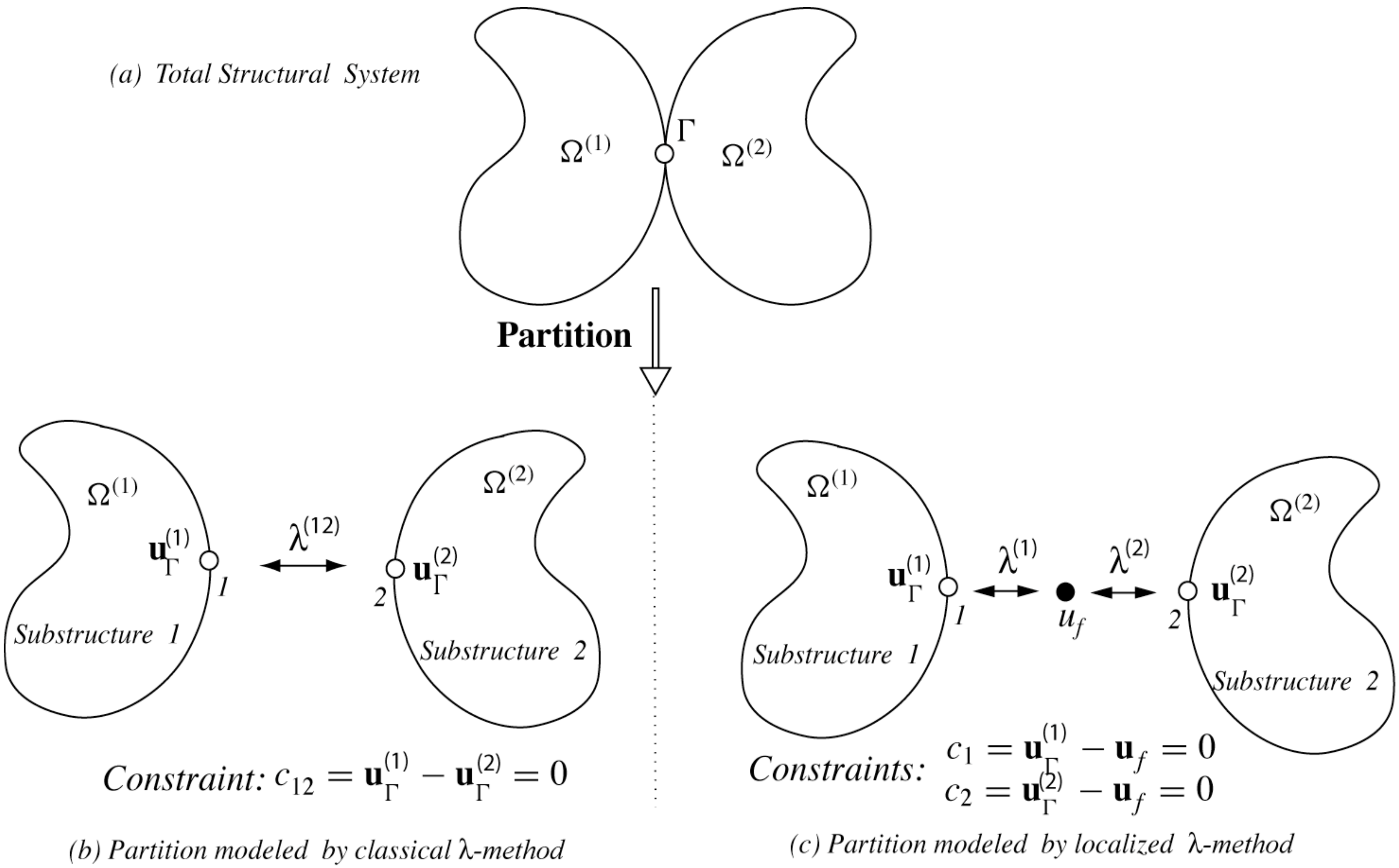
Partitioning of two coupled domains



Functional Relations of Coupling

1. Structure-structure interaction (when interfaces are governed by the same variables):

Partition interface reference variable: $\mathbf{u}_r$

Field A variable: $\mathbf{u}_A$

Field B variable: $\mathbf{u}_B$

$$\Downarrow$$

Interaction Relation: $\begin{Bmatrix} \mathbf{f}(\mathbf{u}_A) \\ \mathbf{g}(\mathbf{u}_B) \end{Bmatrix} - \mathbf{h}(\mathbf{u}_r) = \mathbf{0}$

Functional Relations of Coupling – cont'd

2. Structure-fluid interaction (when interfaces are governed by the different variables):

Interface boundary geometry: $\dot{\Gamma}(\mathbf{X} + \mathbf{u}_s) = \Gamma_f(\dot{\mathbf{u}}_f)$

Variable acting on the fluid is displacement: $\mathbf{u}_s$

Variable acting on the structure is pressure: $\mathbf{p}_f$

$$\Downarrow$$

Interaction Relation: $\mathbf{p}_f = \rho c^2 \, \mathbf{div}(\mathbf{u}_s)$

Classical Treatment of Partitioned Boundaries

Subsystem 1: $\delta\Pi^{(1)} = \delta\mathbf{u}^{(1)^T} [\, \mathbf{A}(\mathbf{u}^{(1)}) - \mathbf{f}^{(1)}]$

Subsystem 2: $\delta\Pi^{(2)} = \delta\mathbf{u}^{(2)^T} [\, \mathbf{A}(\mathbf{u}^{(2)}) - \mathbf{f}^{(2)}]$

