29

Optimal Membrane Triangles with Drilling Freedoms

Source

Technical details in expository paper:

C. A. Felippa, A study of optimal membrane triangles with drilling freedoms, *Comp. Meth. Appl. Mech. Engrg.*, **192**, 2125-2168, 2003

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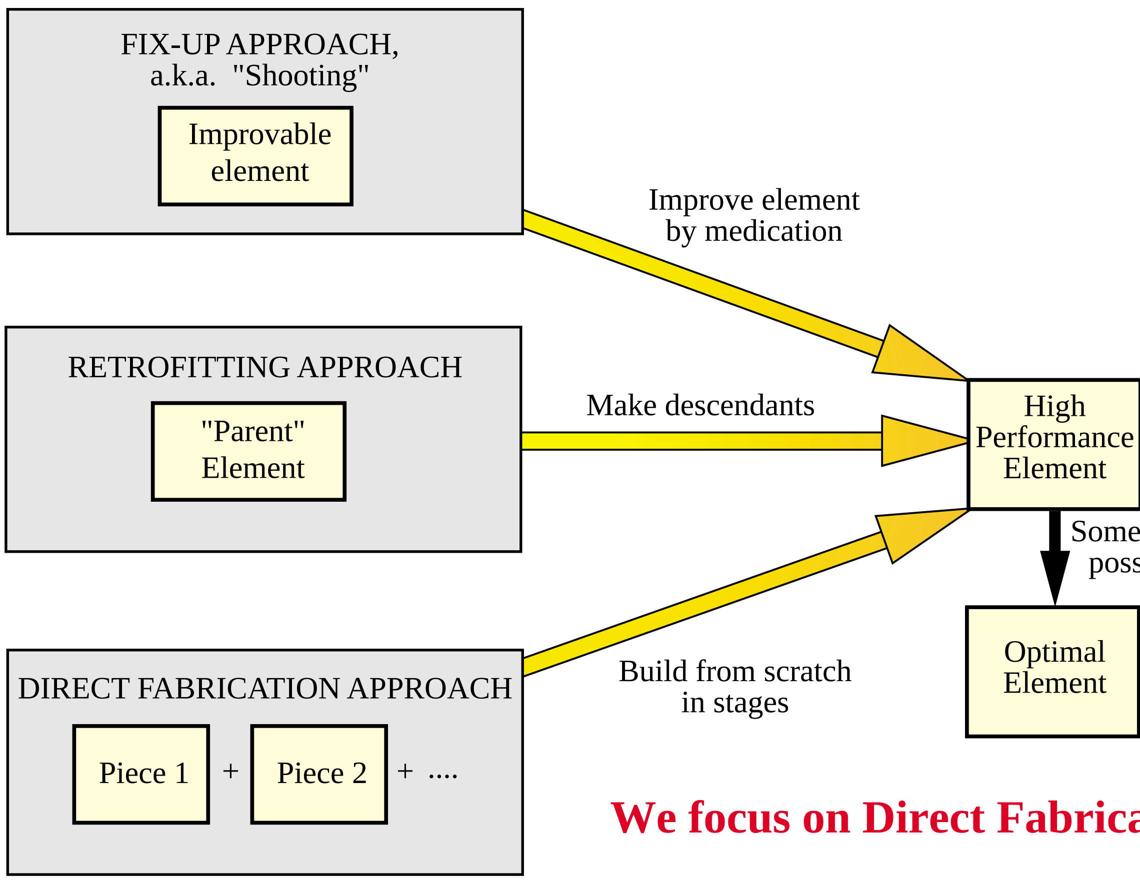
PowerPoint Slides of USNCCM'03 presentation also available on web

Outline

- Why drilling freedoms
- Element development
- Finding the best
- Templates
- Benchmarks & Conclusions

Three Aproaches to Element Development



We focus on Direct Fabrication

High Performance Elements

Simple elements that deliver results of engineering accuracy with arbitrary coarse meshes

HP Elements: Preferences & Properties I

only corner nodes

only physical freedoms

passes patch test on any plane mesh

no condensation: EAS, incompatible modes, considered waste of time

HP Element: Preferences & Properties II

do not assume you can refine the mesh in real engineering systems

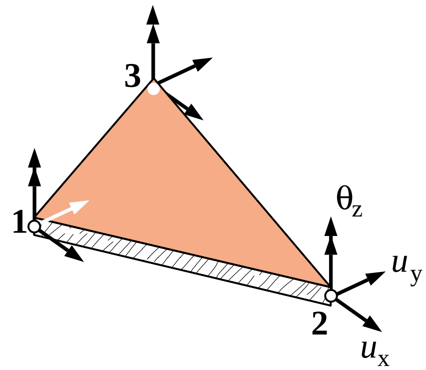
robust when used by inexperienced modelers

