

ASEN 6367 Advanced FEM Fall 09 Midterm Exam 1

Take Home, assigned Th March 5, 2009, due Tu Mar 10, 2009 by class time
(may be submitted earlier at AE 187 or instructor mailbox at AE 196 or OT 635)

Begin each **QUESTION** on a **separate page**. Write your name on each page. Attach this exam as cover.

QUESTION 1. Conceptual: 30 pts=5 pts per item. **Be brief.** References to answers in Notes are OK.

For items (a–c), $y(x)$ is an ordinary function of x whereas $\Pi[y]$ is a functional of $y(x)$.

- (a) Explain the differences between the ordinary differential dy , the variation δy , and the variation $\delta \Pi$.
- (b) Why is $\delta x = 0$?
- (c) Why are variational forms (if they exist) desirable in numeric computation over weak forms?
- (d) How many canonical variational principles of elasticity exist? (Classics only, exclude hybrids) Which ones are useful for FEM work?
- (e) Why do we bother to use variational principles beyond TPE to derive finite elements?
- (f) What two numerical problems must one must watch out for when doing finite element analysis of axisymmetric solids?

QUESTION 2. Analytical + element testing. 70 pts=20+15+20+15.

Formulate and code a 4-node stress-hybrid quadrilateral ring element for axisymmetric solid analysis. The computer work can be done with any language: Matlab, Mathematica, C, Fortran, etc., you feel comfortable with; since steps are either numeric or may be comfortably done by hand, a CAS is not mandatory.[†] Whatever language you use, **be sure to show your code** to get full credit.

The element has the same number of nodes (4) and degrees of freedom (8) of the Quad4 iso-P ring element derived in Chapter 12, but it is formulated by an equilibrium-stress hybrid functional.

To simplify the work, it is sufficient to derive the element for a *rectangular geometry*, with sides parallel to axes $\{r, z\}$. The node coordinates are: $r_1 = r_4 = a$, $r_2 = r_3 = b$, $z_1 = z_2 = 0$, $z_3 = z_4 = h$, where $a \geq 0$, $b > a$ and $h \geq 0$. The ring spans 1 radian circumferentially. The material is isotropic, with modulus E , Poisson's ratio ν and compliance matrix

$$\mathbf{C} = \mathbf{E}^{-1} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 \\ -\nu & 1 & -\nu & 0 \\ -\nu & -\nu & 1 & 0 \\ 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \quad (\text{Q1.1})$$

Body forces are assumed zero: $b_r = b_z = 0$.

- (a) Start from the stress assumption

$$\begin{bmatrix} \sigma_{rr} \\ \sigma_{zz} \\ \sigma_{\theta\theta} \\ \sigma_{rz} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & \bar{r} & \bar{z} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \bar{r} & 0 & 0 & 0 & \bar{z} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \bar{z} & \bar{r} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \bar{r} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{11} \end{bmatrix}. \quad (\text{Q1.2})$$

Here α_1 through α_{11} are 11 stress parameters, while $\bar{r} = r - r_0$, and $\bar{z} = z - z_0$, in which $r_0 = (r_1 + r_2 + r_3 + r_4)/4$ and $z_0 = (z_1 + z_2 + z_3 + z_4)/4$ are the r and z coordinates of the element center,

[†] If you know how to do derivatives symbolically and solve a symbolic system of equations, item (a) in Question 2 could be speeded up by using a CAS such as *Mathematica*.

respectively. Inserting (Q1.2) into the equilibrium equations (10.13) with zero body forces, show that if one imposes that internal equilibrium is to be satisfied strongly, (Q1.2) reduces to

$$\begin{bmatrix} \sigma_{rr} \\ \sigma_{zz} \\ \sigma_{\theta\theta} \\ \sigma_{rz} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \bar{r} & \bar{z} & 0 & 0 \\ 0 & 1 & -\bar{z}/r & 0 & 0 & \bar{r} & -(r + \bar{r})\bar{z}/r \\ 1 & 0 & 0 & r + \bar{r} & \bar{z} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \bar{r} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_7 \end{bmatrix} \quad \text{or} \quad \boldsymbol{\sigma} = \mathbf{S}\mathbf{a}. \quad (\text{Q1.3})$$

This has only 7 independent stress parameters: a_1, \dots, a_7 . Explain why 7 is a good number for this element. (Hint: think about rank).

- (b) Form the 7×7 flexibility matrix

$$\mathbf{F} = \int_{V^{(e)}} \mathbf{S}^T \mathbf{C} \mathbf{S} dV, \quad (\text{Q1.4})$$

where $dV = r dA$ (note: no 2π factor, since we work with a 1-radian ring). The integration may be done by a 2×2 Gauss rule — it can be done analytically, but even for a rectangle it is a bit messy because of $1/r$ terms. Give the numerical expression of \mathbf{F} for

$$a = 4, \quad b = 10, \quad h = 2, \quad E = 1000, \quad \nu = 1/3. \quad (\text{Q1.5})$$

Check the rank of this \mathbf{F} . If rank is 7 and all eigenvalues are positive, things are fine (because \mathbf{F} can be inverted in the way to \mathbf{K}).

- (c) Construct matrices \mathbf{T} and \mathbf{P} as explained in §9.3. Then obtain the 7×8 connection matrix \mathbf{G}

$$\mathbf{G} = \int_{S^{(e)}} \mathbf{T}^T \mathbf{P} dS, \quad (\text{Q1.6})$$

where $dS = r ds$, s being the boundary arclength. Evaluate this integral by 2-point Gauss on each of the 4 element sides. Give the numerical expression of \mathbf{G} for the data (Q1.5).

- (d) Evaluate the 8×8 stiffness matrix

$$\mathbf{K} = \mathbf{G}^T \mathbf{F}^{-1} \mathbf{G} \quad (\text{Q1.7})$$

Give the value of \mathbf{K} for the data (Q1.5). By taking its eigenvalues, check that \mathbf{K} is nonnegative (meaning that there are no negative eigenvalues) and has the correct rank of 7.

Bonus Question. (up to 10 pts). Total may exceed 100.

Rederive the element using the same stress assumption (Q1.3) and a isoparametric bilinear interpolation of displacements, using the Hellinger-Reissner principle. In this case, \mathbf{K} has the same expression as (Q1.7) and \mathbf{F} is the same (so it does not need to be recomputed). However \mathbf{G} is computed by a volume integral instead of a surface integral. Is the resulting \mathbf{G} the same as that of the hybrid element?