Calculus of Variations and Euler-Lagrange Equation

A theoretical derivation of the Euler-Lagrange equations can be carried out by utilizing the calculus of variations of a definite integral. To this end, let us now address how one obtains a stationary value of a function via the calculus of variations.

First, the concept of virtual displacement should not be confused with the concept of the variation of a function. For we have the virtual displacement at our disposal, but the variation of the function is not. Suppose we are to obtain a stationary value of

$$\Pi = \int_a^b G(y, y', x)dx \tag{1}$$

with the boundary conditions

$$y(a) = \alpha, \quad y(b) = \beta \tag{2}$$

Assume that $y = f(x)$ by hypothesis gives a stationary value to Π. One way to prove this to be true is to evaluate the same integral for a slightly modified function $\bar{y}$ and establish that the rate of change of Π due to the change in y is zero (why?). We can thus write

$$\bar{y} = f(x) + \epsilon g(x) = y(x) + \epsilon g(x) \tag{3}$$

where $g(x)$ is an arbitrary function that must be continuous and differentiable as y. Since $g(x)$ is an arbitrary function, this difference is called the variation of the function y and is denoted by δy:

$$\delta y = \bar{y} - y = \epsilon g(x) \tag{4}$$

This is a seemingly trivial expression but there is an important property:

> *The variation of δy refers to an arbitrary infinitesimal change of the value of the dependent variable of y, at the point x. The independent variable, x, does not participate in the process of variation.*

A consequence of the above statement is

$$\delta x = 0 \tag{5}$$

hence

$$\delta y(a) = \delta y(b) = 0 \tag{6}$$

Before we proceed to minimize Π in (1), we need to establish two additional properties of the δ-process. Since $G(y, y', x)$ involves y', we need to know how to express $\delta y'$. To this end, we note from (4)

$$\frac{\partial}{\partial x}\delta y = \frac{\partial}{\partial x}(\bar{y} - y) = \frac{\partial}{\partial x}(\epsilon g(x)) = \epsilon g' \tag{7}$$

which is *the derivative of the variation* δy. On the other hand, for *the variation of the derivative*, we have

$$\delta y' = \bar{y}' - y' = (y + \epsilon g)' - y' = \epsilon g' \tag{8}$$

Equations (7) and (8) give

$$\frac{\partial}{\partial x}\delta y = \delta \frac{\partial y}{\partial x} \tag{9}$$

Hence, the derivative of the variation is the same as the variation of the derivative. Similarly, one can show

$$\delta \int_a^b G(y, y', x)dx = \int_a^b \delta G(y, y', x)dx \tag{10}$$

In other words, variation and differentiation are commutative. Similarly, one can show that variation and integration are also commutative.

We are now ready to minimize Π in (1). First, we have

$$\delta G(y, y', x) = G(y + \epsilon g, y' + \epsilon g', x) - G(y, y', x) \tag{11}$$

$$= \epsilon(\frac{\partial G}{\partial y} g + \frac{\partial G}{\partial y'} g') \tag{12}$$

Now, we have for the variation of the definite integral (2) from (10) and (12):

$$\delta \Pi = \delta \int_a^b G(y, y', x)dx = \int_a^b \delta G(y, y', x)dx \tag{13}$$

$$= \epsilon \int_a^b (\frac{\partial G}{\partial y} g + \frac{\partial G}{\partial y'} g')dx \tag{14}$$

But, via the rule of integration by parts, we have

$$\int_a^b \frac{\partial G}{\partial y'} g' dx = [\frac{\partial G}{\partial y'} g]_a^b - \int_a^b \frac{d}{dx}(\frac{\partial G}{\partial y'})g dx \tag{15}$$

Since the variation of y at $x = a$ and $x = b$ is zero from (6), we have the following variational quantities:

$$\{\delta y(a) = \epsilon g(a) = 0, \quad \delta y(b) = \epsilon g(b) = 0\} => g(a) = g(b) = 0 \tag{16}$$

so that we have

$$[\frac{\partial G}{\partial y'} g(b)] - [\frac{\partial G}{\partial y'} g(a)] = 0 \tag{17}$$

Substituting (15) and (17) into (15), we obtain

$$\delta\Pi = \epsilon \int_a^b (\frac{\partial G}{\partial y} - \frac{d}{dx}\frac{\partial G}{\partial y'})g dx \tag{18}$$

As ϵ is associated with an arbitrary variation of y, the stationary value of Π is

$$\frac{\delta\Pi}{\epsilon} = \int_a^b (\frac{\partial G}{\partial y} - \frac{d}{dx}\frac{\partial G}{\partial y'})g(x) dx = 0 \tag{19}$$

Since $g(x)$ is an arbitrary function, designating the variation of y, we must for the stationary value of Π have

$$\frac{\partial G}{\partial y} - \frac{d}{dx}\frac{\partial G}{\partial y'} = 0 \tag{20}$$

This is the celebrated Euler-Lagrange equation in mechanics when $G(y, y', x)$ is replaced by the Lagrangian function, L , with substitutions of y by q and x by t:

$$G(y, y', x) = L = T(\dot{\mathbf{q}}, \mathbf{q}) - V(\mathbf{q}) \tag{21}$$

so that we have from (20) the final equations of motion

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\mathbf{q}}} - \frac{\partial T}{\partial \mathbf{q}} + \frac{\partial V}{\partial \mathbf{q}} = 0 \tag{22}$$

When G is of the form

$$G = G(y, y', y'', x) \tag{23}$$

the resulting governing equation is given by

$$\frac{\partial G}{\partial y} - \frac{d}{dx}\frac{\partial G}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial G}{\partial y''} = 0 \tag{24}$$

with the boundary conditions given by

$$[(\frac{\partial G}{\partial y'} - \frac{d}{dx}\frac{\partial G}{\partial y''})\delta y + \frac{\partial G}{\partial y''}\delta y']|_a^b = 0 \tag{25}$$

The preceeding equations are applicable for a beam under gravity load for which G becomes

$$G = EI(\frac{\partial^2 w(x)}{\partial x^2})^2 - \rho g w(x) \tag{26}$$