ASEN5519 Topics in Multiphysics Modeling

Structure-Acoustic Interactions

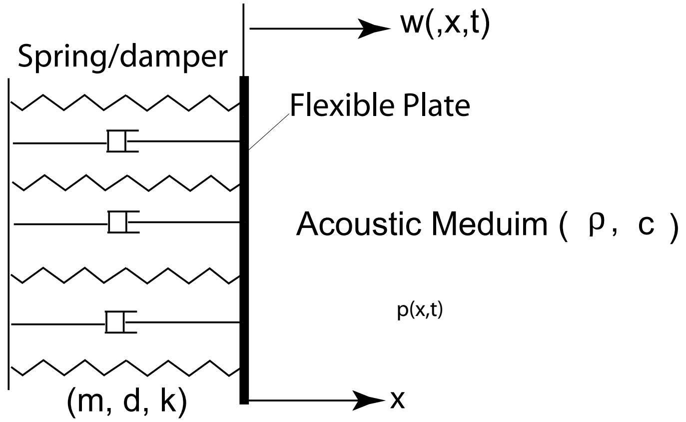
Classical Approach

Approximate Formulation for Three-D Problems

Field Transformation

Analysis of Structure-Acoustic Coupling Problems

A Classical Example:



A simple acoustic-structure interaction model

$$
\begin{aligned}
&m\ddot{w}(x,t) + d\dot{w}(x,t) + kw(x,t) = f(t) - p(0,t) \\
&w(x,0) = w_0,\ \dot{w}(x,0) = \dot{w}_0,\ \dot{w}(0,t) = v(0,t),\ p(x,t) = \rho\dot{\phi}(x,t) \\
&v(x,t) = -\frac{\partial \phi(x,t)}{\partial x},\ \ddot{\phi}(x,t) = c^2 \frac{\partial^2 \phi(x,t)}{dx^2},\ \phi(x,0) = 0
\end{aligned}
\tag{1}
$$

Observation: As the acoustic wave radiates into $x \to \infty$, the acoustic variables (v, p) need to be discretized with domains including the far right into the x-coordinate domain.

Solution: Apply the boundary element method!

An Elementary Boundary Element Solution:

A wave that generates a plane acoustic wave that travels to the right hasthe form of $g(t-x/c)$, whose Laplace transform is given by

$$\Phi(x,s) = A(s)e^{-sx/c} \tag{2}$$

The Laplace transform of $v(x,t) = -\frac{\partial\phi(x,t)}{\partial x}$ yields

$$V(x,s) = (\frac{s}{c})\ A(s)e^{-sx/c} \tag{3}$$

From (2) and (3), and the fifth of (1) we obtain

$$p(x,t) = \rho\dot{\phi}(x,t) \quad \Rightarrow \quad p(0,t) = \rho c v(0,t) \tag{4}$$

which, when substituted into the first of (1), yields

$$m\ddot{w}(x,t) + (d+\rho c)\dot{w}(x,t) + kw(x,t) = f(t) \tag{5}$$

thus obviating the need for modeling the acoustic field far into the x-coordinate axis. In essence, this is one-dimensional version of the boundary element method.

An Approximation of Three Dimensional Acoustic Field Equation

$$c^2 \nabla^2 \phi(x, y, z, t) = \frac{\partial^2 \phi(x, y, z, t)}{\partial t^2} \tag{6}$$

The solution of the above equation along the interface of the structure is one of the features of the BEM. Suffice to say that ample references for both the fundamental BEM method and its application to treat waves in the inifnite domain.

An attrctive approximation of the pressure field equation on the structural boundary is given by

$$\begin{aligned} \text{High frequency limit: } & p(, x, y, z, t) = \rho c v(x, y, z, t) \\ \text{Low frequency limit: } & A p(, x, y, z, t) = M v(x, y, z, t) \end{aligned} \tag{7}$$

where A is the area of the surface element normal to the structural surface, and M is the fluid-mass matrix, often referred to *added mass matrix*. Combing the above two limits, one obtains

$$\mathbf{M}\dot{\mathbf{p}} + \rho c \mathbf{A}\, \mathbf{p} = \rho c \mathbf{M} \mathbf{v} \tag{8}$$

which is known in the literature as *doubly asymptotic approximation*. Note that $(\mathbf{p}, \mathbf{v})$ are values right on the structural interface, thus effectively eliminating the need for discretizing the acoustic fields in teh infinite domain.

Variational Internal Acoustic-Structural Interactions

For structure:

$$\int_{\Omega_s} [k(u, \delta u) + \rho_s \ddot{u}] \, d\Omega_s - \int_\Gamma pn\delta u \, d\Gamma = \int_{\partial\Omega_s \setminus \Gamma} f \cdot \delta u \, d\Gamma \tag{9}$$

For acoustic field (fluid)

$$\frac{1}{\rho_F} \int_{\Omega_F} \nabla p \cdot \nabla \delta p \, d\Omega_F + \frac{1}{\rho_F c^2} \int_{\Omega_F} \ddot{p} \cdot \delta p \, d\Omega_F + \int_\Gamma \ddot{u} \cdot n\delta p \, d\Gamma = 0 \tag{10}$$

For constitutive equation with irrotationality constraint

$$\frac{1}{\rho_F c^2} \int_{\Omega_F} p \, d\Omega_F + \int_\Gamma u \cdot n \, d\Gamma = 0 \tag{11}$$

When discretized, the resulting coupled equations do not lead to a symmetric form of discrete equations!