Interface functional: $\pi_{classic} = [\boldsymbol{\lambda}^{(12)}(\mathbf{u}_\Gamma^{(1)} - \mathbf{u}_\Gamma^{(2)})]$

$$\mathbf{u}_\Gamma^{(1)} = \mathbf{B}^{(1)}\ \mathbf{u}^{(1)}$$

$$\mathbf{u}_\Gamma^{(2)} = \mathbf{B}^{(2)}\ \mathbf{u}^{(2)}$$

Classical partitioning - cont'd

The variational functional

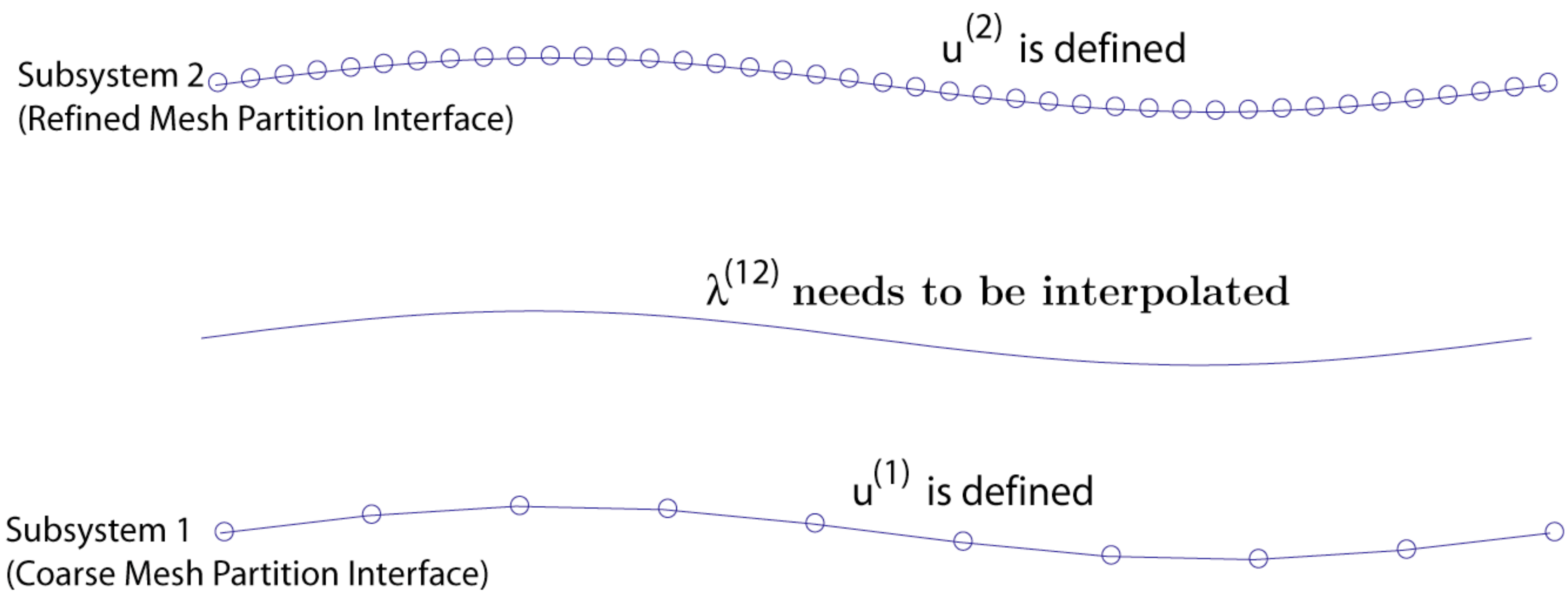
$$\begin{aligned}\delta\Pi_{total} &= \delta\Pi^{(1)} + \delta\Pi^{(2)} + \delta\pi_{classical} \\ &= \delta\mathbf{u}^{(1)^T}\ [\ \mathbf{A}(\mathbf{u}^{(1)}) - \mathbf{f}^{(1)}] + \delta\mathbf{u}^{(2)^T}\ [\ \mathbf{A}(\mathbf{u}^{(2)}) - \mathbf{f}^{(2)}] \\ &\quad \delta[\boldsymbol{\lambda}^{(12)}(\mathbf{B}^{(1)}\ \mathbf{u}^{(1)} - \ \mathbf{B}^{(2)}\ \mathbf{u}^{(2)}]\end{aligned}$$

Partitioned equation set:

