26

Thin Plate Elements: Overview

TABLE OF CONTENTS

TABLE OF CONTENTS

This Chapter presents an overview of finite element models for thin plates usinf the Kirchhoff Plate bending (KPB) model. The derivation of shape functions for the triangle geometry is covered in the next chapter.

§26.1. An Overview of KTP FE Models

The plate domain Ω is subdivided into finite elements in the usual way, as illustrated in Figure 26.1. The most widely used geometries are triangles and quadrilaterals with straight sides. Curved side KPB elements are rare. They are more widely seen in shear-endowed C^0 models derived by the degenerate solid approach.

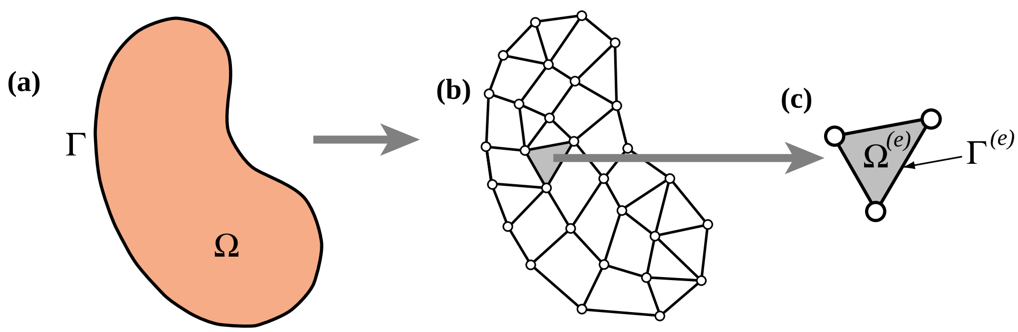

Figure 26.1. A thin plate subdivided into finite elements.

§26.1.1. Triangles

This and following Chapter will focus on KPB *triangular* elements only. These triangles will invariably have straight sides. Their geometry is degined by the position of the three corners as pictured in Figure 26.2(a). The positive sense of traversal of the boundary is shown in Figure 26.2. This sense defines three side directions: s_1, s_2 and s_3, which are aligned with the sides opposite corners 1, 2 and 3, respectively. The *external* normal directions n_1, n_2 and n_3 shown there are oriented at -90° from s_1, s_2 and s_3.[1]

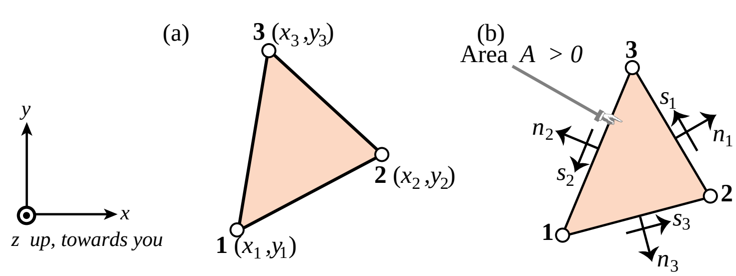

Figure 26.2. Triangular geometry and side-normal directions.

[1] This means that $\{n_i, s_i, z\}$ for $i = 1, 2, 3$ form three right-handed RCC systems, one for each side.

