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Shell Structures: Basic Concepts

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§29.1. GENERAL REMARKS

The word *shell* is an old one and is commonly used to describe the hard covering of eggs, crustacea, tortoises, etc. The dictionary says that the word shell is derived from the Latin *scalus*, as in fish scale. But to us now there is a clear difference between the tough but flexible scaly covering of a fish and the tough but rigid shell of, say, a turtle.

In this course we shall be concerned with man-made shell structures as used in various branches of engineering. There are many interesting aspects of the use of shells in engineering, but one alone stands out as being of paramount importance: the *structural* aspect.

The *theory of structures* tends to deal with a class of idealized mathematical models, stripped of many of the features that make them recognizable as useful object in engineering. Thus a *beam* is often idealized as a line endowed with certain mechanical properties, irrespective of whether it is a large bridge, an aircraft wing, or a flat spring inside a machine. In a similar way, the theory of shell structures deals, for example, with the "cylindrical shell" as an idealized entity: it is a cylindrical surface endowed with certain mechanical properties. The treatment is the same whether the actual structure under study is a gas-transmission pipeline, a grain storage silo, or a steam boiler.

Before entering this realm of geometry, in which shells are classified by their geometry (cylindrical, spherical, etc.) rather than their function, it is desirable to give a glimpse of the wide range of applications of shell structures in engineering practice. Indeed, a list of familar examples will be useful in enabling us to pick up some structural features in a qualitative way to provide an introduction to the theory.

§29.2. SHELL STRUCTURES OVERVIEW

§29.2.1. Applications

It is instructive to assemble a list of applications from a historical point of view, and to take as a connecting theme the way in which the introduction of the *thin shell* as a structural form made an important contribution to the development of several branches of engineering. The following is a brief list, which is by no means complete.

Architecture and building. The development of masonry domes and vaults in the Middle Ages made possible the construction of more spatious buildings. In more recent times the availability of reinforced concrete has stimulated interest in the use of shells for roofing purposes.

Power and chemical engineering. The development of steam power during the Industrial Revolution depended to some extent on the construction of suitable boilers. These thin shells were constructed from plates suitable formed and joined by riveting. More recently the used of welding in preesure vessel construction has led to more efficient designs. Pressure vessels and associated pipework are key components in thermal and nuclear power plants, and in all branches of the chemical and petroleum industries.

Structural engineering. An important problem in the early development of steel for structural purposes was to design compression members against buckling. A striking advance was the use of tubular members in the construction of the Forth railway bridge in 1889: steel plates were riveted together to form reinforced tubes as large as 12 feet in diameter, and having a radius/thickness ratio of between 60 and 180.

Vehicle body structures. The construction of vehicle bodies in the early days of road transport involved a system of structural ribs and non-structural paneling or sheeting. The modern form of vehicle construction, in which the skin plays an importatnt structural part, followed the introduction of sheet-metal components, preformed into thin doubly curved shells by large power presses, and firmly connected to each other by welds along the boundaries. The use of the curved skin of vehicles as a load bearing member has similarly revolutionized the construction of railway carriages and aircraft. In the construction of all kind of spacecraft the idea of a thin but strong skin has been used from the beginning.

Composite construction. The introduction of fiberglass and similar lightweight composite materials has impacted the construction of vehicles ranging from boats, racing cars, fighter and stealth aircraft, and so on. The exterior skin can be used as a strong structural shell.

Miscellaneous Examples. Other examples of the impact of shell structures include water cooling towers for power stations, grain silos, armour, arch dams, tunnels, submarines, and so forth.

§29.2.2. Continuity and Curvature

The essential ingredients of a shell structure in all of the foregoing examples are *continuity* and *curvature*. Thus, a fiberglass hull of a boat is continuous in a way that the overlapping planks of clinker construction are not. A pressure vessel must be obviously constructed to hold a fluid at pressure, although the physical components may be joined to each other by riveting, bolting or welding. On the other hand, an ancient masonry dome or vault is not obviously continuous in the sense that it may be composed of of separate stone subunits or voussoirs not necessarily cemented to each other. But in general domes are in a state of compression throughout, and the subunits are thus held in compressive contact with each other. The important point here is that shells are structurally continuous in the sense that they can transmit forces in a number of different directions in the surface of the shell, as required. These structures have quite a different mode of action from *skeletal structures*, of which simple examples are trusses, frameworks, and trees. These structures are only capable of transmitting forces along their discrete structural members.

The fundamental effect of *curvature* and its effect on the stregth and stiffness of a shell is discussed in §29.4.