can be customized to do multiple duties

can be optimized for specific criteria

need to be developed by symbolic computation

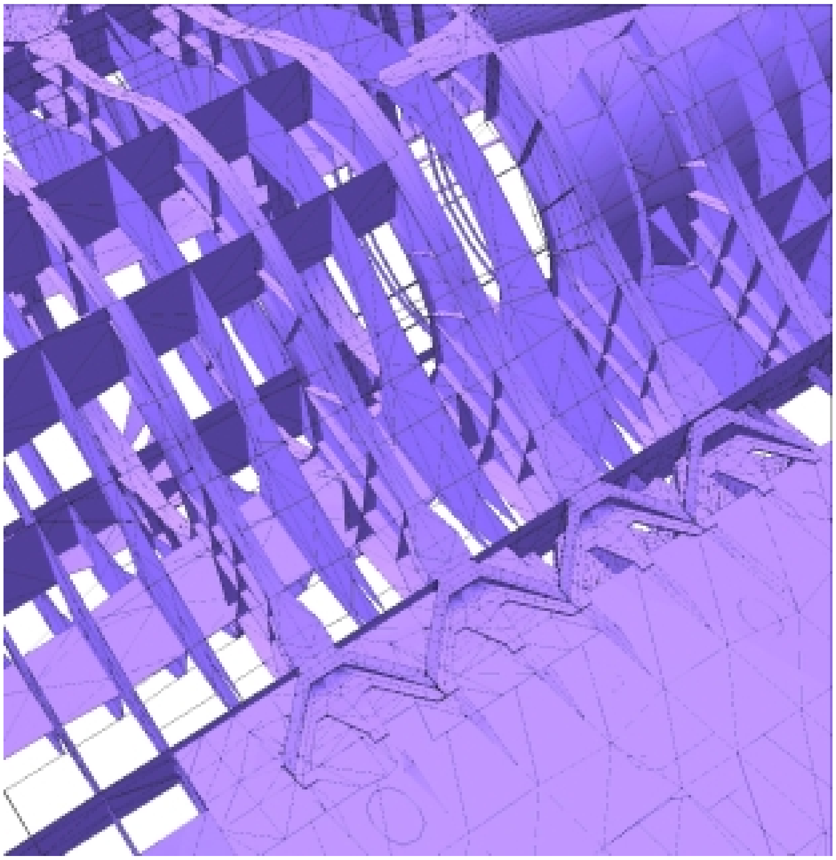
Why Drilling Freedoms



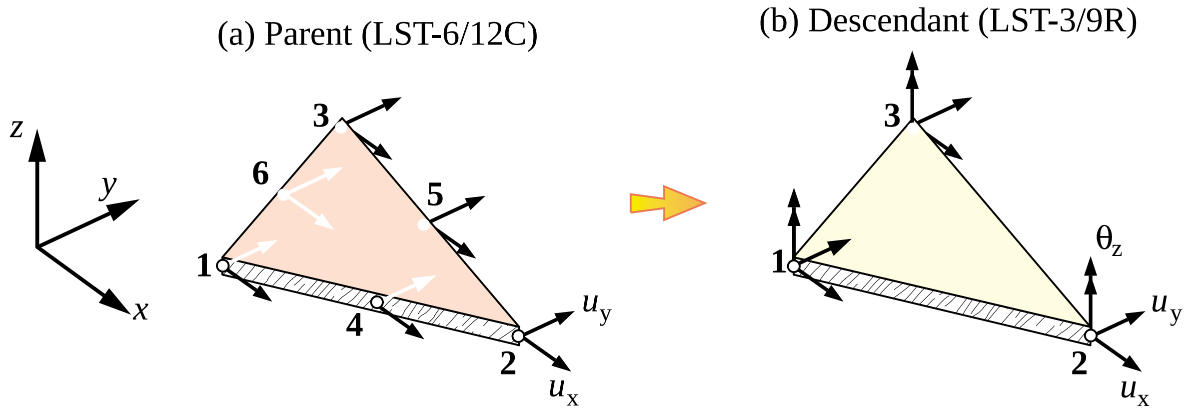
Addition of 3 drilling DOFs at corners offers these advantages:

Performance improved without adding side DOFs
Rotational DOF "free of charge"
Simplifies shell modeling: intersections, stiffeners, ..

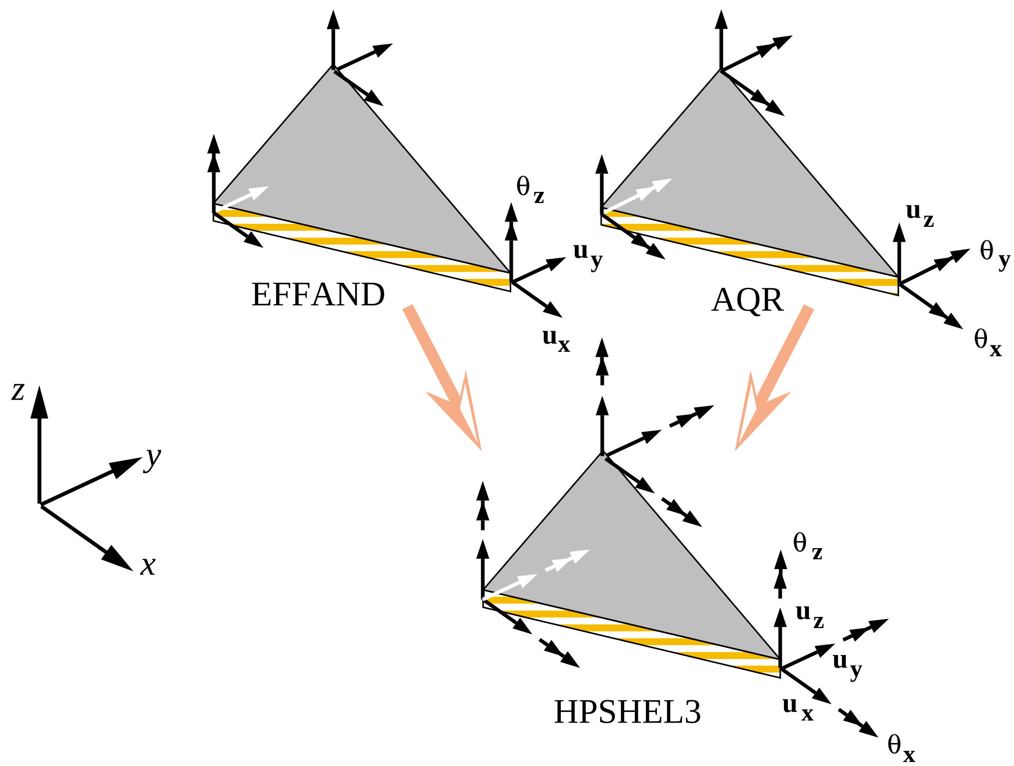
F-16 Fighter: Internal Structure



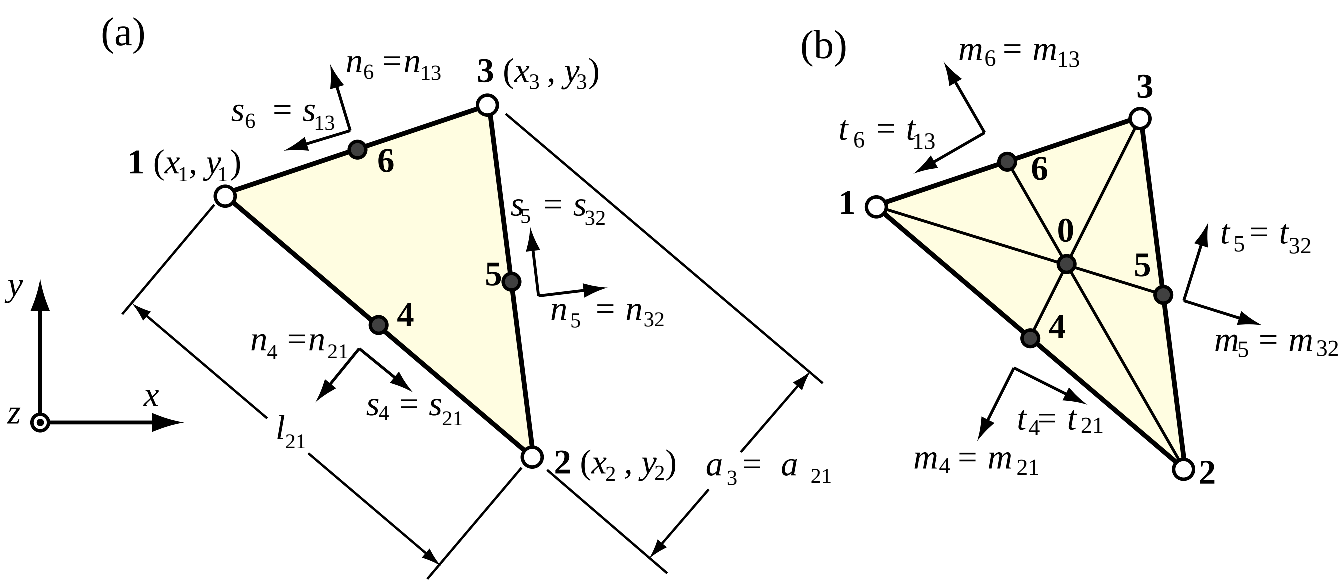
Provenance of Drilling-Freedom MembraneTriangle



