Supplementary Lecture Notes on Kinematics – Descriptions of Angular Orientations

1. Euler's Axis of Rotation

Let's begin with Euler's Theorem:

Two arbitrary oriented dextral basis vectors, $\mathbf{r}$ *and* $\mathbf{r}^*$, *with common origin* $\mathbf{O}$ *can be made to coincide with one another by rotating one of them through a certain angle about an axis which passes through* $\mathbf{O}$.

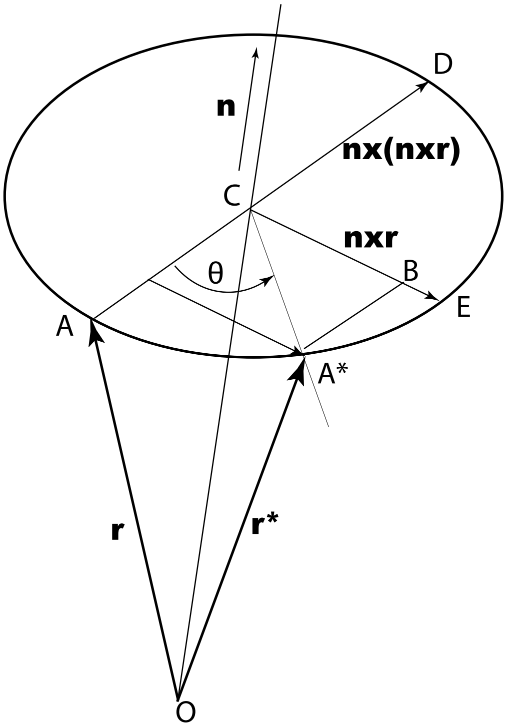

Figure for Euler's Axis of Rotation

Let us construct now the rotational tensor (or operator) $\mathbf{R}$ in terms of the axis of rotation $\mathbf{n}$ and its rotational angle ϕ. As shown in the figure above, let us rotate vector $\mathbf{r}$ around $\mathbf{n}$ by ϕ in a rigid motion into $\mathbf{r}^*$. Thus, we obtain for the new rotated vector $\mathbf{r}^*$ as

$$\mathbf{r}^* \begin{cases} = \mathbf{OC} + \mathbf{CB} + \mathbf{BA}^* \\ = (\mathbf{r}\cdot\mathbf{n})\mathbf{n} - \cos\phi\,\mathbf{n}\times(\mathbf{n}\times\mathbf{r}) + \sin\phi\,\mathbf{n}\times\mathbf{r} \\ = \{\mathbf{nn} + \cos\phi(\mathbf{I}-\mathbf{nn}) + \sin\phi\,\mathbf{n}\times\mathbf{I}\}\cdot\mathbf{r} \\ = \mathbf{R}\cdot\mathbf{r} \end{cases} \tag{1}$$

from which we identify the rotational tensor $\mathbf{R}$ to be

$$\mathbf{R}(\mathbf{n},\phi) = \mathbf{nn} + \cos\phi(\mathbf{I}-\mathbf{nn}) + \sin\phi\,\mathbf{n}\times\mathbf{I} \tag{2}$$

or its matrix equivalent as

$$R^T(n,\phi) = nn^T + \cos\phi(I - nn^T) + \sin\phi\,\tilde{n} \tag{3}$$

$$R(n,\phi) = nn^T + \cos\phi(I - nn^T) - \sin\phi\tilde{n} \tag{4}$$

Note that there are nine direction cosines in **R** for which we must impose six constraints due to orthogonality.

$$\sum_{i=1}^{3} R_{ji}R_{ki} = \delta_{jk} \qquad , \qquad j,k = 1,2,3. \tag{5}$$

This leaves three coordinates to remain independent. Before we specialize **R** to various rotational representations, let us show that it is in fact a rotational operator. To see this, let us apply it to a vector, **r**:

If **r** is parallel to **n** ($\mathbf{r} = \alpha\mathbf{n}$) , then we have

$$\mathbf{R}(\mathbf{n},\phi)\cdot\mathbf{r} = \alpha\mathbf{n} = \mathbf{r} \tag{6}$$

Hence, $\mathbf{R}(\mathbf{n},\phi)$ leaves unchanged all vectors that are parallel to **n**.