Partitioned Formulation of Structural-Acoustic Interaction Problems via the Method of Localized Lagrangian Multipliers (This leads to a symmetric form!)

- Why the localized $\boldsymbol{\lambda}$ method?
  - Although the interface force is the same either acting on the fluid boundary or on wetted structural surface due to Newton's 3rd law, they emanate from completely different physics.

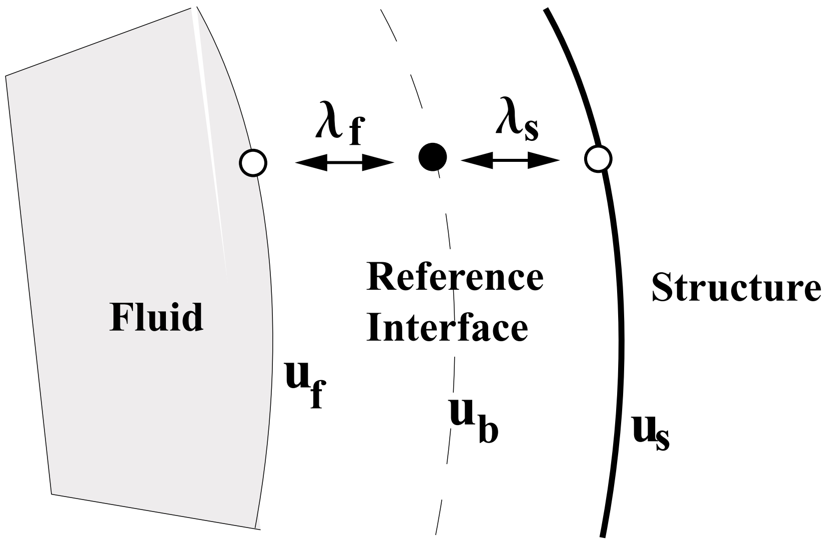
$$\text{For fluid:} \quad \boldsymbol{\lambda}_f = Ap = -\rho_f A\, c_f^2\, \nabla \mathbf{u}_f$$

$$\text{For Structure:} \quad \boldsymbol{\lambda}_s = A\, \boldsymbol{\sigma}\, \mathbf{n}_s$$

  - Use stand-alone software modules: a fluid analyzer, a structural analyzer, and possibly an interface module that accounts for the fluid-structure interaction phenomena.
  - Allow different grid or meshes on the interface.
  - Provide flexibility for eliminating the interface force acting on the fluid boundary $\boldsymbol{\lambda}_f$ or the one acting on the structural wetted boundary $\boldsymbol{\lambda}_s$

Fluid-Structure Interactions by the Localized Lagrangian Multipliers – continued

- One may begin with the displacement formulation and then achieve other formulations either by transformations or one of the two independent Localized Lagrangian multipliers.



Constraint from fluid to reference interface: $c_f = u_f - u_b = 0$

Constraint from structure to reference interface: $c_s = u_s - u_b = 0$

Fluid-Structure Interface Descriptions

Fluid-Structure Interactions by the Localized Lagrangian Multipliers – continued

Step 1: Variational Formalism

The variational formulation of a small-amplitude internal fluid vibro-acoustic problem in terms of displacement variables can be expresses as

$$
\begin{aligned}
&\tfrac{1}{2}\int_{\Omega_s} \boldsymbol{\sigma}^T\, \epsilon \, d\Omega_s + \tfrac{1}{2}\rho_f c_f^2 \int_{\Omega_f} (\nabla \mathbf{u}_f)^2 d\Omega_f \\
&+ \int_{\Omega_s} \mathbf{u}_s^T (\mathbf{f}_s - \rho_s \ddot{\mathbf{u}}_s)\, d\Omega_s + \int_{\Omega_f} \mathbf{u}_f^T (\mathbf{f}_f - \rho_f \ddot{\mathbf{u}}_f)\, d\Omega_f \\
&= \int_{\Gamma_s} \mathbf{u}_s^T \mathbf{T}_s \, d\Gamma + \int_{\Gamma_f} \mathbf{u}_f^T \mathbf{T}_f \, d\Gamma \\
&p = -\rho_f c_f^2 \, \nabla \mathbf{u}_f \\
&\text{With condition: } \mathbf{curl}\, \mathbf{u}_f = 0 \quad \text{in} \quad \Omega_f
\end{aligned}
$$

where (ρ_s, ρ_f) are the density of the structure and fluid; c_f is the sound speed of the fluid; $(\mathbf{u}_s, \mathbf{u}_f)$ are the structural and fluid displacement vectors; $(\boldsymbol{\sigma}, \epsilon)$ are the stress and strain vectors of the structure; $(\mathbf{f}_s, \mathbf{f}_f)$ are the body forces of the structure and fluid; and (Ω_s, Ω_f) designate the interior structure and fluid domains, (Γ_s, Γ_f) represent the physical boundaries of the structure and fluid; and, the superscript dots $\ddot{()}$ designate time differentiation.