§29.2.3. The Empirical Approach

Many of the structures listed in §29.2.1 were constructed long before there was anything like a textbook on the subject of shell structures. The early engineers had a strongly empirical outlook. They could see the advantages of shell construction from simple small-scale models, and clearly understood the practical benefitrs of doing “overload tests” on prototypes or scale models. Much the same brand of empiricism is practice successfully today in the design of motor vehicles, where the geometry of the structure is so complicated as to defy even simple description, let alone calculation. But in other areas of engineering, where precision is needed in the interest of economical design and where the geometry is more straightforward, the theory of shell structures is an important design tool.

§29.2.4. Closed and Open Shells

Before describing the main body of the theory it is useful to discuss quantitatively an important practical point.

Anyone who has built children’s toys from thick paper or thin cardboard will be familiar with the fact that a *closed* box is rigid, whereas an *open* box is easily deformable. Similarly, a chocolate box with the lid open can easily be twisted, yet it is effectively rigid when the lid is closed. The same sort of thing applies to an aluminum can, which may be squashed far more easily after an end has been removed. Again, it is noticeable that a boiled egg will not normally fit snugly into a rigid egg cup until the top of the shell has been removed: the closed egg is so rigid that small deviations from circularity are noticeable; but it becomes flexible enough to adapt to the shape of the egg cup once an opening has been made.

There seems to be a principle here that *closed surfaces are rigid*. This is used in many areas of engineering construction. For example, the deck of a ship is not merely a horizontal surface to walk on: it also closes the hull, making a box-like structure. It is easy to think of many other examples of this form of construction, including aircraft wings, suspension bridge roadway girders, rocket skins and unibody cars.

Conversely, a boat without a deck, such as a small fishing or rowing boat, has little rigidity by virtue of form. It must rely on the provision of ribs and struts for what little rigidity it has.

In practice, of course, it is not usually possible to make completely closed structural boxes. In a ship, for example, there will be various cutouts in the deck for things such as hatches and stairways. It is sometimes possible to close such openings with doors and hatch covers that provide structural continuity. Submarines and aircraft are obvious examples. But this is often not possible and compromise solutions must be adopted. The usual plan is to reinforce the edge of the hole in such a way as to compensate, to a certain extent, for the presence of the hole. The amount of reinforcement that is required depends on the size of the hole, and to what extent the presence of the hole makes the structure an open one. Large openings are essential in some forms of construction, such as cooling towers. A more extreme example is provided by shell roofs in general. Here the shell is usually very open, being merely a "cap" of a shell, and the provision of adequate edge ribs, together with suitable supports, is of crucial importance. A main objective in the design of shell roofs is to eliminate those aspects of behavior that spring from the open nature of the shell.

From the foregoing discussion it is obvaious that although the ideas of open and closed shells, respectively, are fairly clear, it is difficult to quantify intermediate cases. The majority of actual shell structures fall into such a grey area. While the effect of a small cutout on the overall rigidity of a shell structure may be trivial, the effect of a large cutout can be serious. The crux of this problem is to quantify the ideas of "small" and "large" in this context. Unfortunately there is no simple way to do this, because the problem involves the interaction between "global" and "locxal" effects. It is largely for this reason that the subject of shell structures generally is a difficult one.

§29.2.5. A Simple Geometric Approach

The notion that a closed surface is rigid is well known in the field of pure Euclidean geometry. There is a theorem of Cauchy which states that a covex polyhedron is rigid. The concept of rigidity is, of course, hedged around with suitable restrictions, but will be an obvious one to anybody who has made cardboard cutout models of polyhedra. It is significant that the qualifier *convex* appears in the theorem. Although it is possible to demonstrate by means of simple examples that some non-convex polyhedra (that is, polyhedra with regions of non-convexity) are rigid, it is also possible to demonstrate special cases of non-convex polyhedra which are not rigid, and are capable of undergoing infinitesimal distortions at least. This is a difficult area of pure mathematics. For the present purposes we note that convexity guarantees rigidity whereas non-convexity may produce deformability.

§29.2.6. A Disadvantage of Rigidity

While rigidity and strength are in many cases desirable attributes of shell structures, there are some important difficulties that can occur precisely on account of unavoidable rigidity. Aa an example of this consider a chemical plant where two large pressure vessels, firmly mounted on separate foundations, are connected by a length of straight pipe. Thermal expnasion of the vessels can only be occomodated without distortion if the pipe contracts in length. If it also expands thermally very large forces can be set up as a result of the rigidity of the vessels. In cases like this it is often convenient to accomodate expansion by a device such as a *bellows* unit. Alternatively, when the interconnecting pipework has bends, it is sometimes possible to make use of the fact that the bends can be relatively flexible. In the case of bellows and bends the flexibility is to a lrage extent related to the geometry of the respective surfaces. It is significant that both are *non-convex*. Nevertheless this of itself does not constitute a proper explanation of their flexibility.