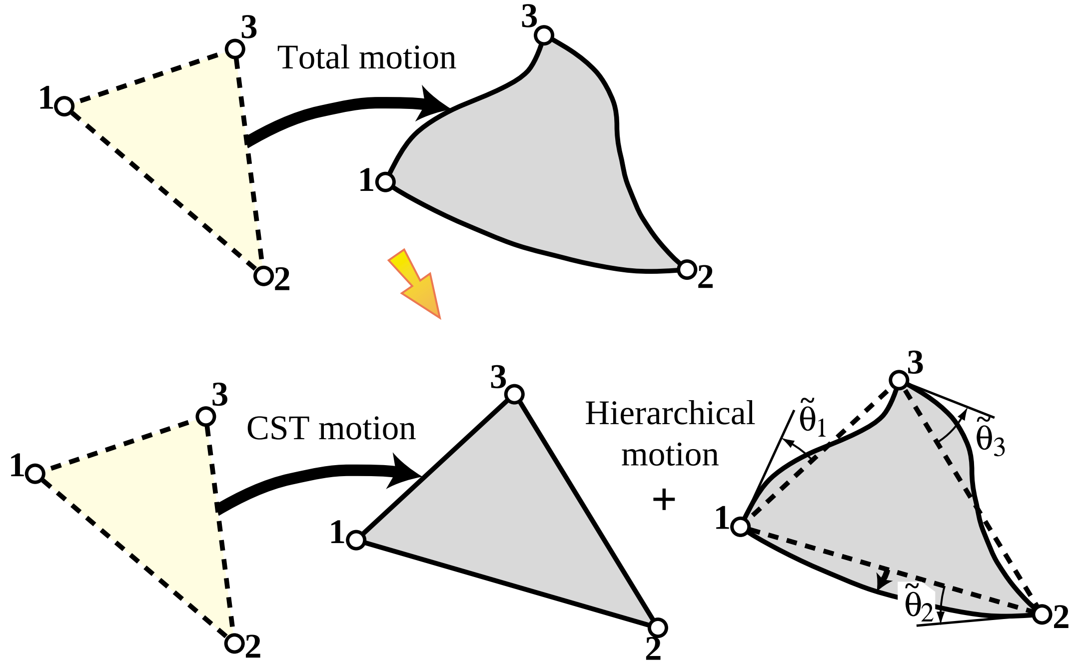
Fabrication of 18 DOF Facet Shell Element



Intrinsic Triangle Dimensions



Decomposition of Element Kinematics



Hierarchical Rotations

$$\tilde{\theta}_i = \theta_i - \theta_0$$

$$\theta_0 = \frac{1}{4A}\left(x_{23}u_{x1} + x_{31}u_{x2} + x_{12}u_{x3} + y_{23}u_{y1} + y_{31}u_{y2} + y_{12}u_{y3}\right)$$

$$\tilde{\boldsymbol{\theta}} = \begin{bmatrix} \tilde{\theta}_1 \\ \tilde{\theta}_2 \\ \tilde{\theta}_3 \end{bmatrix} = \frac{1}{4A}\begin{bmatrix} x_{32} & y_{32} & 4A & x_{13} & y_{13} & 0 & x_{21} & y_{21} & 0 \\ x_{32} & y_{32} & 0 & x_{13} & y_{13} & 4A & x_{21} & y_{21} & 0 \\ x_{32} & y_{32} & 0 & x_{13} & y_{13} & 0 & x_{21} & y_{21} & 4A \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ \theta_1 \\ u_{x2} \\ u_{y2} \\ \theta_2 \\ u_{x3} \\ u_{y3} \\ \theta_3 \end{bmatrix} = \tilde{\mathbf{T}}_{\theta u}\mathbf{u}_R$$

The Basic Stiffness

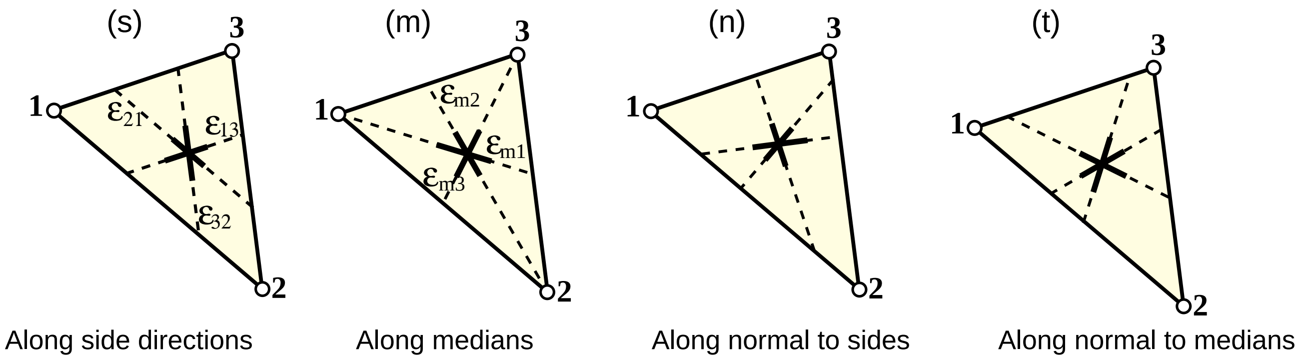
Discovered in 1984 by Pal Bergan and published the following year:

$$\mathbf{K}_b = V^{-1}\mathbf{L}\,\mathbf{E}\,\mathbf{L}^T.$$

where $V = A\,h$ **is the element volume and L is a 3 x 9 matrix that contains a free parameter** α_b

$$\mathbf{L} = \tfrac{1}{2}h \begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ \frac{1}{6}\alpha_b y_{23}(y_{13}-y_{21}) & \frac{1}{6}\alpha_b x_{32}(x_{31}-x_{12}) & \frac{1}{3}\alpha_b(x_{31}y_{13}-x_{12}y_{21}) \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ \frac{1}{6}\alpha_b y_{31}(y_{21}-y_{32}) & \frac{1}{6}\alpha_b x_{13}(x_{12}-x_{23}) & \frac{1}{3}\alpha_b(x_{12}y_{21}-x_{23}y_{32}) \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \\ \frac{1}{6}\alpha_b y_{12}(y_{32}-y_{13}) & \frac{1}{6}\alpha_b x_{21}(x_{23}-x_{31}) & \frac{1}{3}\alpha_b(x_{23}y_{32}-x_{31}y_{13}) \end{bmatrix}$$