Fluid-Structure Interactions by the Localized Lagrangian Multipliers – continued

Step 2: Finite Element Discretization

$$\boxed{\begin{array}{l}
\delta\Pi(\mathbf{u}_g) = \delta\mathbf{u}_g^T\ (\mathbf{K}_g\ \mathbf{u}_g + \mathbf{M}_g\ \ddot{\mathbf{u}}_g - \mathbf{F}_g) \\
\textbf{curl}\ \mathbf{u}_f \approx \mathbf{C}_f^T\ \mathbf{u}_f = 0 \\
\mathbf{K}_g = [\,\mathbf{L}_s^T \quad \mathbf{L}_f^T\,] \begin{bmatrix} \mathbf{K}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_f \end{bmatrix} \begin{bmatrix} \mathbf{L}_s \\ \mathbf{L}_f \end{bmatrix} \\
\mathbf{M}_g = [\,\mathbf{L}_s^T \quad \mathbf{L}_f\,] \begin{bmatrix} \mathbf{M}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_f \end{bmatrix} \begin{bmatrix} \mathbf{L}_s^T \\ \mathbf{L}_f \end{bmatrix} \\
\left\{ \begin{array}{c} \mathbf{u}_s \\ \mathbf{u}_f \end{array} \right\} = \begin{bmatrix} \mathbf{L}_s \\ \mathbf{L}_f \end{bmatrix} \mathbf{u}_g \\
\mathbf{f}_g = [\,\mathbf{L}_s^T \quad \mathbf{L}_f^T\,] \left\{ \begin{array}{c} \mathbf{f}_{\gamma s} + \mathbf{t}_s \\ \mathbf{f}_{\gamma f} + \mathbf{t}_f \end{array} \right\}
\end{array}}$$

where **L** denotes a Boolean matrix that assembles elements to a global matrix, and various matrix and vectorial quantities are given by

$$\delta \int_{\Omega_s} \boldsymbol{\sigma}^T \; \boldsymbol{\epsilon} \; d\Omega \approx \delta \mathbf{u}_s^T \; \mathbf{K}_s \; \mathbf{u}_s$$

$$\rho_f \; c_f^2 \; \delta \int_{\Omega_f} (\nabla \mathbf{u}_f)^2 d\Omega \approx \delta \mathbf{u}_f^T \; \mathbf{K}_f \; \mathbf{u}_f$$

$$\int_{\Omega_s} \rho_s \; \delta \mathbf{u}_s^T \ddot{\mathbf{u}}_s \; d\Omega \approx \delta \mathbf{u}_s^T \; \mathbf{M}_s \; \ddot{\mathbf{u}}_s$$

$$\int_{\Omega_f} \rho_f \; \delta \mathbf{u}_f^T \ddot{\mathbf{u}}_f \; d\Omega \approx \delta \mathbf{u}_f^T \; \mathbf{M}_f \; \ddot{\mathbf{u}}_f$$

$$\delta \int_{\Omega_s} \mathbf{u}_s^T \mathbf{f}_s \; d\Omega \approx \delta \mathbf{u}_s^T \; \mathbf{f}_{\gamma s}$$

$$\delta \int_{\Omega_f} \mathbf{u}_f^T \mathbf{f}_f \; d\Omega \approx \delta \mathbf{u}_f^T \; \mathbf{f}_{\gamma f}$$

$$\delta \int_{\Gamma_s} \mathbf{u}_s^T \mathbf{T}_s \; d\Gamma \approx \delta \mathbf{u}_s^T \; \mathbf{t}_s$$

$$\delta \int_{\Gamma_f} \mathbf{u}_f^T \mathbf{T}_f \; d\Gamma \approx \delta \mathbf{u}_f^T \; \mathbf{t}_f$$

Formulation of Fluid-Structure Interactions – continued

Step 3: Incorporation of Irrotational Condition

Augment $\delta\Pi(\mathbf{u}_g)$ with the irrotational condition:

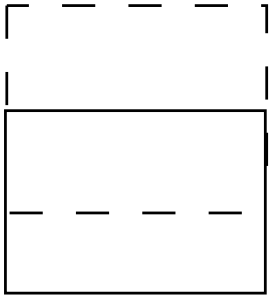
$$\delta\Pi(\mathbf{u}_g, \boldsymbol{\mu}) = \delta\mathbf{u}_g^T \; (\mathbf{K}_g \; \mathbf{u}_g + \mathbf{M}_g \; \ddot{\mathbf{u}}_g - \mathbf{f}_g) + \delta\boldsymbol{\mu}^T \; \mathbf{C}_f^T \; \mathbf{u}_f + \delta\mathbf{u}_f^T \; \mathbf{C}_f \; \boldsymbol{\mu}$$