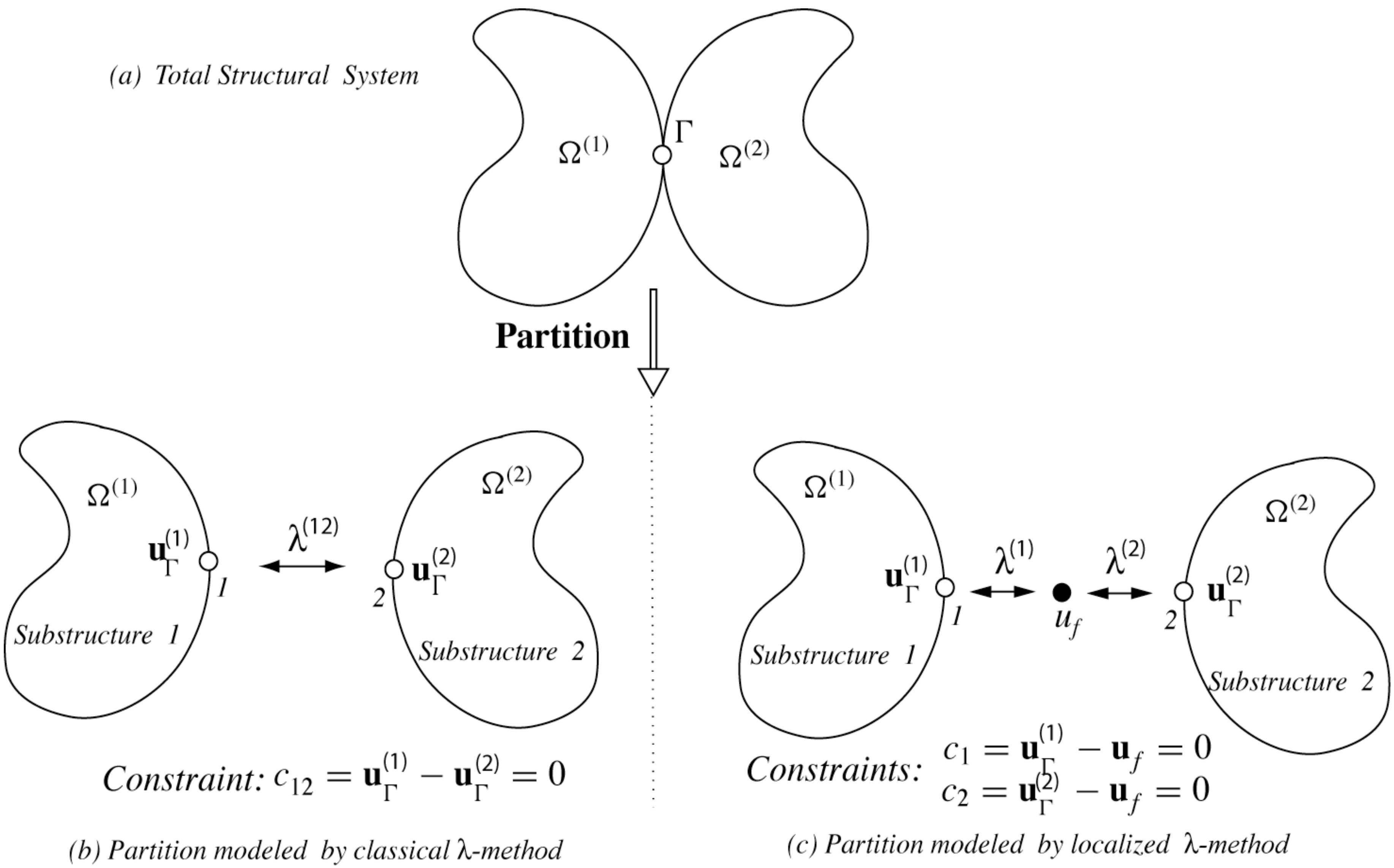
$$\begin{bmatrix} \mathbf{A}^{(1)} & 0 & \mathbf{B}^{(1)} \\ 0 & \mathbf{A}^{(2)} & -\mathbf{B}^{(2)} \\ \mathbf{B}^{(1)^T} & -\mathbf{B}^{(2)^T} & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{u}^{(1)} \\ \mathbf{u}^{(2)} \\ \boldsymbol{\lambda}^{(12)} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}^{(1)} \\ \mathbf{f}^{(2)} \\ \mathbf{0} \end{Bmatrix}$$

Nonmatching Interfaces - Classical form (A mixed interpolation!)

$$\pi_{classic} = \int_{\Gamma} \boldsymbol{\lambda}^{(12)}(\mathbf{x}) \, \{\mathbf{u}_{\Gamma}^{(1)}(\mathbf{x}) - \mathbf{u}_{\Gamma}^{(2)}(\mathbf{x})\} \, d\mathbf{x}$$



Mixed formulations can lead to mixed results!


(a) Total Structural System
$\Omega^{(1)}$
Γ
$\Omega^{(2)}$
Partition
$\Omega^{(1)}$
$\mathbf{u}_\Gamma^{(1)}$
1
$\lambda^{(12)}$
2
$\mathbf{u}_\Gamma^{(2)}$
$\Omega^{(2)}$
Substructure 1
Substructure 2
Constraint: $c_{12} = \mathbf{u}_\Gamma^{(1)} - \mathbf{u}_\Gamma^{(2)} = 0$
(b) Partition modeled by classical λ-method
$\Omega^{(1)}$
$\mathbf{u}_\Gamma^{(1)}$
1
$\lambda^{(1)}$
u_f
$\lambda^{(2)}$
2
$\mathbf{u}_\Gamma^{(2)}$
$\Omega^{(2)}$
Substructure 1
Substructure 2
Constraints: $c_1 = \mathbf{u}_\Gamma^{(1)} - \mathbf{u}_f = 0$
$c_2 = \mathbf{u}_\Gamma^{(2)} - \mathbf{u}_f = 0$
(c) Partition modeled by localized λ-method

Localized treatment of partition boundaries.

