Solution of Homework #7 (ASEN5010, Spring 2005)

(Due at Start of Class on Wednesday, 16 March 2005)

Stability Regions of Kelvin's Gyrostat

Problem 7.1 Derivation of Linearized Equations

Starting with the governing equations of motion in the form:

$$\mathbf{J} \cdot \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\mathbf{J} \cdot \boldsymbol{\omega} + \mathbf{h}_s) = \mathbf{0} \tag{1}$$

with the following conditions

$$\begin{gathered} \boldsymbol{\omega} = \begin{Bmatrix} 0 \\ 0 \\ \Omega_3 \end{Bmatrix} \\ \mathbf{H} = \mathbf{J} \cdot \boldsymbol{\omega} + \mathbf{h}_s = \boldsymbol{\lambda}\boldsymbol{\omega}, \qquad \mathbf{h}_s = h_s\ \mathbf{b}_3 \end{gathered} \tag{2}$$

and utilizing the general linearized equation set handed out previously in carrying out Homework#5, derive the following linearized equation:

$$\begin{aligned} & J_3\ddot{\theta}_3 = 0 \\ & \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \Omega_3 \begin{bmatrix} 0 & -(J_2 + J_1 - \lambda) \\ +(J_2 + J_1 - \lambda) & 0 \end{bmatrix} \begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{Bmatrix} \\ & + \Omega_3^2 \begin{bmatrix} (\lambda - J_2) & 0 \\ 0 & (\lambda - J_1) \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = 0 \end{aligned} \tag{3}$$

(Hint: Show that, for the present problem, λ corresponds to J_3 in the standard linearized equation.)

Problem 7.2: Stability analysis

7.2.1 Obtain the characteristic equation of (3) in the form:

$$s^4 + a_2 s^2 + a_4 = 0 \tag{4}$$

7.2.2 Utilizing the Routh-Hurwitz condition, determine the stability condition as

$$a_2 > 0, \quad a_4 > 0, \quad a_2^2 - 4a_4 > 0 \tag{5}$$

7.2.3 From the stability condition obtained from Problem 7.2.2, express the stability conditions in terms of (*Ignore equation (6.117), elegant but complex, and use your simpler expressions!*)

$$\begin{aligned} & x = (J_3 - J_2)/J_1, \qquad y = (J_3 - J_1)/J_2, \qquad \hat{\Omega} = \frac{h_s/(J_1 J_2)^{1/2}}{\Omega_3} \\ & \frac{J_2}{J_1} = \frac{(1-x)}{(1-y)} \quad (provided \quad J_1 + J_2 - J_3 \neq 0) \end{aligned} \tag{6}$$

where $\lambda = J_3 + h_s/\Omega_3$ is to be utilized.

Hint: Use

$$\begin{aligned} \frac{h_s}{J_2\Omega} &= \frac{h_s}{\sqrt{J_1 J_2}\Omega}\sqrt{\frac{J_1}{J_2}} = \hat{\Omega}\sqrt{\frac{(1-y)}{(1-x)}} \\ \frac{h_s}{J_1\Omega} &= \frac{h_s}{\sqrt{J_1 J_2}\Omega}\sqrt{\frac{J_2}{J_1}} = \hat{\Omega}\sqrt{\frac{(1-x)}{(1-y)}} \end{aligned} \tag{7}$$

7.2.4 For $\hat{\Omega} = [\ -0.5,\ -0.2,\ 0.2,\ 0.5\ ]$, and the ranges of $(-1 < x < 1)$ and $(-1 < y < 1)$, plot the stable regions.(*Stable regions must satisfy all of the above three conditions.*)

Hint: Plot the three conditions for a given $\hat{\Omega}$ in x vs. y on the same figure. Then find the stable regions.

Solutions are in the next two pages!

Problems 7.1 and 7.2.1-7.2.2 have been already discussed in the class.

Solution of Problem 7.2.3

The characteristic equation for the lineaized equation can be expressed as

$$\begin{aligned}
&\det\left|\begin{bmatrix} J_1 s^2 + k_1\Omega^2 & -gs\Omega \\ +gs\Omega & J_2 s^2 + k_2\Omega^2 \end{bmatrix}\right| = 0 \\
&k_1/J_1 = (\lambda - J_2)/J_1 = x + \hat{\Omega} p(x, y) \\
&k_2/J_2 = (\lambda - J_1)/J_2 = y + \hat{\Omega}/p(x, y) \\
&g^2/(J_1 J_2) = (J_3 - J_1 J_2 + h/\Omega)^2/(J_1 J_2) = [(x-1)p(x, y) + \hat{\Omega}]^2 \\
&p(x, y) = \sqrt{\frac{1-y}{1-x}}
\end{aligned} \tag{8}$$

