1) Given: $$\underline{\nabla} = \frac{\partial}{\partial x_1}\underline{i} + \frac{\partial}{\partial x_2}\underline{j} + \frac{\partial}{\partial x_3}\underline{k}$$
$$\underline{u} = u_1\underline{i} + u_2\underline{j} + u_3\underline{k}$$
Perform: $$\underline{\nabla}\underline{u} = grad(\underline{u})$$

$$\Rightarrow \underline{\nabla}\underline{u} = \frac{\partial}{\partial x_1}\left(u_1\underline{i} + u_2\underline{j} + u_3\underline{k}\right)\underline{i} + \frac{\partial}{\partial x_2}\left(u_1\underline{i} + u_2\underline{j} + u_3\underline{k}\right)\underline{j} + \frac{\partial}{\partial x_3}\left(u_1\underline{i} + u_2\underline{j} + u_3\underline{k}\right)\underline{k}$$

$$\Rightarrow \underline{\nabla}\underline{u} = \left(\frac{\partial u_1}{\partial x_1}\underline{i} + \frac{\partial u_2}{\partial x_1}\underline{j} + \frac{\partial u_3}{\partial x_1}\underline{k}\right)\underline{i} + \left(\frac{\partial u_1}{\partial x_2}\underline{i} + \frac{\partial u_2}{\partial x_2}\underline{j} + \frac{\partial u_3}{\partial x_2}\underline{k}\right)\underline{j} + \left(\frac{\partial u_1}{\partial x_3}\underline{i} + \frac{\partial u_2}{\partial x_3}\underline{j} + \frac{\partial u_3}{\partial x_3}\underline{k}\right)\underline{k}$$

$$\Rightarrow \underline{\nabla}\underline{u} = \frac{\partial u_1}{\partial x_1}\underline{i}\,\underline{i} + \frac{\partial u_2}{\partial x_1}\underline{j}\,\underline{i} + \frac{\partial u_3}{\partial x_1}\underline{k}\,\underline{i} + \frac{\partial u_1}{\partial x_2}\underline{i}\underline{j} + \frac{\partial u_2}{\partial x_2}\underline{j}\underline{j} + \frac{\partial u_3}{\partial x_2}\underline{k}\underline{j} + \frac{\partial u_1}{\partial x_3}\underline{i}\,\underline{k} + \frac{\partial u_2}{\partial x_3}\underline{j}\underline{k} + \frac{\partial u_3}{\partial x_3}\underline{k}\underline{k}$$

$$\Rightarrow \underline{\nabla}\underline{u} = \left\{\underline{i} \quad \underline{j} \quad \underline{k}\right\}\begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}\begin{Bmatrix} \underline{i} \\ \underline{j} \\ \underline{k} \end{Bmatrix}$$

2) When the position vector $\underline{r}$ is given by:
$$\underline{r} = r\underline{e}_r + z\underline{k} = x\underline{i} + y\underline{j} + z\underline{k}$$
$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$
Obtain: $$\underline{\nabla}(r,\theta,z) = ?$$

$$\Rightarrow \underline{r} = \sqrt{x^2 + y^2}$$
$$\Rightarrow \begin{bmatrix} \underline{i} \\ \underline{j} \\ \underline{k} \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \underline{e}_r \\ \underline{e}_\theta \\ \underline{k} \end{bmatrix}$$
$$\Rightarrow \frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} = \cos(\theta)\frac{\partial}{\partial r} + \frac{-\sin(\theta)}{r}\frac{\partial}{\partial \theta}$$
$$\Rightarrow \frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} = \sin(\theta)\frac{\partial}{\partial r} + \frac{\cos(\theta)}{r}\frac{\partial}{\partial \theta}$$

$$\Rightarrow \underline{\nabla} = \frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k}$$

$$\Rightarrow \underline{\nabla} = \left(\cos(\theta)\frac{\partial}{\partial r} + \frac{-\sin(\theta)}{r}\frac{\partial}{\partial \theta}\right)\left(\cos(\theta)\underline{e}_r - \sin(\theta)\underline{e}_\theta\right) + \left(\sin(\theta)\frac{\partial}{\partial r} + \frac{\cos(\theta)}{r}\frac{\partial}{\partial \theta}\right)\left(\sin(\theta)\underline{e}_r + \cos(\theta)\underline{e}_\theta\right) + \frac{\partial}{\partial z}\underline{k}$$