Hence, the governing discrete equations of motion subject to the irrotational constraint leads to the following equation:

$$\begin{bmatrix} \mathbf{K}_g + \mathbf{M}_g \, \frac{d^2}{dt^2} & \mathbf{C}_g \\ \mathbf{C}_g^T & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_g \\ \boldsymbol{\mu} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_g \\ \mathbf{0} \end{Bmatrix}, \qquad \mathbf{C}_g^T = [\, \mathbf{0} \quad \mathbf{C}_f^T \,] \begin{bmatrix} \mathbf{L}_s \\ \mathbf{L}_f \end{bmatrix}$$

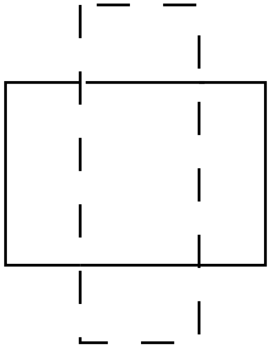
Fluid-Structure Interactions – continued

Step 4: Admissible Rigid Body Modes in Two-Dimensional Irrotational Flow

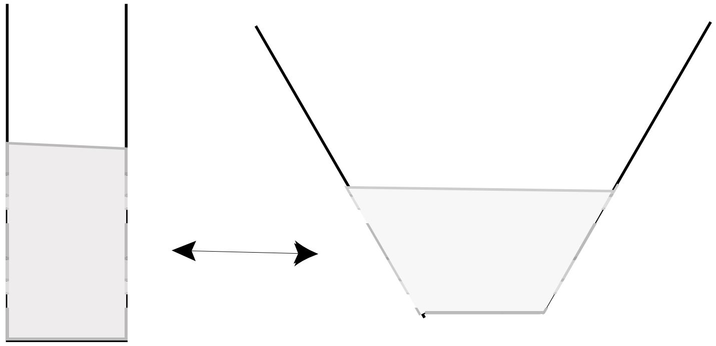


Vertical Rigid Motion

Horizontal Rigid Mode



Filling Mode



When no gravity is present, this filling mode
does not require energy; hence it is
an admissible rigid body mode

Three Admissible Rigid Body Modes

Fluid-Structure Interactions – continued

Step 5: Localized Fluid-Structure Interface Constraints

$$\begin{Bmatrix} \mathbf{u}_{fb} \\ \mathbf{u}_{sb} \end{Bmatrix} - \mathcal{L}_b\, \mathbf{u}_b = 0 \quad \Rightarrow \quad \mathbf{B}_\ell^T \mathbf{u} - \mathcal{L}_b\, \mathbf{u}_b = 0, \quad \mathbf{u} = \begin{Bmatrix} \mathbf{u}_s \\ \mathbf{u}_f \end{Bmatrix}$$

$$\mathcal{L}_b = \mathcal{F}\, \mathbf{B}_\ell^T \mathbf{L} \ = \begin{bmatrix} \mathcal{L}_{bs} \\ \mathcal{L}_{bf} \end{bmatrix}, \quad \mathcal{F} \ = \begin{bmatrix} \mathcal{F}_f & 0 \\ 0 & \mathcal{F}_s \end{bmatrix}, \quad \mathbf{B}_\ell = \begin{bmatrix} \mathbf{B}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_f \end{bmatrix},$$

where

$\mathbf{u}_{fb}$: fluid displacement on the interaction interface

$\mathbf{u}_{sb}$: structural displacement on the interaction interface

$\mathbf{B}_\ell^T$: extracts only the interface portion

$\mathcal{L}_b$: interface reference dofs

$\mathcal{F}$: interpolation functions of interface reference displacement

Formulation of Fluid-Structure Interactions – continued

Step 6: Augmentation of Interface Constraints to Variational Functional

$$
\begin{aligned}
\delta\Pi(\mathbf{u}, \boldsymbol{\lambda}_\ell, \mathbf{u}_b, \boldsymbol{\mu}) = {} & \delta\mathbf{u}^T\ (\mathbf{K}\,\mathbf{u} + \mathbf{M}\,\ddot{\mathbf{u}} - \mathbf{f}) + \delta\boldsymbol{\mu}^T\ \mathbf{C}_f^T\ \mathbf{u}_f + \delta\mathbf{u}_f^T\ \mathbf{C}_f^T\ \boldsymbol{\mu} \\
& + \delta\boldsymbol{\lambda}_\ell^T\ (\mathbf{B}^T\mathbf{u} - \mathcal{L}_b\ \mathbf{u}_g) + (\delta\mathbf{u}^T\mathbf{B}_\ell - \delta\mathbf{u}_b^T\mathcal{L}_b^T\ )\ \boldsymbol{\lambda}_\ell
\end{aligned}
$$

which, when expanded into six-variable functional, becomes:

$$
\begin{aligned}
\delta\Pi(\mathbf{u}_s, \mathbf{u}_f, \boldsymbol{\lambda}_s, \boldsymbol{\lambda}_f, \mathbf{u}_b, \boldsymbol{\mu}) = {} & \delta\mathbf{u}_s^T\ (\mathbf{K}_s\ \mathbf{u}_s + \mathbf{M}_s\ \ddot{\mathbf{u}}_s - \mathbf{f}_s + \mathbf{B}_s\ \boldsymbol{\lambda}_s) + \delta\boldsymbol{\mu}^T\ \mathbf{C}_f^T\ \mathbf{u}_f \\
& + \delta\mathbf{u}_f^T\ (\mathbf{K}_f\ \mathbf{u}_f + \mathbf{M}_f\ \ddot{\mathbf{u}}_f - \mathbf{f}_f + \mathbf{C}_f\ \boldsymbol{\mu} + \mathbf{B}_f\ \boldsymbol{\lambda}_f) \\
& + \delta\boldsymbol{\lambda}_\ell^T\ (\begin{bmatrix} \mathbf{B}_s^T & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_f^T \end{bmatrix} \begin{Bmatrix} \mathbf{u}_s \\ \mathbf{u}_f \end{Bmatrix} - \begin{bmatrix} \mathcal{L}_{bs} \\ \mathcal{L}_{bf} \end{bmatrix}\ \mathbf{u}_b) \\
& - \delta\mathbf{u}_b^T\,[\,\mathcal{L}_{bs}^T \quad \mathcal{L}_{bf}^T\,] \begin{Bmatrix} \boldsymbol{\lambda}_s \\ \boldsymbol{\lambda}_f \end{Bmatrix}, \qquad \boldsymbol{\lambda}_\ell = \begin{Bmatrix} \boldsymbol{\lambda}_s \\ \boldsymbol{\lambda}_f \end{Bmatrix}
\end{aligned}
$$

Formulation of Fluid-Structure Interactions – continued

Step 7: Partitioned Equations of Motion for Coupled Fluid-Structure Systems

$$\begin{bmatrix} \mathbf{K}_s + \frac{d^2}{dt^2}\mathbf{M}_s & 0 & \mathbf{B}_s & 0 & 0 & 0 \\ 0 & \mathbf{K}_f + \frac{d^2}{dt^2}\mathbf{M}_f & 0 & \mathbf{B}_f & \mathbf{C}_f & 0 \\ \mathbf{B}_s^T & 0 & 0 & 0 & 0 & -\mathcal{L}_{bs} \\ 0 & \mathbf{B}_f^T & 0 & 0 & 0 & -\mathcal{L}_{bf} \\ 0 & \mathbf{C}_f^T & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathcal{L}_{bs}^T & -\mathcal{L}_{bs}^T & 0 & 0 \end{bmatrix} \left\{ \begin{matrix} \mathbf{u}_s \\ \mathbf{u}_f \\ \boldsymbol{\lambda}_s \\ \boldsymbol{\lambda}_f \\ \boldsymbol{\mu} \\ \mathbf{u}_b \end{matrix} \right\} = \left\{ \begin{matrix} \mathbf{f}_s \\ \mathbf{f}_f \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix} \right\}$$

Special Case: Pressure-Structural Displacement $(\mathbf{p}, \mathbf{u})$ - Formulation

- Existing formulations suffer from their inability to correctly account for floating conditions, e.g., rockets flying in the air, unless special corrections were introduced (cf., Ohayon, 1995).

- A key idea: express the fluid displacement $\mathbf{u}_f$ in terms of the pressure $\mathbf{p}$ and the proper nullspace (rigid-body modes) $\mathbf{R}$ given by

$$\mathbf{u}_f = \mathbf{D}_p\, \mathbf{p} + \mathbf{R}_f\, \boldsymbol{\alpha}_f$$

  - For each element **e**, discretize: $p = -\rho_f\, c^2\, \nabla \mathbf{u}_f$ to obtain $\mathbf{p}^e = \mathbf{D}_f^e\, \mathbf{u}_f^e$
  - Augment the irrotationality condition $\mathbf{C}_f^e\, \mathbf{u}_f^e = 0$: $\begin{Bmatrix} \mathbf{p}^e \\ \mathbf{0} \end{Bmatrix} = \begin{bmatrix} \mathbf{D}_f^e \\ \mathbf{C}_f^e \end{bmatrix} \mathbf{u}_f^e$
  - Solve for $\mathbf{u}_f^e$: $\mathbf{u}_f^e = \mathbf{D}_p^e\, \mathbf{p}^e + \mathbf{R}_f^e\, \boldsymbol{\alpha}_f^e, \quad \mathbf{D}_p^e = \begin{bmatrix} \mathbf{D}_f^e \\ \mathbf{C}_f^e \end{bmatrix}^{-1}$
  - Sum over all the fluid elements

Special Case: Pressure-Structural Displacement $(\mathbf{p}, \mathbf{u})$ - Formulation