If **r** is perpendicular to **n** ($\mathbf{r}\cdot\mathbf{n} = 0$), we have

$$\mathbf{R}(\mathbf{n},\phi)\cdot\mathbf{r} = \cos\phi\mathbf{r} + \sin\phi\mathbf{n}\times\mathbf{r} \tag{7}$$

Hence **r** has been rotated in its plane through the angle ϕ. In general, if **r** is any vector, its component parallel to **n** is preserved during the rotational operation, $\mathbf{R}(\mathbf{n},\phi)$, and its component perpendicular to **n** will be rotated about **n** through the angle ϕ.

2. Euler Angles

Let us consider a specific orthogonal basis, $\lfloor\mathbf{i}\ \mathbf{j}\ \mathbf{k}\rfloor$ and three successive rotations in the following manner. First, rotate the **k**-axis through angle ψ. The corresponding rotatinal operator from (3.39) is

$$\mathbf{R}(\mathbf{k},\psi) = \mathbf{kk} + cos\psi(\mathbf{I} - \mathbf{kk}) + sin\psi\mathbf{I}\times\mathbf{k} \tag{8}$$

in which

$$\mathbf{I} = \mathbf{ii} + \mathbf{jj} + \mathbf{kk} \tag{9}$$

Second, denote the new rotated **i**-axis as $\mathbf{i}'$. Now, rotate the $\mathbf{i}'$-axis through angle θ such that

$$\mathbf{R}(\mathbf{i}',\theta) = \mathbf{i}'\mathbf{i}' + \cos\theta(\mathbf{I} - \mathbf{i}'\mathbf{i}') + \sin\theta\mathbf{I}\times\mathbf{i}' \tag{10}$$

Third, denote the new rotated **k**-axis as $\mathbf{k}'$ and rotate the $\mathbf{k}'$-axis through angle ϕ so that we have

$$\mathbf{R}(\mathbf{k}',\phi) = \mathbf{k}'\mathbf{k}' + \cos\phi(\mathbf{I} - \mathbf{k}'\mathbf{k}') + \sin\phi\mathbf{I}\times\mathbf{k}' \tag{11}$$

Finally, if we express the compounded rotation to be effectively about the **n**-axis through an angle β, we have

$$\mathbf{R}(\mathbf{n},\beta) = \mathbf{R}(\mathbf{k}',\phi)\cdot\mathbf{R}(\mathbf{i}',\theta)\cdot\mathbf{R}(\mathbf{k},\psi) \tag{12}$$

How do we use (12) to describe the rotation of a vector **r**? The answer is simply

$$\mathbf{r}' = \mathbf{R}(\mathbf{k}', \phi) \cdot \mathbf{R}(\mathbf{i}', \theta) \cdot \mathbf{R}(\mathbf{k}, \phi) \cdot \mathbf{r} \tag{13}$$

Note, however, that the new rotated vector, $\mathbf{r}'$, has its three components in $\mathbf{e} = \lfloor \mathbf{i}\ \mathbf{j}\ \mathbf{k} \rfloor^T$ and not in $\mathbf{b} = \lfloor i'''\ j'''\ k''' \rfloor^T$!

To obtain the matrix expression for (13), we note

$$\mathbf{r} = \{r^e\}^T \mathbf{e} = \{r^b\}^T \mathbf{b} \tag{14}$$

Pre-dot producting by **b**, we obtain

$$\{r^b\} = \mathbf{b} \cdot \mathbf{e}^T \{r^e\} = R^{be}\{r^e\} \tag{15}$$

Since

$$\mathbf{b} = \mathbf{R}(\mathbf{n}, \beta) \cdot \mathbf{e} \tag{16}$$

we have

$$R^{be} = \mathbf{b} \cdot \mathbf{e}^T = (\mathbf{R} \cdot \mathbf{e}) \cdot \mathbf{e}^T$$

or in component form

$$R^{be}_{ij} = \mathbf{e}_j \cdot \mathbf{R}(\mathbf{n}, \beta) \cdot \mathbf{e}_i \tag{17}$$

It can be shown that (12) is identical with the familiar successive projection

$$R^{be} = R_3 R_2 R_1 \tag{18}$$

in which

$$\begin{gathered} R_1 = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ R_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \\ R_3 = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{gathered} \tag{19}$$