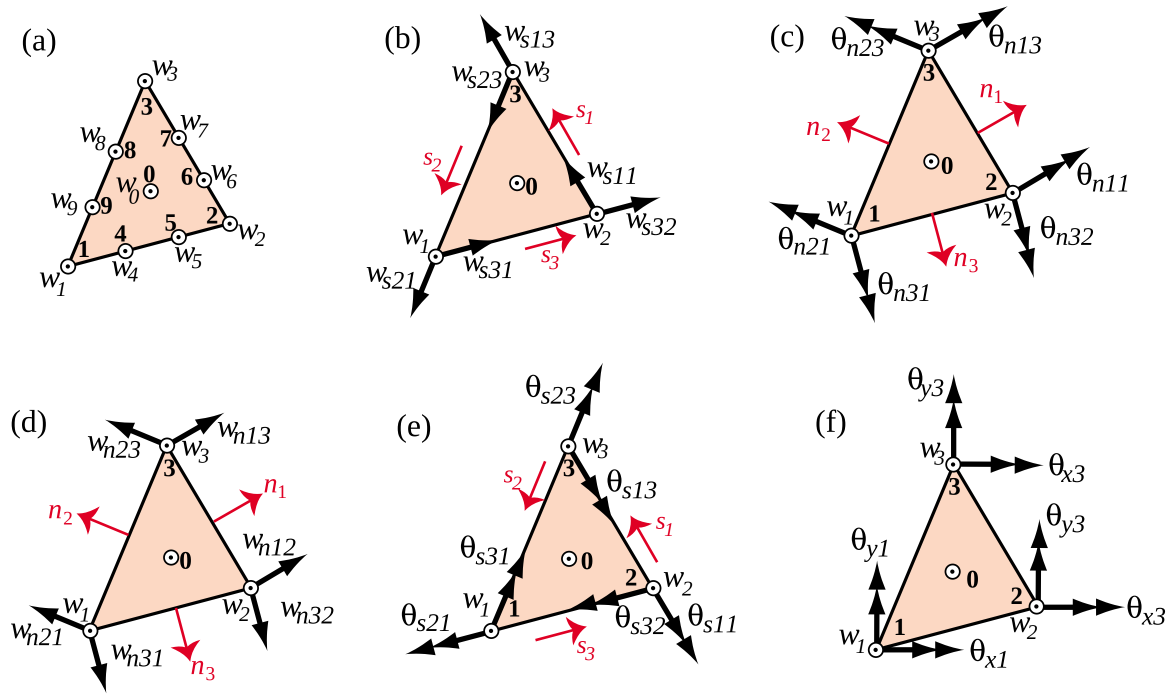

Figure 26.3. Several 10-DOF configurations for expressing the complete cubic interpolation of the lateral deflection w over a KPB triangle. Normal to sides and tangential directions identified by red arrows to avoid confusion with DOFs shown in black.

The area of the triangle, denoted by A, is a signed quantity given by

$$2A = \det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = (x_2 y_3 - x_3 y_2) + (x_3 y_1 - x_1 y_3) + (x_1 y_2 - x_2 y_1). \qquad (26.1)$$

We require that $A > 0$.

§26.1.2. A Potpourri of Freedom Configurations

In KPB elements treated by assumed transverse displacements, the minimum polynomial expansion of w to achieve at least partly the compatibility requirements, is cubic. A complete cubic has 10 terms and consequently can accomodate 10 element degrees of freedom (DOFs). Figure 26.3 shows several 10-DOF configurations from which the cubic interpolation over the complete triangle can be written as an interpolation formula, with shape functions expressed in terms of the geometry data and the triangular coordinates. These interpolation formulas are studied in the next Chapter.

Because a complete polynomial is invariant with respect to a change in basis, all of the configurations depicted in Figure 26.3 are *equivalent* in providing the *same* interpolation over the triangle. They differ, however, when connecting to adjacent elements. Only configuration (f) is practical for connecting elements over arbitrary meshes using the Direct Stiffness Method (DSM). The other configurations are valuable in intermediate derivations, or for various theoretical studies.

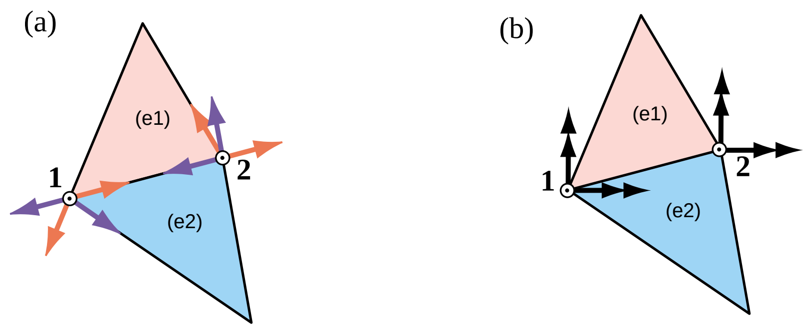

Figure 26.4. Connecting KPB elements.