- Substitute the pressure-to-displacement relation into the variational functional:

$$
\begin{aligned}
\delta\Pi(\mathbf{u}_s, \mathbf{p}, \boldsymbol{\alpha}_f, \boldsymbol{\lambda}_s, \boldsymbol{\lambda}_f, \mathbf{u}_b, \boldsymbol{\mu}) = {} & \delta\mathbf{u}_s^T\ (\mathbf{K}_s\ \mathbf{u}_s + \mathbf{M}_s\ \ddot{\mathbf{u}}_s - \mathbf{f}_s + \mathbf{B}_s\ \boldsymbol{\lambda}_s) \\
& + \delta\boldsymbol{\mu}^T\ \mathbf{C}_f^T\ (\mathbf{D}_p\ \mathbf{p} + \mathbf{R}_f\ \boldsymbol{\alpha}_f) \\
& + \delta(\mathbf{D}_p\ \mathbf{p} + \mathbf{R}_f\ \boldsymbol{\alpha}_f)^T\ [\mathbf{K}_f\ (\mathbf{D}_p\ \mathbf{p} + \mathbf{R}_f\ \boldsymbol{\alpha}_f) \\
& \qquad + \mathbf{M}_f\ (\mathbf{D}_p\ \ddot{\mathbf{p}} + \mathbf{R}_f\ \ddot{\boldsymbol{\alpha}}_f) - \mathbf{f}_f + \mathbf{C}_f\ \boldsymbol{\mu} + \mathbf{B}_f\ \boldsymbol{\lambda}_f] \\
& + \delta\boldsymbol{\lambda}_\ell^T\ (\begin{bmatrix} \mathbf{B}_s^T & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_f^T \end{bmatrix} \left\{ \begin{matrix} \mathbf{u}_s \\ \\ (\mathbf{D}_p\ \mathbf{p} + \mathbf{R}_f\ \boldsymbol{\alpha}_f) \end{matrix} \right\} - \begin{bmatrix} \mathcal{L}_{bs} \\ \mathcal{L}_{bf} \end{bmatrix} \mathbf{u}_b) \\
& - \delta\mathbf{u}_b^T\,[\,\mathcal{L}_{bs}^T \quad \mathcal{L}_{bf}^T\,] \left\{ \begin{matrix} \boldsymbol{\lambda}_s \\ \boldsymbol{\lambda}_f \end{matrix} \right\}, \quad \boldsymbol{\lambda}_\ell = \left\{ \begin{matrix} \boldsymbol{\lambda}_s \\ \boldsymbol{\lambda}_f \end{matrix} \right\}
\end{aligned}
$$

Special Case: Pressure-Structural Displacement $(\mathbf{p}, \mathbf{u})$ - Formulation

- By making use of the relations

$$\begin{aligned} \mathbf{C}_f^T \mathbf{D}_p &= 0 \quad \Leftarrow \quad \text{div} \cdot (\text{curl } \mathbf{u}_f) = 0 \\ \mathbf{C}_f^T \ \mathbf{R}_f &= 0 \quad \text{since rotation is orthogonal to rigid modes.} \\ \mathbf{K}_f \ \mathbf{R}_f &= 0 \quad \text{as } \mathbf{R}_f \text{ is a nullspace of } \mathbf{K}_f \end{aligned}$$

Simplify the preceding variational equation we obtain:

$$\begin{aligned} &\delta \Pi (\mathbf{u}_s, \mathbf{p}, \boldsymbol{\alpha}_f, \boldsymbol{\lambda}_s, \boldsymbol{\lambda}_f, \mathbf{u}_b, \boldsymbol{\mu}) = \\ &\qquad \delta \mathbf{u}_s^T \ (\mathbf{K}_s \ \mathbf{u}_s + \mathbf{M}_s \ \ddot{\mathbf{u}}_s - \mathbf{f}_s + \mathbf{B}_s \ \boldsymbol{\lambda}_s) \\ &\qquad + \delta \mathbf{p}^T \mathbf{D}_p^T \ [\mathbf{K}_f \ \mathbf{D}_p \ \mathbf{p} + \mathbf{M}_f \ (\mathbf{D}_p \ \ddot{\mathbf{p}} + \mathbf{R}_f \ \ddot{\boldsymbol{\alpha}}_f) - \mathbf{f}_f + \mathbf{B}_f \ \boldsymbol{\lambda}_f] \\ &\qquad + \delta \boldsymbol{\alpha}_f^T \mathbf{R}^T \ [\mathbf{M}_f \ (\mathbf{D}_p \ \ddot{\mathbf{p}} + \mathbf{R} \ \ddot{\boldsymbol{\alpha}}_f) - \mathbf{f}_f + \mathbf{B}_f \ \boldsymbol{\lambda}_f] \\ &\qquad + \delta \boldsymbol{\lambda}_\ell^T \ (\begin{bmatrix} \mathbf{B}_s^T & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_f^T \end{bmatrix} \begin{Bmatrix} \mathbf{u}_s \\ (\mathbf{D}_p \ \mathbf{p} + \mathbf{R}_f \ \boldsymbol{\alpha}_f) \end{Bmatrix} - \begin{bmatrix} \mathcal{L}_{bs} \\ \mathcal{L}_{bf} \end{bmatrix} \mathbf{u}_b) \\ &\qquad - \delta \mathbf{u}_b^T \, [\, \mathcal{L}_{bs}^T \quad \mathcal{L}_{bf}^T \,] \begin{Bmatrix} \boldsymbol{\lambda}_s \\ \boldsymbol{\lambda}_f \end{Bmatrix} \end{aligned}$$

Special Case: Pressure-Structural Displacement $(\mathbf{p},\mathbf{u})$ - Formulation