3. Euler Parameters

Let us recall the dyadic operator (2) as

$$\mathbf{R}(\mathbf{n}, \beta) = \mathbf{nn} + \cos\beta(\mathbf{I} - \mathbf{nn}) + \sin\beta\,\mathbf{n} \times \mathbf{I} \tag{20}$$

and introduce

$$q_0 = \cos\frac{\beta}{2}, \qquad \mathbf{q} = \mathbf{n}\sin\frac{\beta}{2}, \qquad \{q\} = \langle\, q_1 \quad q_2 \quad q_3 \,\rangle^T \tag{21}$$

so that

$$q_0^2 + \mathbf{q} \cdot \mathbf{q} = q_0^2 + q_1^2 + q_2^2 + q_3^2 = \cos^2 \frac{\beta}{2} + \sin^2 \frac{\beta}{2} = 1 \tag{22}$$

The four parameters, (q_0, q_1, q_2, q_3), are called the *Euler parameters*. Substituting (21) and by making use of the constraint condition (22), we obtain

$$\mathbf{R}(q_0, \mathbf{q}) = (2q_0^2 - 1)\mathbf{I} + 2\mathbf{qq} + 2q_0(\mathbf{q} \times \mathbf{I}) \tag{23}$$

from which one obtains the corresponding matrix expression

$$R = (2q_0^2 - 1)I + 2\{q\}\{q\}^T - 2q_0\tilde{q} \tag{24}$$

which, when expanded, can be expressed as

$$R = \begin{bmatrix} 2(q_0^2 + q_1^2) - 1 & 2(q_1q_2 + q_0q_3) & 2(q_1q_3 - q_0q_2) \\ 2(q_1q_2 - q_0q_3) & 2(q_0^2 + q_2^2) - 1 & 2(q_2q_3 + q_0q_1) \\ 2(q_1q_3 + q_0q_2) & 2(q_2q_3 - q_0q_1) & 2(q_0^2 + q_3^2) - 1 \end{bmatrix} \tag{25}$$

4. Angular Velocity

Let us assume that **b** is attached to the moving particle, P. The time rate of **b** with respect to the inertially fixed basis vector, **e**, can be obtained from (16)

$$\frac{{}^E d\mathbf{b}}{dt} = \frac{dR}{dt}\mathbf{e} + R\frac{{}^E d\mathbf{e}}{dt} = \frac{dR}{dt}\mathbf{e} = \dot{R}R^T\mathbf{b} \tag{26}$$

We define a skew-symmetric matrix

$$\tilde{\omega} = -\dot{R}R^T = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{27}$$

The three components in the above equation are called the angular velocity components in the **b**-basis with respect to the **e**-basis, which can be written as

$${}^E\dot{\mathbf{b}} = \boldsymbol{\omega} \times \mathbf{b} \tag{28}$$

where the angular velocity vector, $\boldsymbol{\omega}$, can be written as

$$\boldsymbol{\omega} =^b \omega_1\mathbf{b}_1 +^b \omega_2\mathbf{b}_2 +^b \omega_3\mathbf{b}_3 \tag{29}$$

To show that $\dot{R}R^T$ can in fact be written as in (26), we note

$$\mathbf{b} \cdot \mathbf{b}^T = I \tag{30}$$

Time differentiation of the above expression yields

$${}^E\dot{\mathbf{b}} \cdot \mathbf{b}^T + \mathbf{b} \cdot {}^E\dot{\mathbf{b}}^T = 0 \tag{31}$$

Let us assume for the moment

$$^{E}\dot{\mathbf{b}} = \mathbf{C}\mathbf{b} \tag{32}$$

Substitution of (32) into (31) leads to

$$\mathbf{C}\mathbf{b} \cdot \mathbf{b}^T + \mathbf{b} \cdot \mathbf{b}^T \mathbf{C}^T = 0 \qquad \Rightarrow \qquad \mathbf{C} + \mathbf{C}^T = 0 \tag{33}$$

Hence **C** is a skew-symmetric matrix and time differentiation of the body-fixed basis vector **b** yields

$$\begin{aligned} \frac{^{E}d\mathbf{b}_1}{dt} &= \boldsymbol{\omega} \times \mathbf{b}_1 \\ \frac{^{E}d\mathbf{b}_2}{dt} &= \boldsymbol{\omega} \times \mathbf{b}_2 \\ \frac{^{E}d\mathbf{b}_3}{dt} &= \boldsymbol{\omega} \times \mathbf{b}_3 \end{aligned} \tag{34}$$