Variational form:

$$\delta\Pi_{total} = \delta\mathbf{u}^{(1)^T}\,[\,\mathbf{A}(\mathbf{u}^{(1)}) - \mathbf{f}^{(1)}] + \delta\mathbf{u}^{(2)^T}\,[\,\mathbf{A}(\mathbf{u}^{(2)}) - \mathbf{f}^{(2)}]$$
$$+\,\delta\{\boldsymbol{\lambda}^{(1)^T}\,(\mathbf{B}^{(1)}\mathbf{u}^{(1)} - \mathbf{u}_f) + \boldsymbol{\lambda}^{(2)^T}\,(\mathbf{B}^{(2)}\mathbf{u}^{(2)} - \mathbf{u}_f)\}$$

Equation set:

$$\begin{bmatrix} \mathbf{A}^{(1)} & 0 & \mathbf{B}^{(1)} & \mathbf{0} & \mathbf{0} \\ 0 & \mathbf{A}^{(2)} & \mathbf{0} & \mathbf{B}^{(2)} & \mathbf{0} \\ \mathbf{B}^{(1)^T} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{L}_f^{(1)} \\ \mathbf{0} & \mathbf{B}^{(2)^T} & \mathbf{0} & \mathbf{0} & -\mathbf{L}_f^{(2)} \\ \mathbf{0} & \mathbf{B}^{(2)^T} & -\mathbf{L}_f^{(1)^T} & -\mathbf{L}_f^{(2)^T} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{u}^{(1)} \\ \mathbf{u}^{(2)} \\ \boldsymbol{\lambda}^{(1)} \\ \boldsymbol{\lambda}^{(2)} \\ \mathbf{u}_f \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}^{(1)} \\ \mathbf{f}^{(2)} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix}$$

Generalization of the preceding partitioned equations of motion for s-partitions can be expressed as

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{0} \\ \mathbf{B}^T & \mathbf{0} & -\mathbf{L}_f \\ \mathbf{0} & -\mathbf{L}_f{}^T & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \\ \mathbf{u}_f \end{Bmatrix} = \begin{Bmatrix} \mathbf{f} \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix}$$

$$\mathbf{A} = \text{ block diag}(\mathbf{A}^{(1)}, \mathbf{A}^{(2)}, \ldots, \mathbf{A}^{(s)})$$

$$\mathbf{B} = \text{block diag}(\mathbf{B}^{(1)}, \mathbf{B}^{(2)}, \ldots, \mathbf{B}^{(s)})$$

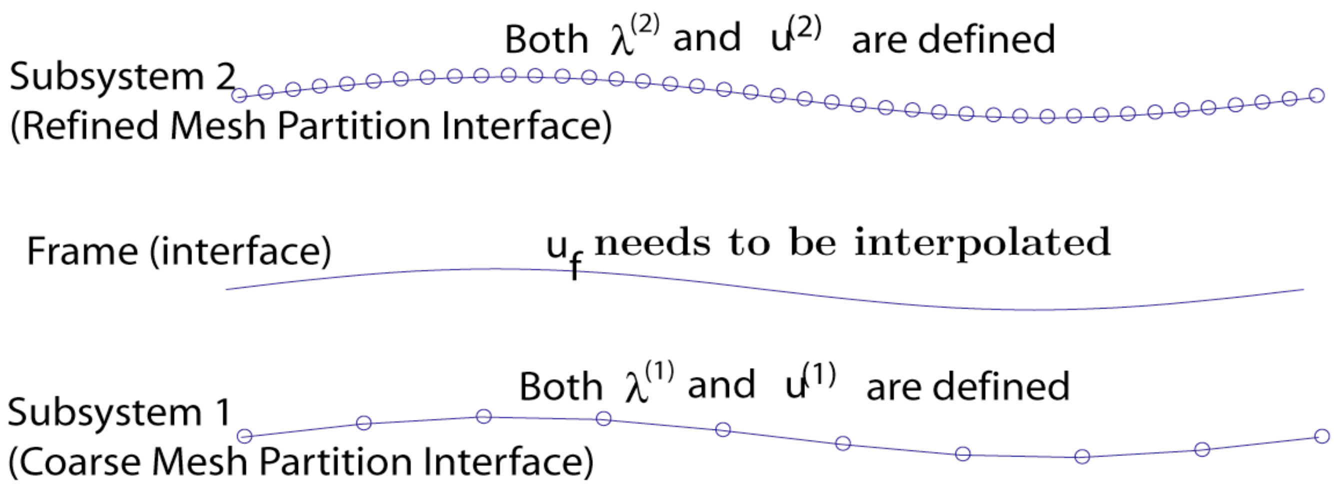
$$\mathbf{L}_f = \langle\, \mathbf{L}_f^{(1)} \quad \mathbf{L}_f^{(2)} \quad \ldots \quad \mathbf{L}_f^{(s)} \,\rangle^T$$

Treatment of Nonmatching Interfaces by Localized λ-Method

$$\pi_{localized} = \int_{\Gamma} \{\boldsymbol{\lambda}^{(1)}(\mathbf{x}) \cdot \ [\mathbf{u}_{\Gamma}^{(1)}(\mathbf{x}) - \mathbf{u}_{f}(\mathbf{x})] + \boldsymbol{\lambda}^{(2)}(\mathbf{x}) \cdot \ [\mathbf{u}_{\Gamma}^{(2)}(\mathbf{x}) - \mathbf{u}_{f}(\mathbf{x}) \ ]\} \, d\mathbf{x}$$