- The general partitioned, coupled pressure-displacement equations are given by:

$$\begin{bmatrix} \mathbf{K}_s + \frac{d^2}{dt^2}\mathbf{M}_s & 0 & 0 & \mathbf{B}_s & 0 & 0 \\ 0 & \mathbf{K}_p + \frac{d^2}{dt^2}\mathbf{M}_p & \frac{d^2}{dt^2}\mathbf{M}_{p\alpha} & 0 & \mathbf{D}_p^T\mathbf{B}_f & 0 \\ 0 & \frac{d^2}{dt^2}\mathbf{M}_{p\alpha}^T & \frac{d^2}{dt^2}\mathbf{M}_\alpha & 0 & \mathbf{R}_f^T\mathbf{B}_f & 0 \\ \mathbf{B}_s^T & 0 & 0 & 0 & 0 & -\mathcal{L}_{bs} \\ 0 & \mathbf{B}_f^T\mathbf{D}_p & \mathbf{B}_f^T\mathbf{R}_f & 0 & 0 & -\mathcal{L}_{bf} \\ 0 & 0 & 0 & -\mathcal{L}_{bs}^T & -\mathcal{L}_{bf}^T & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{u}_s \\ \mathbf{p} \\ \boldsymbol{\alpha}_f \\ \boldsymbol{\lambda}_s \\ \boldsymbol{\lambda}_f \\ \mathbf{u}_b \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_s \\ \mathbf{D}_p^T\mathbf{f}_f \\ \mathbf{R}_f^T\mathbf{f}_f \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\mathbf{K}_p = \mathbf{D}_p^T\mathbf{K}_f\mathbf{D}_p, \quad \mathbf{M}_p = \mathbf{D}_p^T\mathbf{M}_f\mathbf{D}_p, \quad \mathbf{M}_{p\alpha} = \mathbf{D}_p^T\mathbf{M}_f\mathbf{R}_f, \quad \mathbf{M}_\alpha = \mathbf{R}_f^T\mathbf{M}_f\mathbf{R}_f$$

- Approximations: $\mathbf{M}_{p\alpha} \approx \mathbf{0}$

- Several specializations can be derived from this equation set. We will dwell on only a few here.

What happens if the rigid-body modes (α) are eliminated in the pressure-structural displacement $(\mathbf{p}, \mathbf{u})$-formulation?

- In that case the general partitioned equation of the coupled pressure-displacement equations becomes:

$$\ddot{\boldsymbol{\alpha}}_f = -\mathbf{M}_{\alpha\alpha}^{-1}\,\mathbf{R}_f^T\,(\mathbf{f}_f - \mathcal{B}_f \boldsymbol{\lambda}_f) \tag{12}$$

Substituting this into the fifth equation of (11), we obtain

$$\begin{bmatrix} \mathbf{K}_s + \frac{d^2}{dt^2}\mathbf{M}_s & 0 & 0 & \mathcal{B}_s & 0 \\ 0 & \mathbf{K}_p + \frac{d^2}{dt^2}\mathbf{M}_p & 0 & \mathbf{D}_p^T \mathcal{B}_f & 0 \\ \mathcal{B}_s^T & 0 & 0 & 0 & -\mathcal{L}_{bs} \\ 0 & \mathcal{B}_f^T \mathbf{D}_p \frac{d^2}{dt^2} & 0 & -\mathcal{B}_f^T \mathbf{R}_f \mathbf{M}_{\alpha\alpha}^{-1} \mathbf{R}_f^T \mathcal{B}_f & -\mathcal{L}_{bf} \frac{d^2}{dt^2} \\ 0 & 0 & -\mathcal{L}_{bs}^T & -\mathcal{L}_{bf}^T & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{u}_s \\ \mathbf{p} \\ \boldsymbol{\lambda}_s \\ \boldsymbol{\lambda}_f \\ \mathbf{u}_b \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_s \\ \mathbf{D}_p^T \mathbf{f}_f \\ 0 \\ -\mathcal{B}_f^T \mathbf{R}_f \mathbf{M}_{\alpha\alpha}^{-1} \mathbf{R}_f^T \mathbf{f}_f \\ 0 \end{Bmatrix} \tag{13}$$

- Oberve that the resulting equation loses symmetry, akin to the classical formulation. Hence, we conclude that, unless one ignores the rigid body modes, the elimination of rigid-body modes destroys the symmetry of the system.

Vibration Analysis by $(\mathbf{u}_s, \mathbf{p}_f, \boldsymbol{\lambda}_s, \mathbf{u}_b)$-Formulation

Of several possible further eliminations of the system variables for vibration analysis, we present the case of eliminating $\boldsymbol{\lambda}_f$ below. Solving for $\boldsymbol{\lambda}_f$ with $\frac{d^2}{dt^2} = -\omega^2$ and $(\mathbf{f}_s = 0, \mathbf{f}_f = 0)$, we obtain

$$\begin{aligned} \boldsymbol{\lambda}_f &= \mathcal{M}_\alpha^{(b)} \, [-\boldsymbol{\omega}^2 \mathcal{B}_f^T \, \mathbf{D}_p \, \mathbf{p}_f + \boldsymbol{\omega}^2 \mathcal{L}_{bf} \mathbf{u}_b] \\ \mathcal{M}_\alpha^{(b)} &= [\mathcal{B}_f^T \mathbf{R}_f \mathbf{M}_{\alpha\alpha}^{-1} \mathbf{R}_f^T \mathcal{B}_f]^{-1} \end{aligned} \tag{14}$$