5. Angular Velocity in Terms of Euler Angles

The Euler angles have been derived in Section 2. The derivation of the angular velicity vector $\boldsymbol{\omega}$ in terms of the Euler angles can be carried out in several different ways. In this course we will employ both the rotation tensor (2) and the basic definition of the angular velocity components (27). First, we notice that the angular velocity due to the rotation, ψ, around the **k**-axis is

$$^{e}\boldsymbol{\omega}_{\psi} = \dot{\psi}\mathbf{k} \tag{35}$$

Second, the angular velocity due to the rotation, θ, around the $\mathbf{i}'$-axis after the rotation of ψ around the **k**-axis is

$$^{e}\boldsymbol{\omega}_{\theta} = \mathbf{R}(\mathbf{k}, \psi) \cdot \dot{\theta}\mathbf{i} = \dot{\theta}\mathbf{i}' \tag{36}$$

Third, the angular velocity due to the rotation, ϕ, around the $\mathbf{k}^{''}$-axis after the two preceding two successive rotation is

$$^{e}\boldsymbol{\omega}_{\phi} = \mathbf{R}(\mathbf{k}, \psi) \cdot \mathbf{R}(\mathbf{i}, \theta) \cdot \dot{\phi}\mathbf{k} = \dot{\phi}\mathbf{k}'' \tag{37}$$

Finally, the compounded angular velocity vector is obtained by summing the above three components:

$$\begin{aligned} \boldsymbol{\omega} &=^{e} \boldsymbol{\omega}_{\psi} +^{e} \boldsymbol{\omega}_{\theta} +^{e} \boldsymbol{\omega}_{\phi} \\ &= \dot{\psi}\mathbf{k} + \dot{\theta}(\cos\psi\mathbf{i} + \sin\psi\mathbf{j}) + \dot{\phi}(\cos\theta\mathbf{k} + \sin\theta \cdot \sin\psi\mathbf{i} - \sin\theta\cos\psi j) \end{aligned} \tag{38}$$

or in matrix forms

$$\begin{Bmatrix} ^{e}\omega_i \\ ^{e}\omega_j \\ ^{e}\omega_k \end{Bmatrix} = \begin{bmatrix} 0 & \cos\psi & \sin\theta\sin\psi \\ 0 & \cos\psi & -\sin\theta\cos\psi \\ 1 & 0 & \cos\theta \end{bmatrix} \begin{Bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{Bmatrix} \tag{39}$$

Notice that the above resulting angular velocity vector $\boldsymbol{\omega}$ is expressed in the **e**-system. The body-fixed components of $\boldsymbol{\omega}$ can be obtained via the tensor transformation (4) to give

$$\left\{\begin{array}{c} {}^b\omega_i \\ {}^b\omega_j \\ {}^b\omega_k \end{array}\right\} = \begin{bmatrix} \sin\theta\sin\phi & \cos\psi & 0 \\ \cos\theta\sin\phi & -\sin\psi & 0 \\ \cos\theta & 0 & 1 \end{bmatrix} \left\{\begin{array}{c} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{array}\right\} \tag{40}$$

6. Angular Velocity in Terms of the Euler Parameters

The angular velocities in terms of the Euler parameters can be obtained from (24) and (27). First we differentiate (24) to obtain

$$\dot{R} = 4\dot{q}_0 q_0 I + 2\{\dot{q}\}\{q\}^T + 2\{q\}\{\dot{q}\}^T - 2\dot{q}_0\tilde{q} - 2q_0\dot{\tilde{q}} \tag{41}$$

Substituting the above equation and (24) into (27), one can obtain the corresponding expression for the angular velocity as follows:

$$\{\omega\} = 2q_0\{\dot{q}\} - 2\dot{q}_0\{q\} - 2\tilde{q}\{\dot{q}\} \tag{42}$$

As we must satisfy the constraint condition (22), we append its differential form

$$q_0\dot{q}_0 + \{q\}^T\{\dot{q}\} = 0 \tag{43}$$

to form a (4 × 4) equation:

$$\left\{\begin{array}{c} 0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{array}\right\} = 2\begin{bmatrix} q_0 & q_1 & q_2 & q_3 \\ -q_1 & q_0 & q_3 & -q_2 \\ -q_2 & -q_3 & q_0 & q_1 \\ -q_3 & q_2 & -q_1 & q_0 \end{bmatrix} \left\{\begin{array}{c} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{array}\right\} \tag{44}$$

Inverting we obtain:

$$\left\{\begin{array}{c} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{array}\right\} = \frac{1}{2}\begin{bmatrix} 0 & -\{\omega\}^T \\ \{\omega\} & -[\omega] \end{bmatrix} \left\{\begin{array}{c} q_0 \\ q_1 \\ q_2 \\ q_3 \end{array}\right\} \tag{45}$$