The 10-node configuration (a) specifies the cubic by the 10 values w_i, $i = 1, \ldots 10$, of the deflection at corners, thirdpoints of sides, and centroid. This is a useful starting point because the associated shape functions can be constructed directly using the technique explained in Chapter 17 of IFEM. The resulting plate element is useless, however, because it does not enforce interelement C^1 continuity at any boundary point.

From (a) one can pass to any of (b) through (d), the choice being primarily a matter of taste or objectives. Configurations (b) and (d) use the six corner partial derivatives of w along the side directions or the normal to the sides, respectively. The notation is $w_{sij} = (\partial w/\partial s_i)_j$ and $w_{nij} = (\partial w/\partial n_i)_j$, where i is the side index and j the corner index. (Sides are identified by the number of the opposite corner.) For example, $w_{s21} = (\partial w/\partial s_2)_1$. These partials are briefly called *side slopes* and *normal slopes*, respectively, on account of their physical meaning.

According to the fundamental kinematic assumption of the KPB model, a w slope along a midsurface direction is equivalent for small deflections to a *midsurface rotation* about a line perpendicular to and forming a -90° angle with that direction. Rotations about the s_i and n_i directions are called *side rotations* and *normal rotations*, respectively, for brevity.[2]

For example at corner 1, normal rotation θ_{n21} equals side slope w_{s21}. Replacing the six side-slope DOF w_{sij} of Figure 26.3(b) by the normal rotations θ_{nij} produces configuration (c). Similarly, replacing the six normal-slope DOF w_{nij} of Figure 26.3(d) by the side rotations θ_{sij} produces configuration (e). Note that the positive sense of the θ_{sij}, viewed as vectors, is opposite that of s_i; this is a consequence of the sign conventions and positive-rotation rule.

§26.1.3. Connectors

If corner slopes along two noncoincident directions are given, the slope along any other corner direction is known. The same is true for corner rotations. It follows that for any of the configurations of Figure 26.3(b) through (e), *the deflection and tangent plane at the 3 corners are known*. However that information cannot be readily communicated to adjacent elements.

The difficulties are illustrated with Figure 26.4(a), which shows two adjacent triangles: red element $(e1)$ and blue element $(e2)$, possessing the DOF configuration of Figure 26.3(b). The deflections w_1 and w_2 match without problems because direction z is shared. But the color-coded side slopes

[2] Note that in passing from slopes to equivalent rotations, the qualifiers “side” and “normal” exchange.

do not match.[3] For more elements meeting at a corner the result is chaotic.

To make the element suitable for implementation in a DSM-based program, it is necessary to transform slope or rotational DOFs to *global directions*. The obvious choices are the midsurface axes $\{x, y\}$. Most FEM codes use rotations instead of slopes since that simplifies connection of different element types (e.g., shells to beams) in three dimensions. Choosing corner rotations θ_{xi} and θ_{yi} as DOF we are led to the configuration of Figure 26.3(f). As illustrated in Figure 26.4(b), the connection problem is solved and the elements are now suitable for the DSM.

§26.2. Convergence Conditions

The foregoing exposition has centered on displacement assumed elements where w is the master field. Element stiffness equations are obtained through the Total Potential Energy (TPE) variational principle presented in the previous Chapter. The completeness and continuity requirements are summarized in §23.3 on the basis of a variational index $m_w = 2$. These are now studied in more detail for cubic triangles.

§26.2.1. Completeness

The TPE variational index $m_w = 2$ requires that all w-polynomials of order 0, 1 and 2 in $\{x, y\}$ be exactly represented over each element. Constant and linear polynomials represent rigid body motions, whereas quadratic polynomials represent constant curvature states.