Natural Strains: 4 Possibilities



For choice (s)

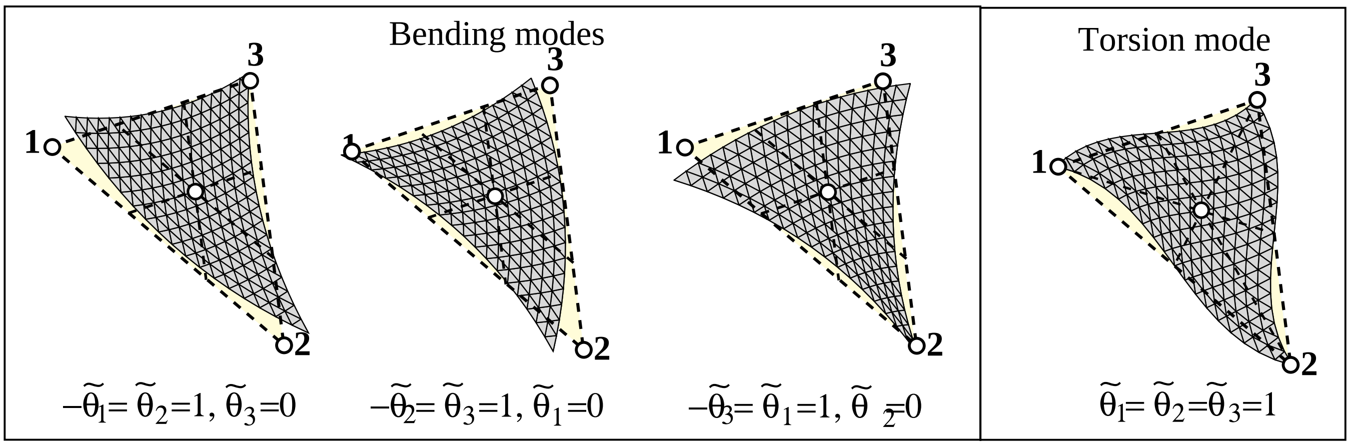
$$\boldsymbol{\epsilon} = [\, \epsilon_{21} \quad \epsilon_{32} \quad \epsilon_{13}\,]^T$$

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{12} \\ \epsilon_{23} \\ \epsilon_{31} \end{bmatrix} = \begin{bmatrix} x_{21}^2/\ell_{21}^2 & y_{21}^2/\ell_{21}^2 & x_{21}y_{21}/\ell_{21}^2 \\ x_{32}^2/\ell_{32}^2 & y_{32}^2/\ell_{32}^2 & x_{32}y_{32}/\ell_{32}^2 \\ x_{13}^2/\ell_{13}^2 & y_{13}^2/\ell_{13}^2 & x_{13}y_{13}/\ell_{13}^2 \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix} = \mathbf{T}_e^{-1}\mathbf{e}$$

$$\begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix} = \frac{1}{4A^2} \begin{bmatrix} y_{23}y_{13}\ell_{21}^2 & y_{31}y_{21}\ell_{32}^2 & y_{12}y_{32}\ell_{13}^2 \\ x_{23}x_{13}\ell_{21}^2 & x_{31}x_{21}\ell_{32}^2 & x_{12}x_{32}\ell_{13}^2 \\ .\,y_{23}x_{31} + x_{32}y_{13}/\ell_{21}^2 & .\,y_{31}x_{12} + x_{13}y_{21}/\ell_{32}^2 & .\,y_{12}x_{23} + x_{21}y_{32}/\ell_{13}^2 \end{bmatrix} \begin{bmatrix} \epsilon_{12} \\ \epsilon_{23} \\ \epsilon_{31} \end{bmatrix}$$

$$\mathbf{E}_{nat} = \mathbf{T}_e^T \mathbf{E} \mathbf{T}_e$$

Natural Higher Order Modes



Higher Order Stiffness

Relate corner natural strains to hierarchical rotations:

$$\mathbf{Q}_1 = \frac{2A}{3}\begin{bmatrix} \frac{\beta_1}{\ell_{21}^2} & \frac{\beta_2}{\ell_{21}^2} & \frac{\beta_3}{\ell_{21}^2} \\ \frac{\beta_4}{\ell_{32}^2} & \frac{\beta_5}{\ell_{32}^2} & \frac{\beta_6}{\ell_{32}^2} \\ \frac{\beta_7}{\ell_{13}^2} & \frac{\beta_8}{\ell_{13}^2} & \frac{\beta_9}{\ell_{13}^2} \end{bmatrix} \quad \mathbf{Q}_2 = \frac{2A}{3}\begin{bmatrix} \frac{\beta_9}{\ell_{21}^2} & \frac{\beta_7}{\ell_{21}^2} & \frac{\beta_8}{\ell_{21}^2} \\ \frac{\beta_3}{\ell_{32}^2} & \frac{\beta_1}{\ell_{32}^2} & \frac{\beta_2}{\ell_{32}^2} \\ \frac{\beta_6}{\ell_{13}^2} & \frac{\beta_4}{\ell_{13}^2} & \frac{\beta_5}{\ell_{13}^2} \end{bmatrix} \quad \mathbf{Q}_3 = \frac{2A}{3}\begin{bmatrix} \frac{\beta_5}{\ell_{21}^2} & \frac{\beta_6}{\ell_{21}^2} & \frac{\beta_4}{\ell_{21}^2} \\ \frac{\beta_8}{\ell_{32}^2} & \frac{\beta_9}{\ell_{32}^2} & \frac{\beta_7}{\ell_{32}^2} \\ \frac{\beta_3}{\ell_{13}^2} & \frac{\beta_1}{\ell_{13}^2} & \frac{\beta_2}{\ell_{13}^2} \end{bmatrix}$$

Evaluate at midpoints

$$\mathbf{Q}_4 = \tfrac{1}{2}(\mathbf{Q}_1 + \mathbf{Q}_2), \quad \mathbf{Q}_5 = \tfrac{1}{2}(\mathbf{Q}_2 + \mathbf{Q}_3), \quad \mathbf{Q}_6 = \tfrac{1}{2}(\mathbf{Q}_3 + \mathbf{Q}_1)$$

Integrate over triangle:

$$\mathbf{K}_\theta = h\left(\mathbf{Q}_4^T \mathbf{E}_{nat} \mathbf{Q}_4 + \mathbf{Q}_5^T \mathbf{E}_{nat} \mathbf{Q}_5 + \mathbf{Q}_6^T \mathbf{E}_{nat} \mathbf{Q}_6\right)$$

