nabla^2 g_{\omega}(\mathbf{r}, \mathbf{r}_0) dV &= \int_S \frac{\partial}{\partial R} g_{\omega}(\mathbf{r}, \mathbf{r}_0) dS \\ &= \int_S \left[ -\frac{e^{ik\epsilon}}{4\pi\epsilon^2} + \frac{ike^{ik\epsilon}}{4\pi\epsilon} \right] dS \\ &= 4\pi\epsilon^2 \left[ -\frac{e^{ik\epsilon}}{4\pi\epsilon^2} + \frac{ike^{ik\epsilon}}{4\pi\epsilon} \right] \rightarrow_{\epsilon \rightarrow 0} -1 \end{aligned}$$

*Reciprocity*

$$g_{\omega}(\mathbf{r}, \mathbf{r}_0) = g_{\omega}(\mathbf{r}_0, \mathbf{r}),$$



# Green's Theorem



## Source in Bounded Medium

### Inhomogeneous Helmholtz Equation

$$[\nabla^2 + k^2] \psi(\mathbf{r}) = f(\mathbf{r}) .$$

### General Green's Function

$$G_\omega(\mathbf{r}, \mathbf{r}_0) = g_\omega(\mathbf{r}, \mathbf{r}_0) + H_\omega(\mathbf{r}) ,$$

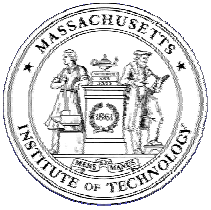
$$[\nabla^2 + k^2] H_\omega(\mathbf{r}) = 0 .$$

$\Rightarrow$

$$[\nabla^2 + k^2] G_\omega(\mathbf{r}, \mathbf{r}_0) = -\delta(\mathbf{r} - \mathbf{r}_0) .$$

### Green's Theorem

$$\psi(\mathbf{r}) = \int_S \left[ G_\omega(\mathbf{r}, \mathbf{r}_0) \frac{\partial \psi(\mathbf{r}_0)}{\partial \mathbf{n}_0} - \psi(\mathbf{r}_0) \frac{\partial G_\omega(\mathbf{r}, \mathbf{r}_0)}{\partial \mathbf{n}_0} \right] dS_0 - \int_V f(\mathbf{r}_0) G_\omega(\mathbf{r}, \mathbf{r}_0) dV_0 ,$$



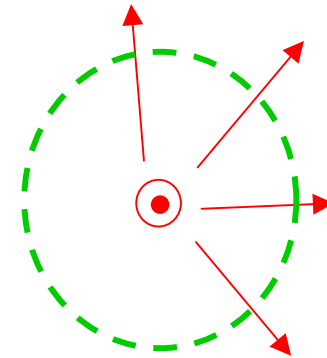
# Source in Infinite Medium

$$\psi(\mathbf{r}) = - \int_V f(\mathbf{r}_0) g_\omega(\mathbf{r}, \mathbf{r}_0) dV_0 .$$

For any imaginary surface enclosing the sources:

$$\int_S \left[ g_\omega(\mathbf{r}, \mathbf{r}_0) \frac{\partial \psi(\mathbf{r}_0)}{\partial \mathbf{n}_0} - \psi(\mathbf{r}_0) \frac{\partial g_\omega(\mathbf{r}, \mathbf{r}_0)}{\partial \mathbf{n}_0} \right] dS_0 = 0 .$$

$$i \Rightarrow \int_S \frac{e^{ikR}}{4\pi R} \left[ \frac{\partial \psi(\mathbf{r}_0)}{\partial R} - ik \psi(\mathbf{r}_0) \right] dS_0 = 0 .$$



## Radiation Condition

$$R \left[ \frac{\partial}{\partial R} - ik \right] \psi(\mathbf{r}_0) \rightarrow 0, \quad R = |\mathbf{r} - \mathbf{r}_0| \rightarrow \infty .$$