§29.2.7. Catastrophic Failures

The property of closed shell structures being rigid and strong is of great practical values. But it should not be in ignorance of a well known design principle: *efficient structures may fail catatrophically*. Here the term "efficient" describes the consequences of using the closed shell principle. By designing a shell structure as a closed box rather than an open one we may be able to use thinner sheet material and hence produce a more efficient design.

On the other hand, thin shells under compressive membrane forces are prone to *buckling* of a particularly unstable kind. The rapid change in geometry after buckling and consequent decrease of load capacity leads to catastrophic collapse. This is illustrated by the well known experience of "crumpling" of thin wall cylinders like soda cans, under axial compression. The crumpling of a thin convex shell is accompanied by a loss of convexity, which partly explains why the post-buckling rigidity is so low.

§29.3. MATHEMATICAL MODELS OF SHELLS

§29.3.1. The Governing Equations

Mathematical models of shells are constructed in principle following the same general idea used for flat plates. The actual shell, which is a three dimensional object, is replaced by a surface endowed with certain mechanical properties. The transfer of such properties onto the surface is done by kinematic and statical assumptions similar to those made in the theory of plates.

As a result of these assumptions one obtains a set of mechanical properties expressed, in the case of an elastic material, in the form of a generalized Hooke's law relating the deformation of a small material element to the stresses applied to it. These are called the *constitutive equations*. Subsequent steps in the theory of shells are similar to those followed for beams and plates: obtain *equilibrium equations* relating the stress resultants in the structure to the applied external forces, and *kinematic relations*, also called *compatibility equations*, that connect strains and displacements. Where dynamic effects are important, as in vibration problems, the equilibrium equations include inertial and possibly damping terms.

In *linear* shell theory the equations of equilibrium and compatibility are written in terms of the initial geometry of the structure. This assumption necessarily restricts the displacements to be very small. The buckling analysis of shells requires consideration of first-order nonlinear effects, and thus are beyond the scope of this course.

The three sets of field equations: kinematic, constitutive and equilibrium, along with appropriate boundary conditions, comprise the governing equations of the mathematical model. What makes the shell problem more complex is the fact that the equations have to be set up with respect to a generally curved surface in three dimensional space. By comparison, in the theory of flat plates the governing equations are set up merely over a planar surface.

§29.3.2. The Dominating Effect of Geometry

In the mechanics of deformable solids the character of the resulting mathematical problem (a system of partial differential equations) is usually determined by the material properties only. For example, one major difference between the theories of elasticity and plasticity is while in elasticity the governing differential equations are elliptic, in plasticity they are sometimes hyperbolic and thus demand different approaches for their solution. In the linear theory of shells the governing equations may be rendered hyperbolic as a consequence of *geometrical* properties of the shell surface.

§29.3.3. Interaction of Bending and Stretching Effects

The mechanical properties of a shell element describe its resistance to deformation in terms of separable stretching and bending effects. Loads applied to the shell are carried in general through a combination of bending and stretching actions, which generally vary from point to point. One of the major difficulties in the theory of shells is to find a relatively simple way of describing the interaction between the two effects. This aspect of the theory has been troublesome from the beginning. Rayleigh[1] argued that the deformation of a thin hemispherical bowl would be primarily inextensional, and accordingly he developed a special method of analysis which took into account only the bending energy of the shell. On the other hand Love[2] argued that for thin shells stretching was the dominant effect. At that time Love had not grasped the strong contrast between open and closed shells. The controversy was resolved by Lamb[3] and Basset[4], who solve Love's general equations for a cylindrical shell and demonstrated the possibility of a narrow *boundary layer* in which there was a rapid transition between bending and stretching effects. The width of the layer was determined by the interaction between those effects.

§29.3.4. Classification in Terms of Thickness Ratio

As in the case of plates, one can classify shell mathematical models in terms of the ratio of the thickness to a characteristic dimension:

- Very thick: 3D effects
- Thick: stretching, bending and higher order transverse shear
- Moderately thick: streching, bending and first order transverse shear
- Thin shells: stretching and bending energy considered but transverse shear neglected
- Very thin shells: dominated by stretching effects. Also called membranes.

The main difference from flat plates is that the determination of characteristic dimensions is more complex.

[1] Lord Rayleigh, On the infinitesimal bending of surfaces of revolution, *Proc. London Math. Soc.*, **13**, 4–16, 1881. Also *The Theory of Sound*, Vol. 1, Ch. 10, MacMillan, London, 1884.

[2] A. E. H. Love, On the small free vibrations and deformations of thin elastic shells, *Phil. Trans. Royal Soc. London*, series A, **179**, 491-546, 1888.

[3] H. Lamb, On the determination of an elastic shell, *Proc. London Math. Soc.*, **21**, 119-146, 1890.

[4] A. B. Basset, On the extension and flexure of cylindrical and spherical thin shells, *Phil. Trans. Royal Soc. London*, Series A, **181**, 433-480, 1890.