Finally:

$$\mathbf{K}^e(\alpha_b, \beta_0, \beta_1, \ldots \beta_9) = \mathbf{K}_b + \tfrac{3}{4}\beta\, \tilde{\mathbf{T}}_{\theta u}^T \mathbf{K}_\theta \tilde{\mathbf{T}}_{\theta u}$$

10 free parameters, reduced to 7 by observer invariance

Template "Genetics"

The set of free parameters is the template signature

The number of free parameters can be reduced by applying behavioral constraints to produce element families

Specific elements instances are obtained by assigning numerical values to the free parameters of a family

Elements with the same signature, possibly derived through different methods, are called clones

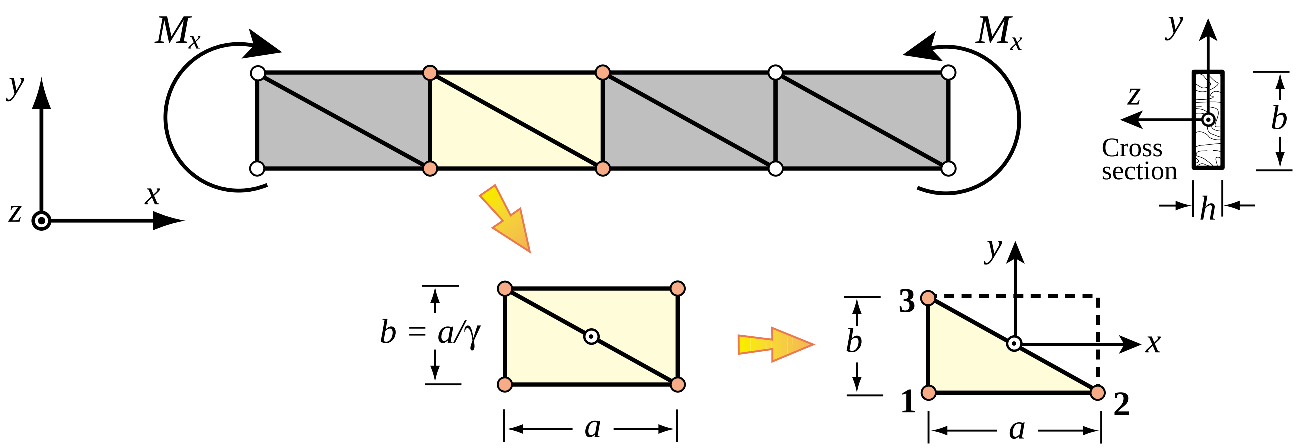
Template Instance Identifiers

Name	Description	See
ALL-3I	Allman 88 element integrated by 3-point interior rule.	Section 8
ALL-3M	Allman 88 element integrated by 3-midpoint rule.	Section 8
ALL-EX	Allman 88 element, exactly integrated	Section 8
ALL-LS	Allman 88 element, least-square strain fit.	Section 8
CST	Constant strain triangle CST-3/6C.	Ref. [7]
FF84	1984 Free Formulation element of Bergan and Felippa.	Ref. [21]
LST-Ret	Retrofitted LST with $\alpha_b = 4/3$.	Section 7
OPT	Optimal ANDES Template.	Sections 5.2 and 5.3

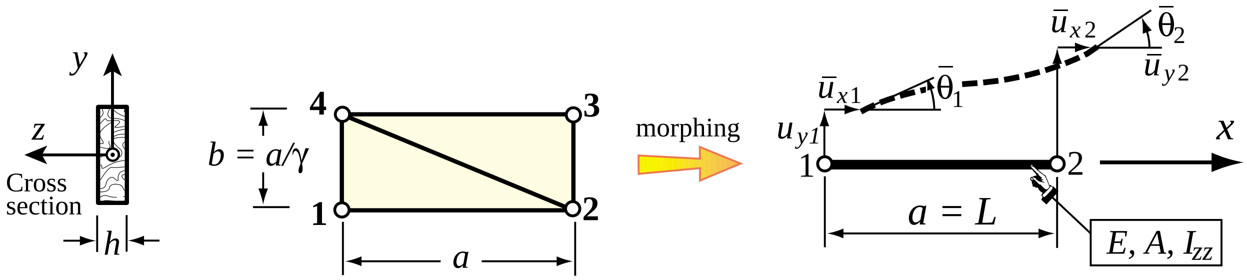
Template Instance Signatures

Name	α_b	β_0	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9
ALL-3I	1	4/9	1/12	5/12	1/2	0	1/3	−1/3	−1/12	−1/2	−5/12
ALL-3M	1	4/9	1/4	5/4	3/2	0	1	−1	−1/4	−3/2	−5/4
ALL-EX	not an instance of ANDES template										
ALL-LS	1	4/9	3/20	3/4	9/10	0	3/5	−3/5	−3/20	−9/10	−3/4
CST	0	any	0	0	0	0	0	0	0	0	0
FF84	not an instance of ANDES template										
LST-Ret	4/3	1/2	2/3	−2/3	0	0	−4/3	4/3	−2/3	0	2/3
OPT	3/2	see §5.2	1	2	1	0	1	−1	−1	−1	−2

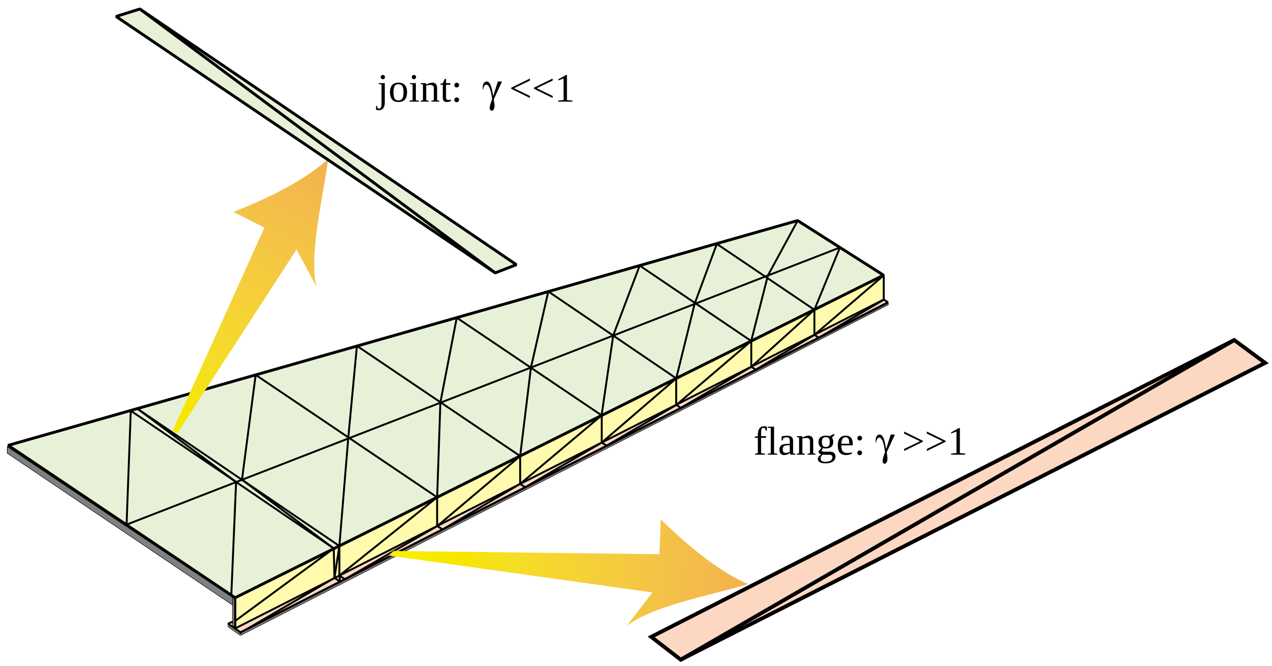
In-Plane Bending Optimality Test
(a Higher Order Patch Test)