Now if w is interpolated by a complete cubic, the ten terms $\{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3\}$ are automatically present for any freedom configuration. This appears to be more than enough. Nothing to worry about, right? Wrong. Preservation of such terms over each triangle is guaranteed only if full C^1 continuity is verified. But, as discussed below, attaining C^1 continuity continuity is difficult. To get it one while sticking to cubics one must make substantial changes in the construction of w. Since those changes do not necessarily preserve completeness, that requirement appears as an *a posteriori* constraint. Alternatively, to make life easier C^1 continuity may be abandoned except at corners. If so completeness may again be lost, for example by a seemingly harmless static condensation of w_0. Again this has to be kept as a constraint.

The conclusion is that *completeness cannot be taken for granted* in displacement-assumed KPB elements. Gone is the “IFEM easy ride” of isoparametric elements for variational index 1.

§26.2.2. Continuity Games

To explain what C^1 interelement continuity entails, it is convenient to break this condition into two levels:[4]

C^0 *Continuity*. The element is C^0 compatible if w over any side is completely specified by DOFs on that side.

C^1 *Continuity*. The element is C^1 compatible if it is C^0 compatible, *and* the normal slope $\partial w/\partial n$ over any side is completely specified by DOFs on that side.

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[3] Positive slopes along the common side point in opposite directions.

[4] It is tacitly assumed that the condition is satisfied inside the element.

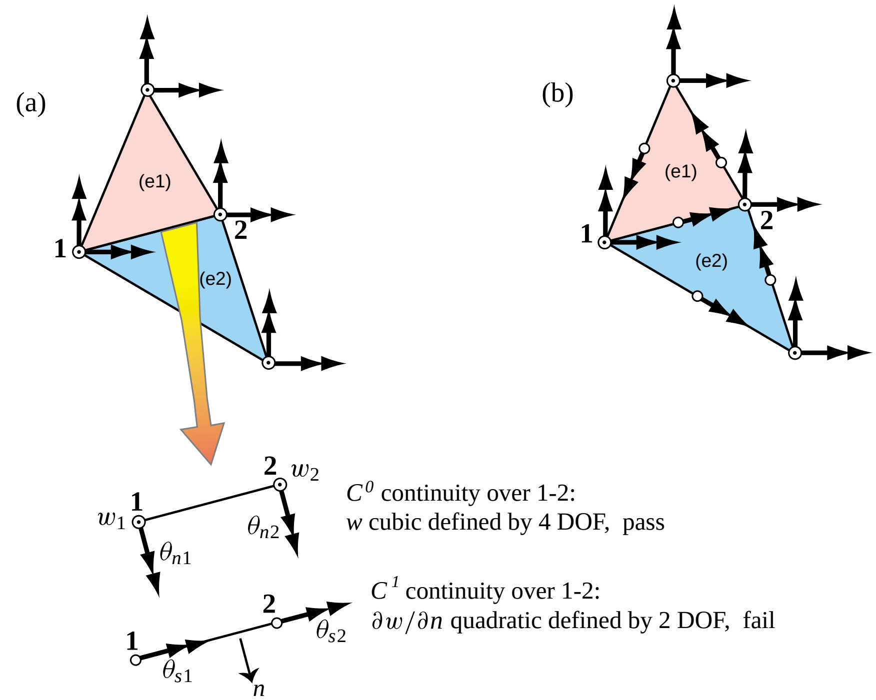

Figure 26.5. Checking interelement continuity of KPB triangles along common side 1-2.

The first level: C^0 continuity, is straightforward. In the IFEM course, which for 2D problems deals with variational index $m = 1$ only, the condition is easily achieved with the isoparametric formulation. The second level is far more difficult. Attaining it is the exception rather than the rule, and elements that make it are not necessarily the best performing ones. Nevertheless it is worth studying since so many theory advances in FEM: hybrid principles, the patch test, etc., came as a result of research in C^1 plate elements.