If the localized Lagrange multipliers $(\boldsymbol{\lambda}^{(1)}, \boldsymbol{\lambda}^{(2)})$ are collocated with each of the interface nodes, $(\mathbf{u}_{\Gamma}^{(1)}, \mathbf{u}_{\Gamma}^{(2)})$, the first and third term in the above constraint functional becomes a simple discrete inner product:

$$\int_{\Gamma} \boldsymbol{\lambda}^{(1)}(\mathbf{x}) \cdot \ \mathbf{u}_{\Gamma}^{(1)}(\mathbf{x}) \, d\mathbf{x} = \boldsymbol{\lambda}^{(1)^T} \cdot \mathbf{u}_{\Gamma}^{(1)}$$
$$\int_{\Gamma} \boldsymbol{\lambda}^{(2)}(\mathbf{x}) \cdot \ (\mathbf{u}_{\Gamma}^{(2)}(\mathbf{x}) \, d\mathbf{x} = \boldsymbol{\lambda}^{(2)^T} \cdot \mathbf{u}_{\Gamma}^{(2)}$$



Localized Treatment of Nonmatching Interfaces - Concluded

Note that the need to interpolate the Lagrange multipliers is obviated as can be seen from the expression:

$$\pi_{localized} = \boldsymbol{\lambda}^{(1)^T}\ \mathbf{u}_\Gamma^{(1)} + \boldsymbol{\lambda}^{(2)^T}\ \mathbf{u}_\Gamma^{(2)} - \int_\Gamma \{\boldsymbol{\lambda}^{(1)}(\mathbf{x})\cdot\ \mathbf{u}_f(\mathbf{x}) + \boldsymbol{\lambda}^{(2)}(\mathbf{x})\cdot\ \mathbf{u}_f(\mathbf{x})\}\, d\mathbf{x}$$

Since the conjugate pairs of $(\boldsymbol{\lambda}^{(k)}, \mathbf{u}^{(k)})$ for subsystem, k, are collocated, the localized Lagrange multipliers $\boldsymbol{\lambda}^{(k)}$ can be expressed in terms of Direct delta functions:

$$\boldsymbol{\lambda}^{(k)} = \boldsymbol{\lambda}^{(k)}(\mathbf{x}_j)\ D(\mathbf{x} - \mathbf{x}_j)$$

$$D(\mathbf{x} - \mathbf{x}_j) = \begin{cases} 1 & \text{if } \mathbf{x} = \mathbf{x}_j \\ 0 & \text{otherwise} \end{cases}$$

The only interface variable to be interpolated is the frame displacement!

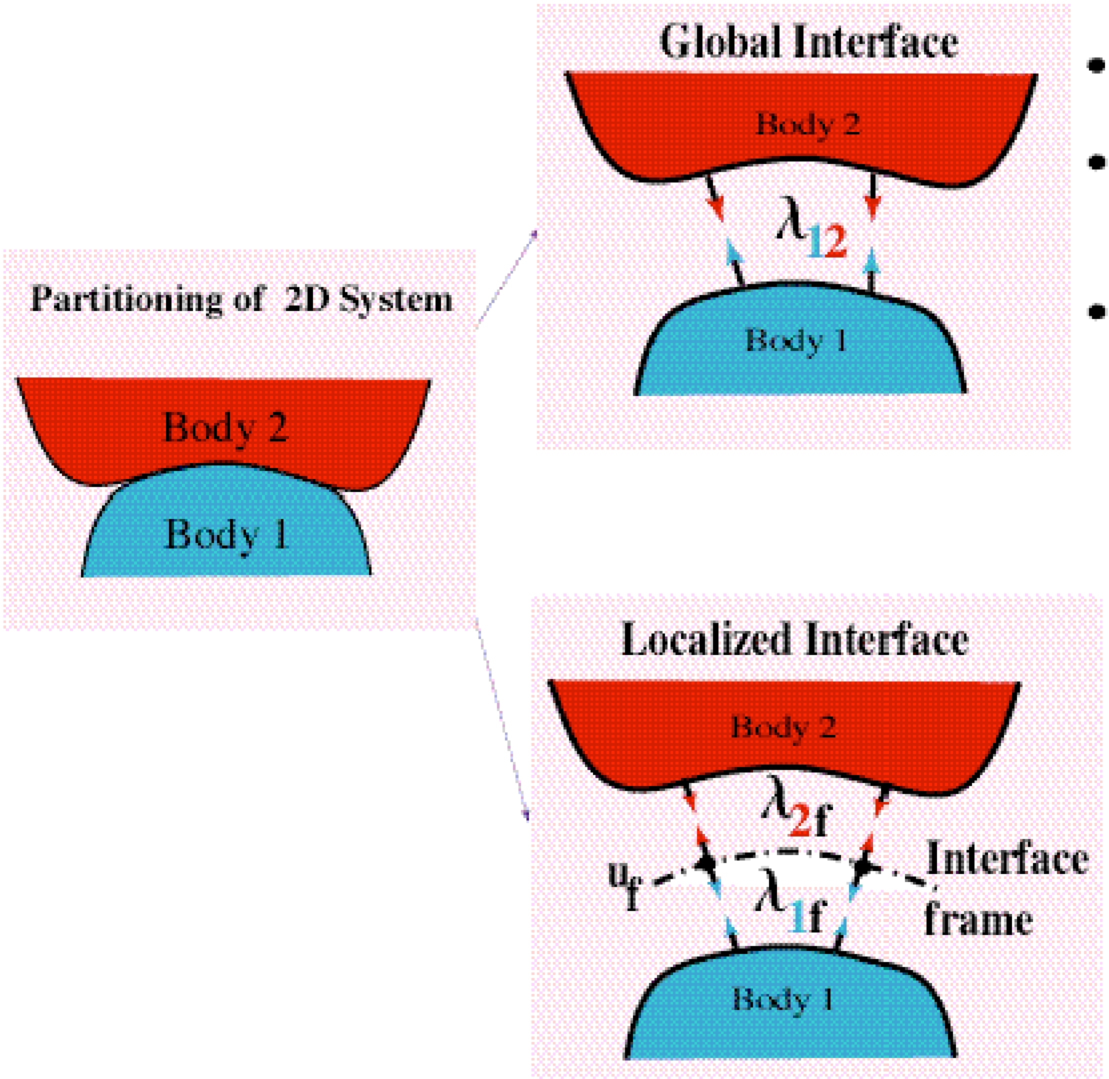
Localized Lagrange Multipliers

The localized modeling, as in Hamilton's canonical equations, also increases the interface unknowns from N to 2N multipliers.