Equation(13) can now be reduced to read as:

$$\begin{bmatrix} \mathbf{K}_s - \boldsymbol{\omega}^2 \mathbf{M}_s & 0 & \mathcal{B}_s & 0 \\ 0 & \mathbf{K}_p - \boldsymbol{\omega}^2 \hat{\mathbf{M}}_p & 0 & \boldsymbol{\omega}^2 \hat{\mathcal{L}}_{bf} \\ \mathcal{B}_s^T & 0 & 0 & -\mathcal{L}_{bs} \\ 0 & \boldsymbol{\omega}^2 \hat{\mathcal{L}}_{bf}^T & -\mathcal{L}_{bs}^T & -\boldsymbol{\omega}^2 \mathcal{L}_{bf}^T \mathcal{M}_\alpha^{(b)} \, \mathcal{L}_{bf}^T \end{bmatrix} \begin{Bmatrix} \mathbf{u}_s \\ \mathbf{p}_f \\ \boldsymbol{\lambda}_s \\ \mathbf{u}_b \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{15}$$

$$\hat{\mathbf{M}}_p = \mathbf{M}_p - \mathbf{D}_p^T \, \mathcal{B}_f \, \mathcal{M}_\alpha^{(b)} \, \mathcal{B}_f^T \, \mathbf{D}_p, \qquad \hat{\mathcal{L}}_{bf} = \mathbf{D}_p^T \, \mathcal{B}_f \, \mathcal{M}_\alpha^{(b)} \, \mathcal{L}_{bf}$$

What if the dynamics of sound sources, viz., the vibration of the structure $\mathbf{u}_s$, is ignored?

- Let us eliminate $\mathbf{u}_s$ from the $(\mathbf{u}_s, \mathbf{p}, \boldsymbol{\lambda}_s, \mathbf{u}_b)$-equation to obtain:

$$\begin{bmatrix} \mathbf{K}_p - \omega^2 \hat{\mathbf{M}}_p & 0 & \omega^2 \hat{\mathcal{L}}_{bf} \\ 0 & -\mathbf{B}_s^T \left(\mathbf{K}_s - \omega^2 \mathbf{M}_s\right)^{-1} \mathbf{B}_s & -\mathcal{L}_{bs} \\ \omega^2 \hat{\mathcal{L}}_{bf}^T & -\mathcal{L}_{bs}^T & -\omega^2 \mathcal{L}_{bf}^T \mathcal{M}_\alpha^{(b)} \mathcal{L}_{bf} \end{bmatrix} \begin{Bmatrix} \mathbf{p}_f \\ \boldsymbol{\lambda}_s \\ \mathbf{u}_b \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

- Physically, the term $(\mathbf{B}_s^T \left(\mathbf{K}_s - \omega^2 \mathbf{M}_s\right)^{-1} \mathbf{B}_s)$ corresponds to the *dynamic flexibility of the sound source* and the term $(\mathcal{L}_{bf}^T \mathcal{M}_\alpha^{(b)} \mathcal{L}_{bf})$ may be viewed as the *added mass of the sound source oscilliating rigidly*.

Hence, the determination of the sound pressure $\mathbf{p}$ is equivalent to ignoring both the sound-source dynamic flexibility and added mass.

Finally, the $(\mathbf{p}, \mathbf{u}_b)$-equation!

$$\begin{bmatrix} \mathbf{K}_p - \omega^2 \hat{\mathbf{M}}_p & \omega^2 \hat{\mathcal{L}}_{bf} \\ \omega^2 \hat{\mathcal{L}}_{bf}^T & \mathcal{L}_{bs}^T [\mathbf{B}_s^T \ (\mathbf{K}_s - \omega^2 \mathbf{M}_s)^{-1} \ \mathbf{B}_s]^{-1} \mathcal{L}_{bs} - \omega^2 \mathcal{L}_{bf}^T \mathcal{M}_\alpha^{(b)} \mathcal{L}_{bf} \end{bmatrix} \begin{Bmatrix} \mathbf{p}_f \\ \mathbf{u}_b \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Since $\mathbf{u}_b$ is the displacement of the acoustic medium boundary displacement, the $\mathbf{u}_b$ operator, viz., $\{\mathcal{L}_{bs}^T [\mathbf{B}_s^T \ (\mathbf{K}_s - \omega^2 \mathbf{M}_s)^{-1} \ \mathbf{B}_s]^{-1} \mathcal{L}_{bs} - \omega^2 \mathcal{L}_{bf}^T \mathcal{M}_\alpha^{(b)} \mathcal{L}_{bf}\} \approx [\mathcal{K}_b - \omega^2 \mathcal{M}_b]$, truly captures the sound-source dynamics which is coupled to the acoustic pressure via $\hat{\mathcal{L}}_{bf}$.