From which we obtain

$$\begin{aligned}
a_2 &= [(x + \hat{\Omega} p(x, y)) + (y + \hat{\Omega}/p(x, y)) + [(x-1)p(x, y) + \hat{\Omega}]^2 \\
a_4 &= (x + \hat{\Omega} p(x, y)) * (y + \hat{\Omega}/p(x, y))
\end{aligned} \tag{9}$$

Observe that from (8) we obtain

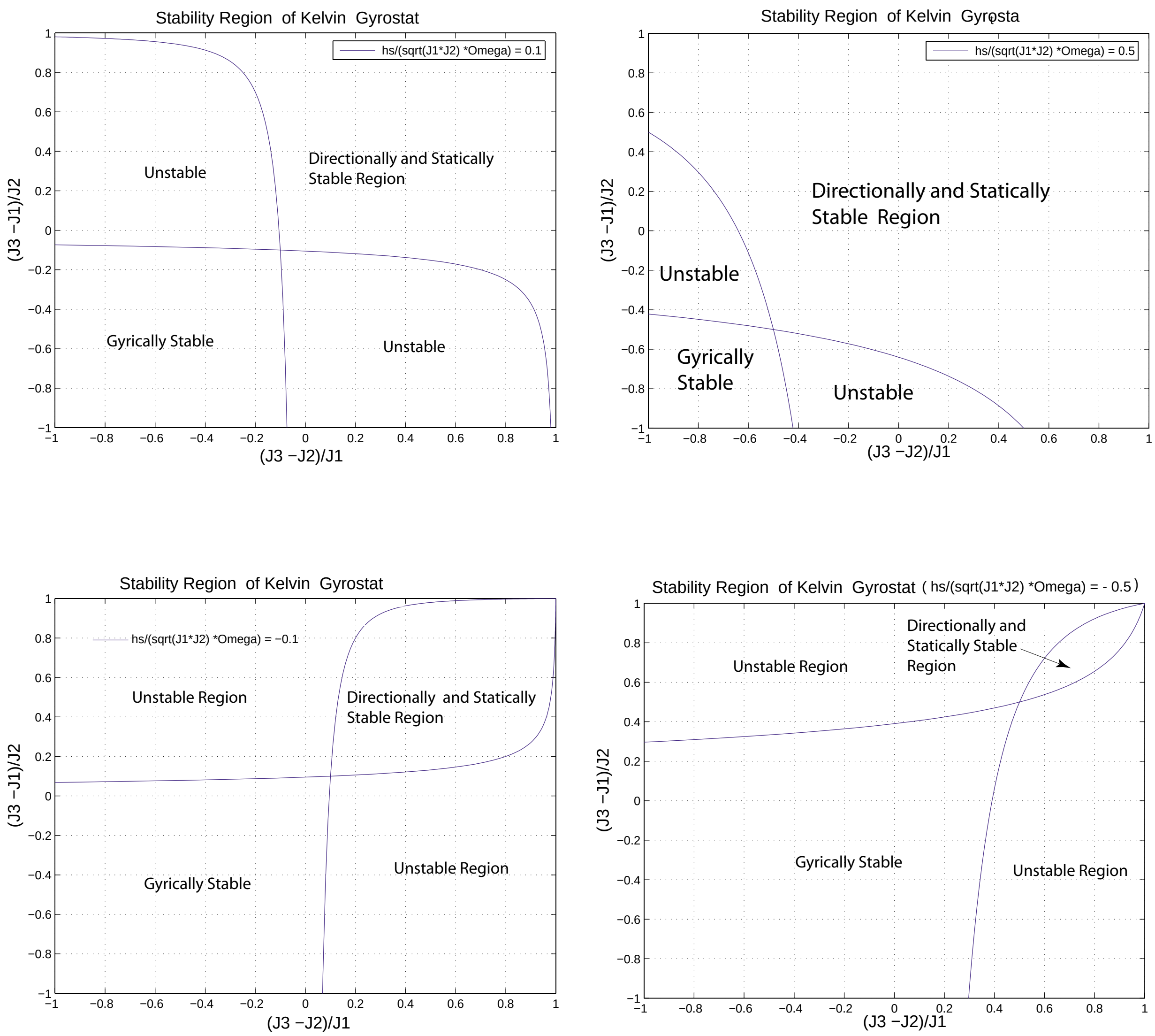
$$\begin{aligned}
&a_2^2 - 4a_4 = (X - Y)^2 + 2(X + Y)[(x-1)p(x, y) + \hat{\Omega}]^2 + [(x-1)p(x, y) + \hat{\Omega}]^4 \\
&X = (x + \hat{\Omega} p(x, y)) \\
&Y = (y + \hat{\Omega}/p(x, y))
\end{aligned} \tag{10}$$

which is satisfied as long as

$$0 \leq X = (x + \hat{\Omega} p(x, y)), \quad 0 \leq Y = (y + \hat{\Omega}/p(x, y)) \tag{11}$$

Solution of Problem 7.2.4:

A Matlab is programmed and the results are as follows:



Stability Regions of Spinning Kelvin Gyrostat