Now the question is:

Is the theory of localized modeling just an intellectual pleasure or does it offer also a practical facility?

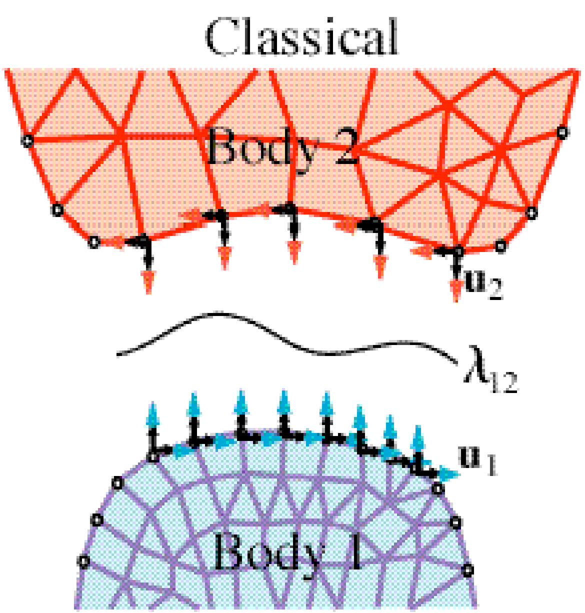
Two or Three Dimensional Interfaces



- Leads to global coupling.
- Mortar-like methods that may not be symmetric
- Requires modification of each equil. equations

- Leads to localized treatment of coupling
- Leads to symmetric interface attributes
- Separate stand-alone module takes care of all the interface attributes

Difficulties in Classical Approach



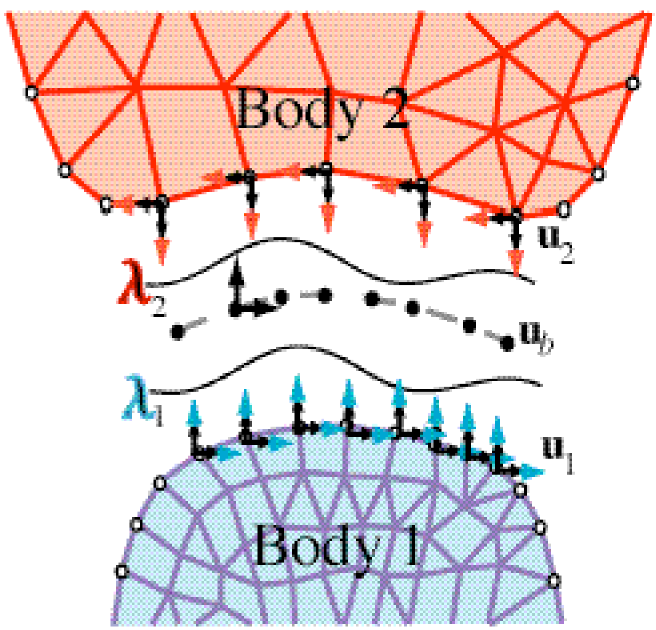
- The constraint functional for a two-partition case is given by

$$G(\boldsymbol{\lambda}_{12}, \mathbf{u}_1, \mathbf{u}_2) = \int_{\Gamma} \boldsymbol{\lambda}_{12} (\mathbf{u}_1 - \mathbf{u}_2)\, d\Gamma$$

- This is a typical mixed variational problem for which Strang once labeled "mixed methods yield mixed success," because:

- There has been no unique way of discretizing $\boldsymbol{\lambda}_{12}$

What Does the Localized Approach Offer?