Optimality Confirmed by Morphing Macroelement to Beam



Motivation for Bending Test: Avoiding In-Plane Bending Locking as Aspect Ratio Increases



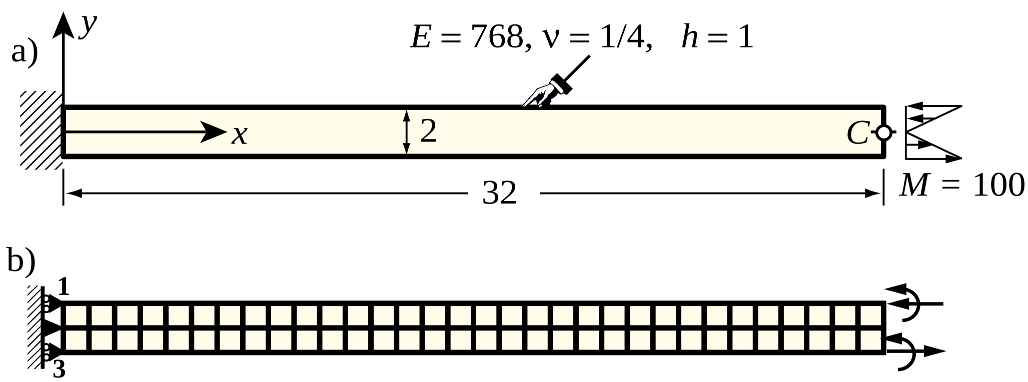
Bending Energy Ratio Test Results

Instance	Bending ratio r for isotropic material	$\gamma=0$, $\nu=\frac{1}{4}$	$\gamma=1$, $\nu=\frac{1}{4}$	$\gamma=4$, $\nu=\frac{1}{4}$	$\gamma=16$, $\nu=\frac{1}{4}$
ALL-3I	$\frac{584+(79-91\nu)\gamma^2+6\gamma^4}{432(1-\nu^2)}$	1.442	1.595	7.457	1007.901
ALL-3M	$\frac{24+(5-9\nu)\gamma^2+2\gamma^4}{16(1-\nu^2)}$	1.600	1.916	38.667	8786.667
ALL-EX	$\frac{84+(15-19\nu)\gamma^2+2\gamma^4}{60(1-\nu^2)}$	1.493	1.711	13.511	2378.311
ALL-LS	$\frac{1672+(263-371\nu)\gamma^2+54\gamma^4}{1200(1-\nu^2)}$	1.486	1.686	16.196	3185.956
CST	$\frac{6+3(1-\nu)\gamma^2}{2(1-\nu^2)}$	3.200	4.400	22.400	310.400
FF84	$\frac{3}{4}+\frac{13}{96(1-\nu)}+\frac{13+54\gamma^2+119\gamma^4+70\gamma^6+13\gamma^8}{96(1+3\gamma^2+\gamma^4)^2\,(1+\nu)}$	1.039	1.020	1.035	1.039
LST-Ret	$\frac{34+5(1-\nu)\gamma^2}{27(1-\nu^2)}$	1.343	1.491	3.714	39.269
OPT	1	1.000	1.000	1.000	1.000

Stiffness Template Module

```
LST39RMembTemplateStiffness [xycoor_,Emat_,h_,fpars_]:=Module[
 {x1,y1,x2,y2,x3,y3,x12,y12,x21,y21,x23,y23,x32,y32,x31,y31,
 x13,y13,A,A4,V,LL21,LL32,LL13,αb,β0,β1,β2,β3,β4,β5,β6,β7,β8,β9,
 Te,Tθu,Q1,Q2,Q3,Q4,Q5,Q6,c,L,Enat,Kθ,Kh,Kb,Ke},
 {{x1,y1},{x2,y2},{x3,y3}}=xycoor;
 {αb,β0,β1,β2,β3,β4,β5,β6,β7,β8,β9}=fpars;
 x12=x1-x2; x23=x2-x3; x31=x3-x1; x21=-x12; x32=-x23; x13=-x31;
 y12=y1-y2; y23=y2-y3; y31=y3-y1; y21=-y12; y32=-y23; y13=-y31;
 A=(y21*x13-x21*y13)/2;  A2=2*A; A4=4*A;
 L= {{y23,0,x32},{0,x32,y23},
     {y23*(y13-y21),x32*(x31-x12),(x31*y13-x12*y21)*2}*αb/6,
     {y31,0,x13},{0,x13,y31},
     {y31*(y21-y32),x13*(x12-x23),(x12*y21-x23*y32)*2}*αb/6,
     {y12,0,x21},{0,x21,y12},
     {y12*(y32-y13),x21*(x23-x31),(x23*y32-x31*y13)*2}*αb/6}*h/2;
 Kb=(L.Emat.Transpose[L])/(h*A);
 Tθu={{x32,y32,A4,x13,y13, 0,x21,y21, 0},
      {x32,y32, 0,x13,y13,A4,x21,y21, 0},
      {x32,y32, 0,x13,y13, 0,x21,y21,A4}}/A4;
 LL21=x21^2+y21^2; LL32=x32^2+y32^2; LL13=x13^2+y13^2;
 Te={{y23*y13*LL21,   y31*y21*LL32,   y12*y32*LL13},
     {x23*x13*LL21,   x31*x21*LL32,   x12*x32*LL13},
     {(y23*x31+x32*y13)*LL21,(y31*x12+x13*y21)*LL32,
      (y12*x23+x21*y32)*LL13}}/(A*A4);
 Q1={{β1,β2,β3}/LL21,{β4,β5,β6}/LL32,{β7,β8,β9}/LL13}*A2/3;
 Q2={{β9,β7,β8}/LL21,{β3,β1,β2}/LL32,{β6,β4,β5}/LL13}*A2/3;
 Q3={{β5,β6,β4}/LL21,{β8,β9,β7}/LL32,{β2,β3,β1}/LL13}*A2/3;
 Q4=(Q1+Q2)/2; Q5=(Q2+Q3)/2; Q6=(Q3+Q1)/2;
 Enat=Transpose[Te].Emat.Te;
 Kθ=(3/4)*β0*h*A*(Transpose[Q4].Enat.Q4+Transpose[Q5].Enat.Q5+
                 Transpose[Q6].Enat.Q6);
 Kh=Transpose[Tθu].Kθ.Tθu;
 Return[{Kb,Kh}]];
```