7. Velocity and Acceleration of a Rigid Body

Having discussed several means for representing the rotational operator and angular velocity vector in the previous sections, let us consider a position vector **z** which consists of

$$\mathbf{z} = \mathbf{r}_o + \mathbf{r} \tag{46}$$

where $\mathbf{r}_o$ is the position vector from the origin of the inertial frame, O, in the **e**-basis to a body-fixed origin, o, in the **b**-basis, and **r** is the position vector from the body-fixed origin o to a particle point, P in a rigid body. The velocity at the point P is then fobtained as:

$${}^E\dot{\mathbf{z}} ={}^E \dot{\mathbf{r}}_o +{}^B \dot{\mathbf{r}} +{}^B \boldsymbol{\omega} \times {}^B\mathbf{r} \tag{47}$$

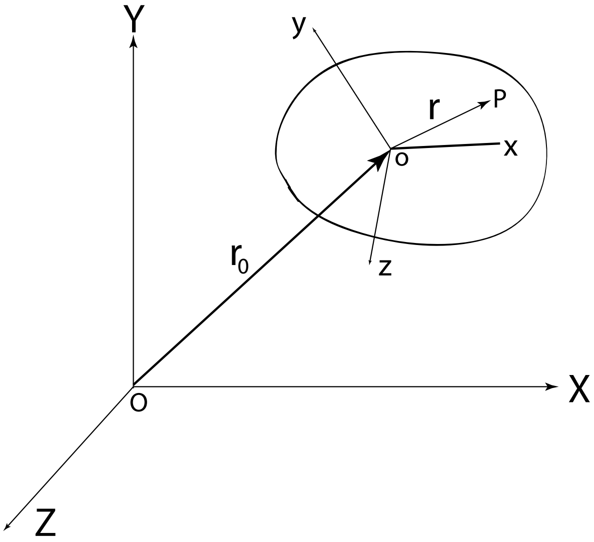

Figure for Euler's Axis of Rotation

The acceleration at the particle position P is then given by the wellknown formula:

$$ {}^E\ddot{\mathbf{z}} = {}^E\ddot{\mathbf{r}}_o + {}^B\ddot{\mathbf{r}} + 2\,{}^B\boldsymbol{\omega} \times {}^B\dot{\mathbf{r}} + {}^B\dot{\boldsymbol{\omega}} \times {}^B\mathbf{r} + {}^B\boldsymbol{\omega} \times {}^B\boldsymbol{\omega} \times {}^B\mathbf{r} \tag{48} $$

For a rigid body, we must have $\ddot{\mathbf{r}} = 0$ and $\dot{\mathbf{r}} = 0$. We will treat the case of deformable bodies when these two qantities play a pivotal role. For now we restrict ourselves to the case of rigid bodies:

$$ {}^E\ddot{\mathbf{z}} = {}^E\ddot{\mathbf{r}}_o + {}^B\boldsymbol{\omega} \times {}^B\boldsymbol{\omega} \times {}^B\mathbf{r} + {}^B\dot{\boldsymbol{\omega}} \times {}^B\mathbf{r} \tag{49} $$

In evaluating the acceleration, the translational acceleration, ${}^E\ddot{\mathbf{z}}_o$, requires no special attention. However, we still have to obtain the angular acceleration, $\dot{\boldsymbol{\omega}}$, which is expressed in terms of the Euler parameters below. To this end, we express the angular velocity in the form

$$ \{\omega\} = T\{\dot{\bar{\mathbf{q}}}\}, \qquad \{\bar{\mathbf{q}}\}^T = \langle\, q_0 \quad q_1 \quad q_2 \quad q_3 \,\rangle \tag{50} $$

where

$$ T = 2\begin{bmatrix} -q_1 & q_0 & q_3 & -q_2 \\ -q_2 & -q_3 & q_0 & q_1 \\ -q_3 & q_2 & -q_1 & q_0 \end{bmatrix} \tag{51} $$

and by differentiating (51), we obtain the angular acceration in temrs of the Euler parameters as

$$ \{\dot{\omega}\} = T\{\ddot{\bar{\mathbf{q}}}\} + \dot{T}\{\dot{\bar{\mathbf{q}}}\} \tag{52} $$

Observe from the above expression that the computations of $\{\dot{\omega}\}$ is greatly simplified when it is expressed in terms of Euler parameters (21) instead of the Euler angles (18) and (19).