To appreciate the difficulties in attaining C^1 continuity consider two cubic triangles with the freedom configuration of Figure 26.3(f) connected as shown in Figure 26.5(a). At corners 1 and 2 the rotational freedoms are rotated to align with common side 1-2 and its normal as shown underneath Figure 26.5(a). Over side 1-2 the deflection w varies cubically. This variation is defined by four DOF on that wide: w_1, w_2, θ_{n1} and θ_{n2}. Consequently C^0 continuity holds. Over side 1-2 the normal slope $\partial w/\partial n$ varies quadratically since it comes from differentiating w once. A quadratic is defined by three values; but there are only two DOF that can control the normal slope: θ_{s1} and θ_{s2}. Consequently C^1 continuity is violated between corner points.

To control a quadratic variation of $\partial w/\partial n = \theta_s$ an additional DOF on the side is needed. The simplest implementation of this idea is illustrated in Figure 26.5(b): add a side rotation DOF at the midpoint. But this increases the total number of DOF of the triangle to at least 12: 9 at the corners and 3 at the midpoints.[5] Because a cubic has only 10 independent terms, terms from a quartic polynomial are needed if we want to keep just a polynomial expansion over the full triangle. But that raises the side variation orders of w and $\partial w/\partial n$ to 4 and 3, respectively, and again we are short. Limitation Theorem III given below state that it is impossible to "catch up" under these conditions.

[5] The number climbs to 13 if the centroid deflection w_0 is kept as a DOF.

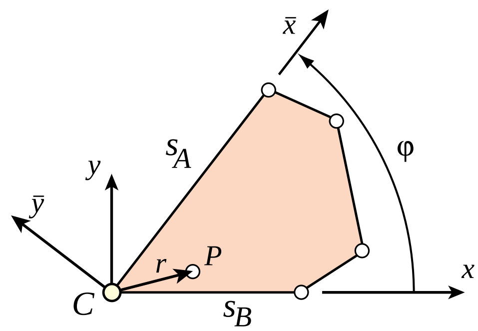

Figure 26.6. A corner C of a polygonal KPB element.

§26.3. *Kinematic Limitation Principles

This section examines *kinematic limitation principles* that place constraints on the construction of KPB displacement-assumed elements. The principles are useful in ruling out once and for all the easy road to constructing such elements, and in explaining why researchers turned their attention elsewhere.

Limitation principles 1 and 2 are valid for an arbitrary element polygonal shape as illustrated in Figure 26.6, that has only corner DOF on its boundary.[6]

Select a corner C bounded by sides s_A and s_B, which subtend angle φ. We use the abbreviations $s_\varphi = \sin\varphi$ and $c_\varphi = \cos\varphi$. Select a rectangular Cartesian coordinate (RCC) system: $\{x, y\}$ with origin at C and x along side s_A. Another RCC system $\{\bar{x}, \bar{y}\}$ is placed with $\bar{x}$ along side s_B. The systems are related by $\{\bar{x} = x c_\varphi + y s_\varphi,\ \bar{y} = -x s_\varphi + y c_\varphi\}$ and $\{x = \bar{x} c_\varphi - \bar{y} s_\varphi,\ y = \bar{x} s_\varphi + \bar{y} c_\varphi\}$

We focus on limitations related to assuming that w has *continuous second derivatives* at C. That is, the following Taylor expansion holds at a point $P(x, y)$ at a distance r from C:

$$w = a_0 + a_1\, x + a_2\, y + a_3\, x^2 + a_4\, xy + a_5\, y^2 + O(r^3) \tag{26.2}$$

We need the following results derivable from (26.2). The lateral deflections over s_A and s_B are

$$\begin{aligned} w_A &= a_0 + a_1 x + a_3 x^2 + O(r^3),\\ w_B &= a_0 + (a_1 c_\varphi + a_2 s_\varphi)\bar{x} + (a_3 c_\varphi^2 + a_4 s_\varphi c_\varphi + a_5 s_\varphi^2)\bar{x}^2 + O(r^3). \end{aligned} \tag{26.3}$$