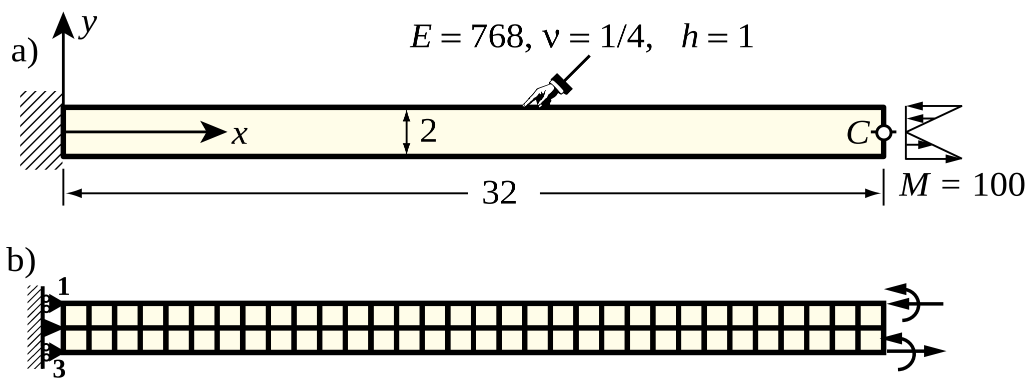
End-Moment-Loaded Thin Cantilever Beam Benchmark



Sample Element Generation

```
xycoor=N[{{0,0},{102/25,-86/25},{85/25,285/250}}];
Print["xycoor=",xycoor];
Em=120; ν=1/4; h=1/8;
Emat=Em/(1-ν^2)*{{1,ν,0},{ν,1,0},{0,0,(1-ν)/2}};
fpars=  LST39RANDESTemplateSignature["OPT", Emat];
{Kb,Kh}=LST39RMembTemplateStiffness [xycoor,Emat,h,fpars];
Print["Kb=",SetPrecision[N[Kb],5]//MatrixForm];
Print["Kh=",SetPrecision[N[Kh],5]//MatrixForm];
Print["Ke=",SetPrecision[N[Kb+Kh],5]//MatrixForm];
Print["eigs of Ke=",Chop[Eigenvalues[N[Kb+Kh]]]];
```

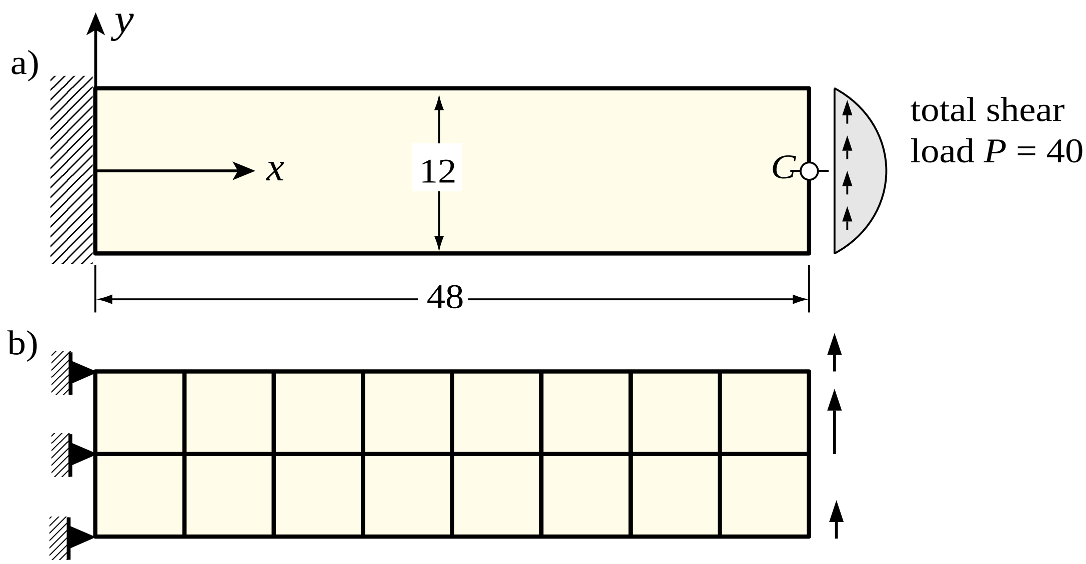
End-Moment-Loaded Thin Cantilever Beam Benchmark



Tip Deflections (Exact =100) for Thin Cantilever Beam under End Moment

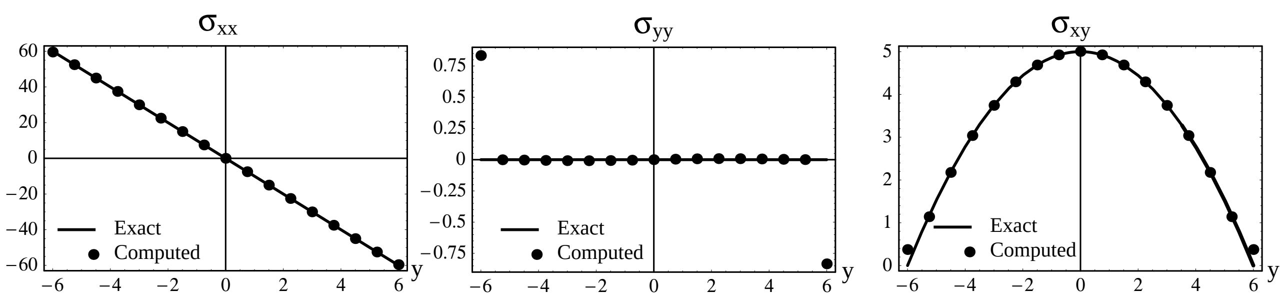
Element	Load Lumping	Mesh: x-subdivisions × y-subdivisions				
		32×2 ($\gamma = 1$)	16×2 ($\gamma = 2$)	8×2 ($\gamma = 4$)	4×2 ($\gamma = 8$)	2×2 ($\gamma = 16$)
ALL-3I	EBQ	87.08	76.48	38.32	5.42	0.39
ALL-3M	EBQ	81.36	53.57	9.59	0.71	0.04
ALL-EX	EBQ	84.90	69.09	24.23	2.47	0.16
ALL-LS	EBQ	85.36	68.25	20.83	1.89	0.12
CST	LI	54.05	36.36	15.75	4.82	1.28
FF84	EBQ	98.36	97.17	96.58	96.34	96.27
LST-Ret	EBQ	89.05	81.04	59.58	28.93	9.46
OPT	EBQ	99.99	99.99	99.99	99.96	100.07