$$\Rightarrow \underline{\nabla} = \left(\cos(\theta)\frac{\partial}{\partial r} + \frac{-\sin(\theta)}{r}\frac{\partial}{\partial \theta}\right)\left(\cos(\theta)\underline{e}_r - \sin(\theta)\underline{e}_\theta\right) + \left(\sin(\theta)\frac{\partial}{\partial r} + \frac{\cos(\theta)}{r}\frac{\partial}{\partial \theta}\right)\left(\sin(\theta)\underline{e}_r + \cos(\theta)\underline{e}_\theta\right) + \frac{\partial}{\partial z}\underline{k}$$

Expand to cancel terms and arrive at:

$$\Rightarrow \underline{\nabla}(r,\theta,z) = \frac{\partial}{\partial r}\underline{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\underline{e}_\theta + \frac{\partial}{\partial z}\underline{k}$$

3) Using the result of problem 2, obtain $\underline{\nabla}\underline{u}$ where:

$$\underline{u} = u_r\underline{e}_r + u_\theta\underline{e}_\theta + u_z\underline{e}_z \qquad \underline{e}_z = \underline{k}$$

Also extract the linear strains.

$$\frac{d\underline{e}_r}{d\theta} = \underline{e}_\theta \quad \frac{d\underline{e}_\theta}{d\theta} = -\underline{e}_r$$

$$\Rightarrow \underline{\nabla}\underline{u} = \frac{\partial}{\partial r}\left(u_r\underline{e}_r + u_\theta\underline{e}_\theta + u_z\underline{k}\right)\underline{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\left(u_r\underline{e}_r + u_\theta\underline{e}_\theta + u_z\underline{k}\right)\underline{e}_\theta + \frac{\partial}{\partial z}\left(u_r\underline{e}_r + u_\theta\underline{e}_\theta + u_z\underline{k}\right)\underline{k}$$

$$\Rightarrow \underline{\nabla}\underline{u} = \left(\frac{\partial u_r}{\partial r}\underline{e}_r + \frac{\partial u_\theta}{\partial r}\underline{e}_\theta + \frac{\partial u_z}{\partial r}\underline{k}\right)\underline{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial u_r}{\partial \theta}\underline{e}_r + u_r\frac{\partial \underline{e}_r}{\partial \theta} + \frac{\partial u_\theta}{\partial \theta}\underline{e}_\theta + u_\theta\frac{\partial \underline{e}_\theta}{\partial \theta} + \frac{\partial u_z}{\partial \theta}\underline{k}\right)\underline{e}_\theta + \left(\frac{\partial u_r}{\partial z}\underline{e}_r + \frac{\partial u_\theta}{\partial z}\underline{e}_\theta + \frac{\partial u_z}{\partial z}\underline{k}\right)\underline{k}$$

$$\Rightarrow \underline{\nabla}\underline{u} = \begin{bmatrix}\underline{e}_r & \underline{e}_\theta & \underline{k}\end{bmatrix}\begin{bmatrix} \frac{\partial u_r}{\partial r} & \frac{\partial u_\theta}{\partial r} & \frac{\partial u_z}{\partial r} \\ \frac{1}{r}\left(\frac{\partial u_r}{\partial \theta} - u_\theta\right) & \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right) & \frac{1}{r}\left(\frac{\partial u_z}{\partial \theta}\right) \\ \frac{\partial u_r}{\partial z} & \frac{\partial u_\theta}{\partial z} & \frac{\partial u_z}{\partial z} \end{bmatrix}\begin{bmatrix}\underline{e}_r \\ \underline{e}_\theta \\ \underline{k}\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} e_{rr} & e_{r\theta} & e_{rz} \\ e_{\theta r} & e_{\theta\theta} & e_{\theta z} \\ e_{zr} & e_{z\theta} & e_{zz} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_r}{\partial r} & \frac{1}{2}\left(\frac{1}{r}\left(\frac{\partial u_r}{\partial \theta} - u_\theta\right) + \frac{\partial u_\theta}{\partial r}\right) & \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \\ \frac{1}{2}\left(\frac{1}{r}\left(\frac{\partial u_r}{\partial \theta} - u_\theta\right) + \frac{\partial u_\theta}{\partial r}\right) & \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right) & \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\left(\frac{\partial u_z}{\partial \theta}\right)\right) \\ \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) & \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\left(\frac{\partial u_z}{\partial \theta}\right)\right) & \frac{\partial u_z}{\partial z} \end{bmatrix}$$