The along-the-side slopes over s_A and s_B are obtained by evaluating $\partial w/\partial x$ at $y = 0$ and $\partial w/\bar{x}$ at $\bar{y} = 0$:

$$\begin{aligned} w_{sA} &= a_1 + 2a_3\, x + O(r^2),\\ w_{sB} &= a_1 c_\varphi + a_2 s_\varphi + 2(a_3 c_\varphi^2 + a_4 s_\varphi c_\varphi + a_5 c_\varphi^2)\bar{x} + O(r^2). \end{aligned} \tag{26.4}$$

The normal slopes over s_A and s_B are obtained by evaluating $-w_y = -\partial w/\partial y = -a_2 - a_4\, x - 2a_5\, y + O(r^2)$ at $y = 0$, and $w_{\bar{y}} = \partial w/\partial \bar{y} = -(a_1 + 2a_3\, x + a_4\, y)s_\varphi + (a_2 + a_4\, x + 2a_5\, y)\, c_\varphi + O(r^2)$ at $\bar{y} = 0$. This gives

$$\begin{aligned} w_{nA} &= -a_2 - a_4\, x + O(r^2),\\ w_{nB} &= -a_1\, s_\varphi + a_2\, c_\varphi + \big(a_4(c_\varphi^2 - s_\varphi^2) - 2(a_3 - a_5)s_\varphi c_\varphi\big)\bar{x} + O(r^2) \end{aligned} \tag{26.5}$$

Assume that the element satisfies the following four assumptions.

[6] The presence of internal DOFs is not excluded.

(I) The Taylor series (26.2) at C is valid; thus the deflection w has second derivatives at C.

(II) Three nodal values are chosen at C: $w_C = a_0$, $\theta_{xC} = (\partial w/\partial y)_C = a_1$ and $\theta_{yC} = -(\partial w/\partial x)_C = -a_2$. This is the standard choice for plate elements.

(III) Completeness is satisfied in that the six states $w = \{1, x, y, x^2, xy, y^2\}$ are exactly representable over the element.

(IV) The variation of the normal slope $\partial w/\partial n$ along the element sides is linear.

§26.3.1. Limitation Theorem I

A KPB element cannot satisfy (I), (II), (III) and (IV) simultaneously.

Proof. Choose three set of corner DOF at C to satisfy:

$$\begin{aligned}
\text{Set 1:} \qquad & w_C = 1, \quad \left(\frac{\partial w}{\partial n_A}\right)_C = 0, \quad \left(\frac{\partial w}{\partial n_B}\right)_C = 0,\\
\text{Set 2:} \qquad & w_C = 0, \quad \left(\frac{\partial w}{\partial n_A}\right)_C = 1, \quad \left(\frac{\partial w}{\partial n_B}\right)_C = 0,\\
\text{Set 3:} \qquad & w_C = 0, \quad \left(\frac{\partial w}{\partial n_A}\right)_C = 0, \quad \left(\frac{\partial w}{\partial n_B}\right)_C = 1.
\end{aligned} \tag{26.6}$$

while all other DOF are set to zero.

Set 1 imposes $a_0 = 1$ and $a_1 = a_2 = 0$. Both normal slopes at C are zero, and so are at other corners. Because of the linear variation assumption (IV), $w_{nA} = \partial w/\partial n_A \equiv 0$ and $w_{nB} = \partial w/\partial n_B \equiv\!= 0$. Expressions (26.5) require $a_4 = 0$ and $a_3 = a_5$.

Set 2 imposes $a_0 = 0$, $a_1 = 1$ and $a_2 = c_\varphi/s_\varphi$. Now $w_{nB} \equiv 0$. This requires $a_4 = 0$ and $a_3 = a_5$, as above. Replacing gives $w_{nA} = 1$, which contradicts (IV).

Set 3 imposes $a_0 = a_1 = 0$, and $a_2 = 1$. Now $w_{nA} = 0$ identically, which forces $a_4 = 0$.