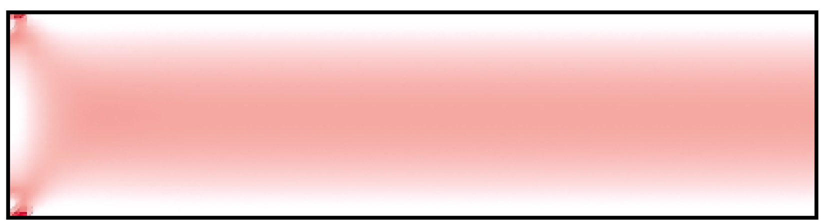
Berkeley Cantilever Benchmark



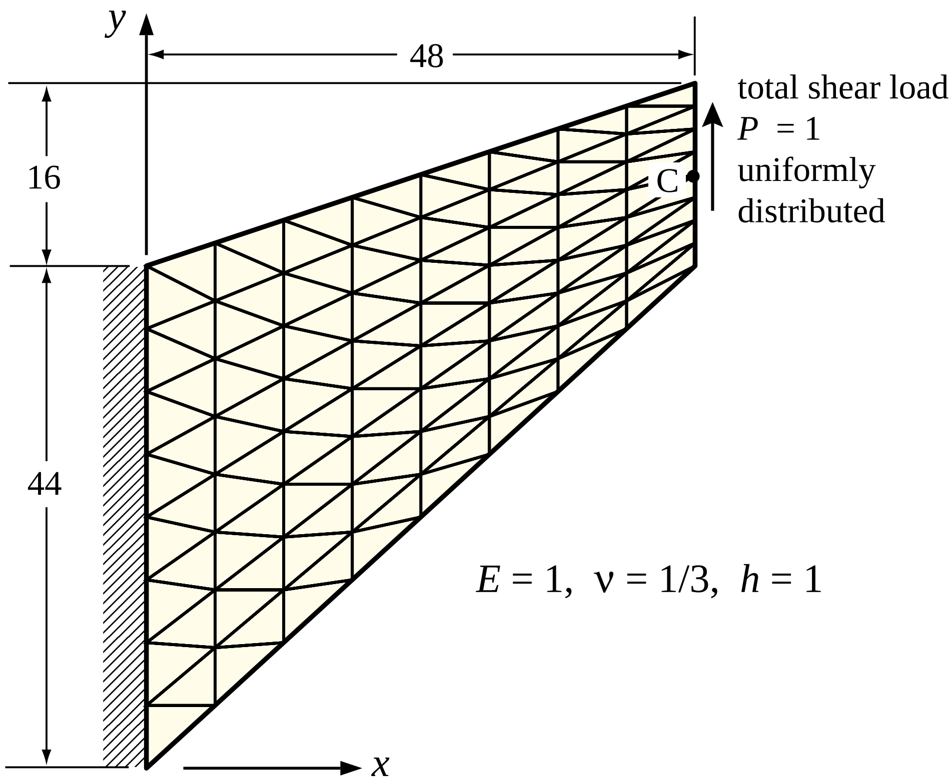
Berkeley Cantilever: Stress Results Using Optimal Triangle

(for deflection results see paper)





Cook's Trapezoidal Plate Benchmark

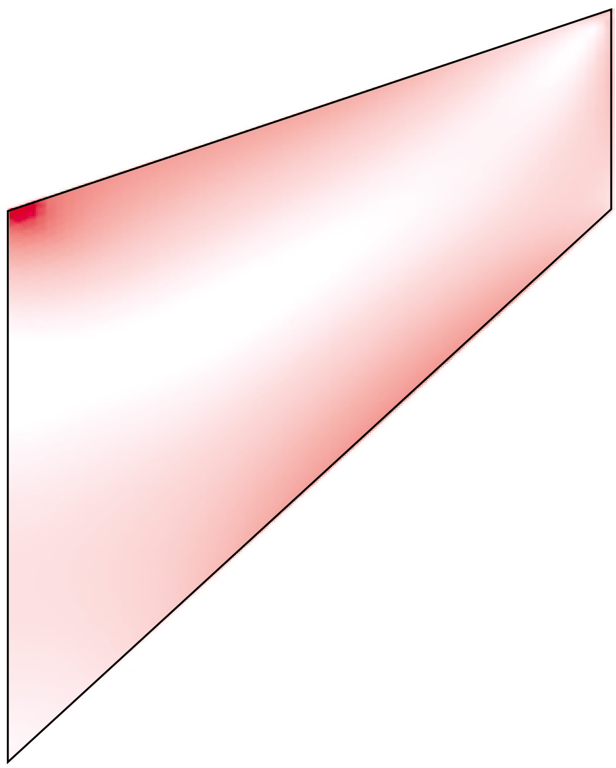


Results for Cook's Benchmark Problem

	Vertical deflection at C for subdivision					
Element	2×2	4×4	8×8	16×16	32×32	64×64
ALL-3I	21.61	23.00	23.66	23.88	23.94	
ALL-3M	16.61	21.05	23.02	23.69	23.87	
ALL-EX	19.01	21.83	23.43	23.81	23.91	
ALL-LS	19.43	22.32	23.44	23.82	23.92	
CST	11.99	18.28	22.02	23.41		
FF84	20.36	22.42	23.41	23.79	23.91	
LST-Ret†	19.82	22.62	23.58	23.86	23.94	
OPT	20.56	22.45	23.43	23.80	23.91	23.95
FFQ	21.66	23.11	23.79	23.88	23.94	
HL	18.17	22.03		23.81		
HG	22.32	23.23		23.91		
Q4	11.85	18.30		23.43		
Q6	22.94	23.48				
QM6	21.05	23.02				

† Requires one drilling DOF to be fixed, else stiffness is singular

Cook's Problem: von Mises Stress



Conclusions: Template Advantages

One form generates an infinite # of instances

Instances are not necessarily obtainable by conventional (variational based) formulations

Unified implementation (input: signature) weeds out clones, simplifies benchmarking

Can be customized, or optimized, for specific configurations or needs