Homework #7 (ASEN5010, Spring 2006)

(Due at Start of Class on Thursday, 23 March 2006)

Problem 7. Stability Analysis of Linearized Euler's Equations of Motion

The perturbed angular velocity and angular acceleration become from the class notes

$$\begin{aligned}\boldsymbol{\omega} &= \boldsymbol{\omega}_0 + \dot{\boldsymbol{\theta}} + [\tilde{\omega}_0]\boldsymbol{\theta} \\ \dot{\boldsymbol{\omega}} &= \ddot{\boldsymbol{\theta}} + [\tilde{\omega}_0]\,\dot{\boldsymbol{\theta}}\end{aligned} \tag{1}$$

Let's recall the Euler's equation under gravity gradient derived in (6.146) of the text as

$$[\mathbf{J}]\,\{\dot{\omega}\} + [\tilde{\omega}]\,[\mathbf{J}]\,\{\omega\} = 3n^2\,[\tilde{\mathbf{R}}_c][\mathbf{J}]\,\{\mathbf{R}_c\}/|\mathbf{R}_c|^2, \quad n^2 = \frac{\mu}{|\mathbf{R}_c|^3} \tag{2}$$

where the term inside [ ] are matrices and the terms inside { } are vectors; and, $\mathbf{R}_c$ is the distance from the satellite to the Earth expressed along the unit vector direction $\mathbf{a}_3$, viz., $\mathbf{R}_c = R_c\mathbf{a}_3$. (See Fig. 6.8 of the text)

It was derived in the class that the above equation can be linearized via (1) as

$$[\mathbf{J}]\,\{\ddot{\boldsymbol{\theta}}\} + [\tilde{\mathbf{D}}_g\ \{\dot{\boldsymbol{\theta}}\} + [\mathbf{K}_c]\,\{\boldsymbol{\theta}\} = 3n^2\,[\tilde{\mathbf{R}}_c][\mathbf{J}]\,[\mathbf{R}_c]/|\mathbf{R}_c|^2 \tag{3}$$

$$\begin{aligned}\tilde{\mathbf{D}}_g &= \mathbf{J}\tilde{\omega}_0 + \tilde{\omega}_0\mathbf{J} - (\widetilde{\mathbf{J}\boldsymbol{\omega}_0}) \\ \mathbf{K}_c &= \tilde{\omega}_0\mathbf{J}\tilde{\omega}_0 - (\widetilde{\mathbf{J}\boldsymbol{\omega}_0})\tilde{\omega}_0\end{aligned} \tag{4}$$

When (3) and (4) are specialized to our problem, we have:

$$\boxed{\begin{aligned}&[\mathbf{J}]\,\{\ddot{\boldsymbol{\theta}}\} + [\tilde{\mathbf{D}}_g\ \{\dot{\boldsymbol{\theta}}\} + [\mathbf{K}_c]\,\{\boldsymbol{\theta}\} = \frac{3n^2}{|\mathbf{R}_c|^2}\,[\tilde{\mathbf{R}}_c][\mathbf{J}]\,\{\mathbf{R}_c\}/|\mathbf{R}_c|^2 \\ &\tilde{\mathbf{D}}_g = \mathbf{J}\tilde{\omega}_0 + \tilde{\omega}_0\mathbf{J} - (\widetilde{\mathbf{J}\boldsymbol{\omega}_0}) \\ &\mathbf{K}_c = \tilde{\omega}_0\mathbf{J}\tilde{\omega}_0 - (\widetilde{\mathbf{J}\boldsymbol{\omega}_0})\tilde{\omega}_0 \\ &\{\omega_0\} = \begin{Bmatrix} 0 \\ -n \\ 0 \end{Bmatrix}, \qquad [\tilde{\omega}_0] = n\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\ &\{\mathbf{R}_c\} = R_c\begin{Bmatrix} -\theta_2 \\ \theta_1 \\ 1 \end{Bmatrix}, \qquad [\mathbf{J}] = \begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix}\end{aligned}} \tag{5}$$

7.1 Obtain $[\tilde{\mathbf{D}}_g]$, $[\mathbf{K}_c]$ and the gravity gratient torque.

7.2 Derive the characteristic equation of the linearized system.

7.3 Construct the Hurwitz matrix (refer to pages 96-97 of the text)

Help: For system of the form

$$A(s) = a_0 s^4 + a_2 s^2 + a_4 = 0$$

take the derivative of $A(s)$ with respect to s, Replace the row of requiring a_1 and a_3 with the coefficients of $dA(s)/ds$.

7.4 Construct the Hurwitz matrix (refer to pages 96-97 of the text) and determine the relationship of the three moments of inertia, (J_1, J_2, J_3), for the stability of the spacecraft.