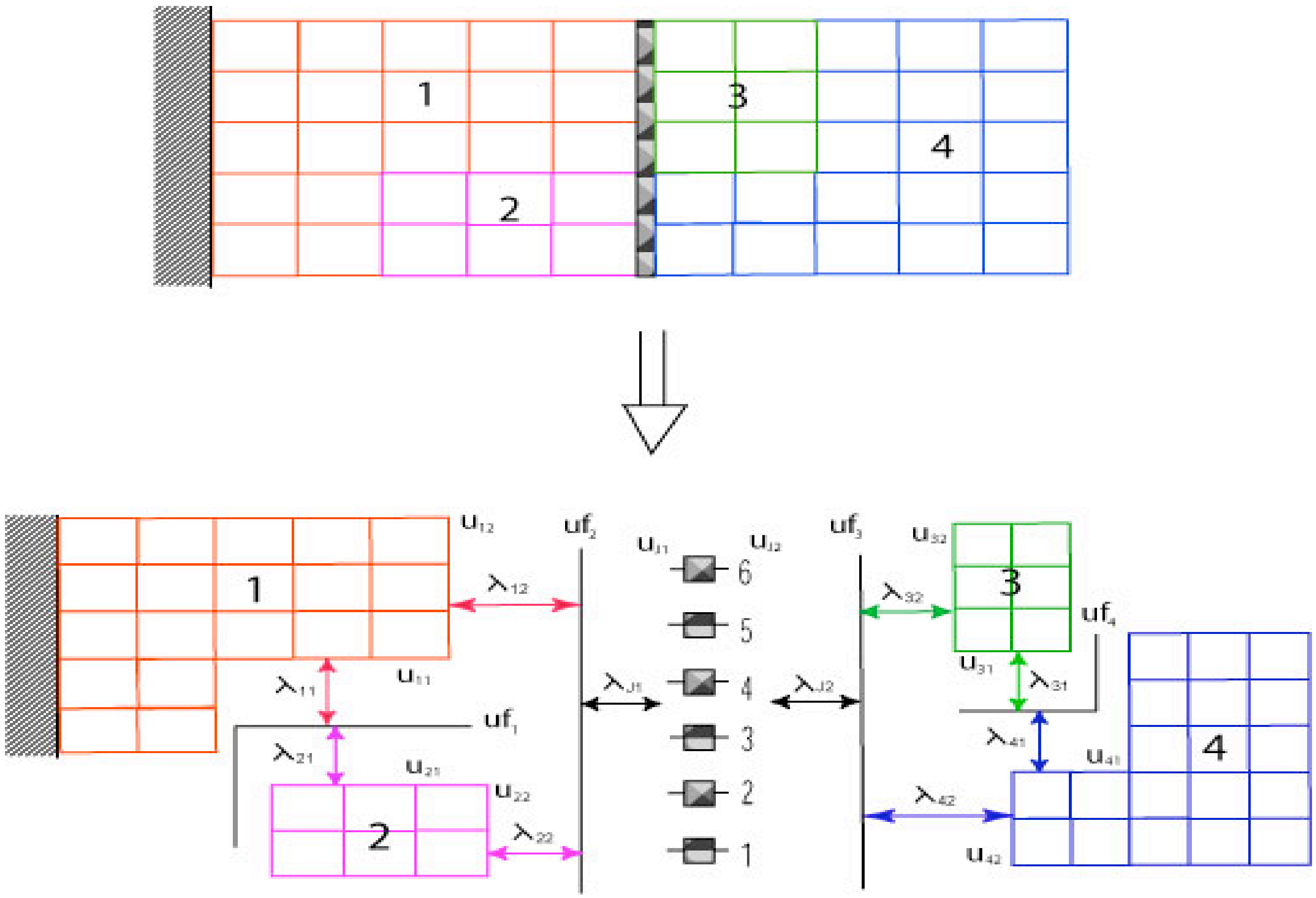
- The localized constraint functional for a two-partition case is given by

$$G(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_b) = \int_\Gamma \boldsymbol{\lambda}_1 (\mathbf{u}_1 - \mathbf{u}_b)\, d\Gamma + \int_\Gamma \boldsymbol{\lambda}_2 (\mathbf{u}_2 - \mathbf{u}_b)\, d\Gamma$$

- Hence, if $\boldsymbol{\lambda}_j$ is colocated with $\mathbf{u}_j$, then the frame displacement $\mathbf{u}_b$ is interpolated. Thus, no mixed interpolation is required.

Example: Heterogeneities of Partitioned Plane-Stress Structure

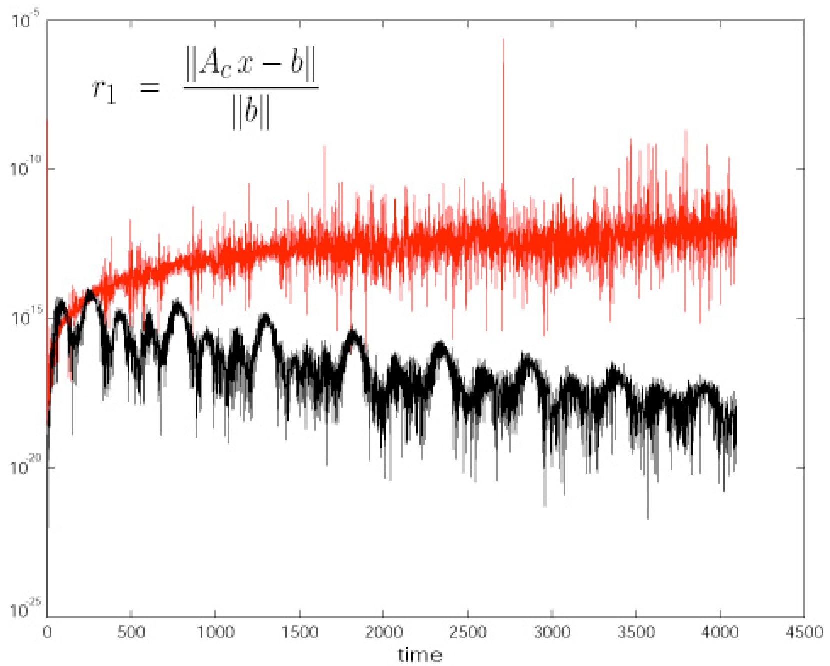
— joint elements in horizontal direction