4) Determine the stress function $\phi(x,y)$ that satisfies the following boundary conditions: (See Review Homework questions for diagram)

$$\sigma_{xx}\big|_{x=0} = -a_1 yx$$
$$\sigma_{yy} = 0$$
$$\sigma_{xy} = -a_2 y^2$$

$$\Rightarrow \sigma_{xy} = -\frac{d^2\phi}{dxdy} \Rightarrow \phi = -\frac{a_2}{3} y^3 x$$

$$\Rightarrow \sigma_{xx} = \frac{d^2\phi}{dy^2} \Rightarrow \phi = -\frac{a_1}{6} y^3 x$$

assuming $\frac{1}{6}a_1 = \frac{1}{3}a_2 = c$

$$\Rightarrow \phi = -cy^3 x$$

5) For linear elastic problems, equation (8.2.12) of Lecture 08 may be expressed as:

$$\Pi = \begin{array}{l} \frac{1}{2}\int_V \left(\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \sigma_{zz}\varepsilon_{zz} + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz}\right)dV \\ -\int_V \left(F_x u + F_y v + F_z w\right)dV - \int_s \left(\overset{r}{T_x^*}\, u + \overset{r}{T_y^*}\, v + \overset{r}{T_z^*}\, w\right)ds \end{array}$$

Apply the above equation to obtain for the torsion problem given in the Review Homework Questions.

Since there is only torsion:

$$\Rightarrow \Pi = \frac{1}{2}\int_z \int_A \left(\tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz}\right)dAdz - \int_s \left(\overset{r}{T_x^*}\, u + \overset{r}{T_y^*}\, v + \overset{r}{T_z^*}\, w\right)ds$$

$$\Rightarrow \Pi = \frac{z}{2G}\iint_{xy} \left(\tau_{xz}^2 + \tau_{yz}^2\right)dxdy - \int_s \left(\overset{r}{T_x^*}\, u + \overset{r}{T_y^*}\, v + \overset{r}{T_z^*}\, w\right)ds$$

$$\Rightarrow \Pi = \frac{z}{2G}\iint_{xy} \left(\left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial x}\right)^2\right)dxdy - \int_s \left(\overset{r}{T_x^*}\, u + \overset{r}{T_y^*}\, v + \overset{r}{T_z^*}\, w\right)ds$$

$$\Rightarrow \Pi = \frac{z}{2G}\iint_{xy} \left(\left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial x}\right)^2\right)dxdy - \iint_{xy} \left(v_3\left(\sigma_{xz}^* u + \sigma_{yz}^* v\right) + \left(\sigma_{xz}^* v_1 + \sigma_{yz}^* v_2\right)w\right)dxdy$$

$$\Rightarrow \Pi = \frac{z}{2G}\iint_{xy} \left(\left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial x}\right)^2\right)dxdy - \iint_{xy} \left(v_3\left(-\sigma_{xz}^*\theta zy + \sigma_{yz}^*\theta zx\right)\right)dxdy$$

$$\Rightarrow \Pi = \frac{z}{2G}\iint_{xy} \left(\left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial x}\right)^2\right)dxdy - z\iint_{xy} \left(\left(\frac{\partial\phi}{\partial y}\theta y + \frac{\partial\phi}{\partial x}\theta x\right)\right)dxdy$$

$$\Rightarrow \Pi = \frac{z}{G}\iint_{xy}\left(\frac{1}{2}\left(\left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial x}\right)^2\right) - 2G\theta\phi\right)dxdy$$