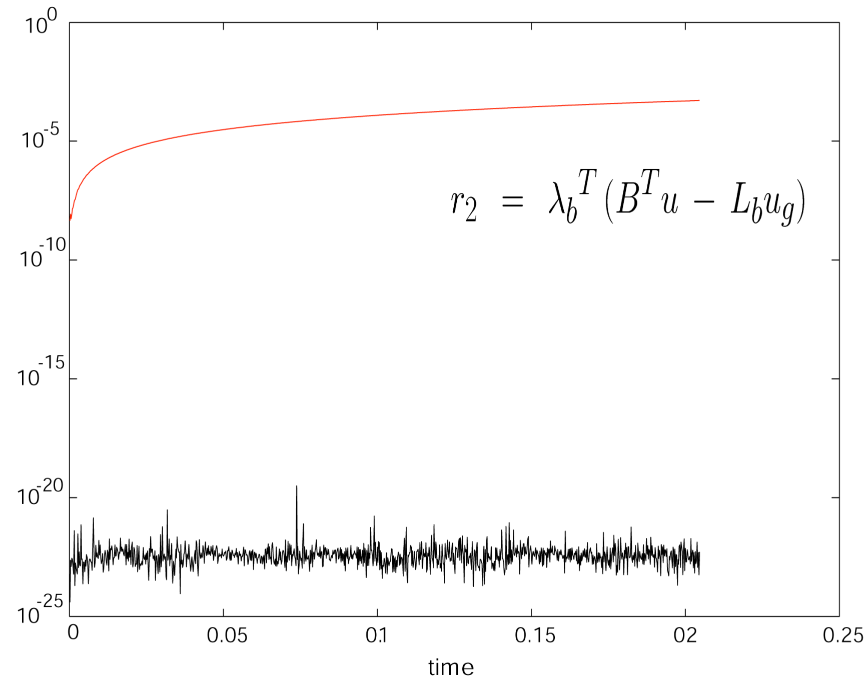
Maximum Difference of Elastic Modulus is 10^5

Residuals Comparison

equilibrium residual comparison

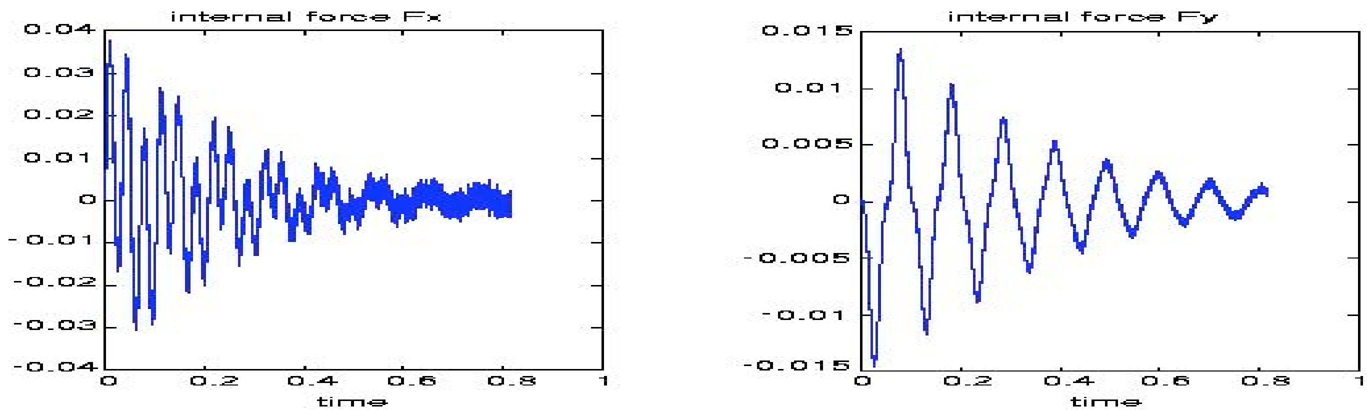


constraint residual comparison

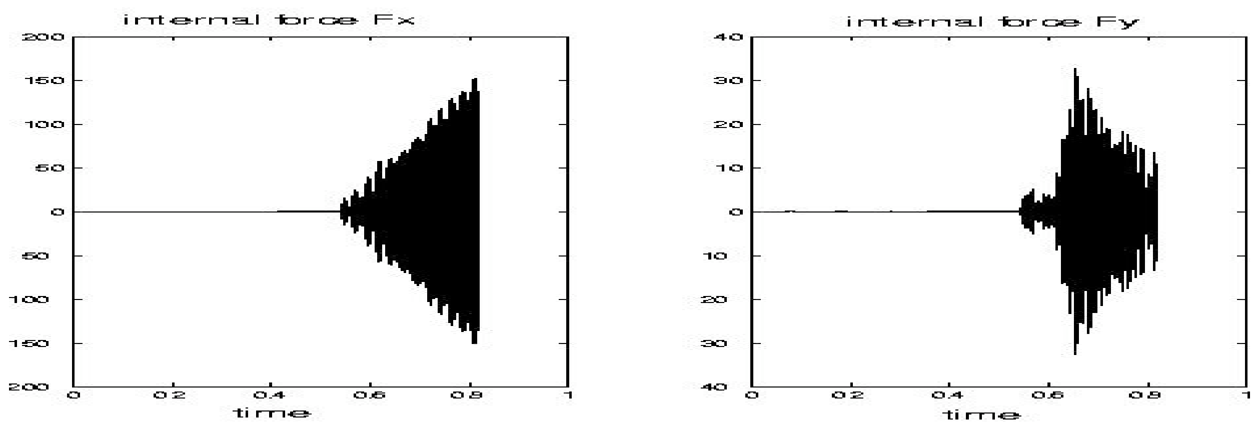


red ----- no scale black ----- $\lambda - scale$

Lagrange Multiplier Time History



$$\lambda - scale$$



no scale