An arbitrary set of values for the DOFs at C can be written as a linear combination of (26.6). But any such combination requires $a_4 = 0$, making the twist term vanish identically in (26.1). Thus the assumption (IV) of completeness cannot be satisfied.

Oddly enough the proof needs no assumption about how w varies along the sides; that is, C^0 compatibility. Just the assumption that the normal slope varies linearly is enough to kill completeness.

This theorem says that to get a C^1 compatible element while retaining assumptions (I), (II) and (III) the normal slope variation cannot be linear. Such conforming elements can be constructed, for example, using product of cubic Hermitian functions along side directions with sutable damping factors along the other directions. But this approach runs into serious trouble as shown by the next limitation principle.

§26.3.2. Limitation Theorem II

Any C^1-compatible, non rectangular KPB element that satisfies conditions (I) and (II) cannot represent exactly all constant curvature states.

Proof. If the element is exactly in a constant curvature state, the deflection w must be quadratic in $\{x, y\}$. Hence the normal slope variation must be linear. But according to Limitation Theorem I the element cannot represent the constant twist state.

This theorem shows that (I), (II) and (III) are incompatible. A more detailed study shows that for a C^1 compatible rectangular element with sides aligned with $\{x, y\}$ only the twist state is lost but that x^2 and y^2 can be exactly represented. For non-rectangular geometries all constant curvature states are lost.

If one insists in C^1 continuity there are two ways out:

Abandon (I): Keep a single polynomial over the element but admit higher order derivatives as corner degrees of freedom.

Abandon (II): Permit discontinuous second derivatives at corners through the use of non-polynomial assumptions, or macroelements.

Both techniques have been tried with success. The use of second derivatives as DOFs, if (I) is abandoned, is forced by the next limitation principle.

§26.3.3. Limitation Theorem III

Suppose that a simple complete polynomial expansion of order $n \geq 3$ is assumed for w over a triangle. At each corner i the deflection w_i, the slopes w_{xi}, w_{yi} and all midsurface derivatives up to order $m \geq 1$ are taken as degrees of freedom. Then C^1 continuity requires $m \geq 2$ and $n \geq 5$.

Proof. Proven in the writer's thesis.[7]

Here is an informal summary of the proof. The total number of DOFs for a complete polynomial is $P_n = (n+1)(n+2)/2 = F_n + B_n$. Of these $F_n = \min\big((n+1)(n+2)/2,\, 6n-9\big)$ are called *fundamental freedoms* in the sense that they affect interelement compatibility. The $B_n = \max\big(0,\, (n-5)(n-4)/2\big)$ are called *bubble freedoms*, which have zero value and normal slopes along the three sides. Bubbles occur only if $n \geq 6$.

Over each side the variation of w is of order n and that of $w_n = \partial w/\partial n$ of order $n-1$. This requires $n+1$ and n control DOF on the side, respectively. The number of corner freedoms is $N_c = (m+1)(m+2)/2$, which provides $2(m+1)$ and $2m$ controls on w and w_n, respectively. Within the side (e.g. at a midpoint) one need to add $N_{ws} = n+1-2(m+1) \geq 0$ control DOFs for C^0 continuity in w and $N_{wns} = n-2m \geq 0$ control DOFs for C^1 continuity in w_n. The grand total of boundary DOFs is $N_b = 3(N_c + N_{ws} + N_{wns})$. This has to be equal to the number of fundamental freedoms F_n. Here is a tabulation for various values of n and m. N/A means that the interpolation order is not applicable as being too low as it gives $N_{ws} < 0$ and/or $N_{wns} < 0$.

	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
	$F_n=10$	$F_n=15$	$F_n=21$	$F_n=27$	$F_n=33$
$m=1$	$N_b=12$	$N_b=18$	$N_b=24$	$N_b=30$	$N_b=36$
$m=2$	N/A	N/A	$\boxed{N_b=21}$	$N_b=27$	$N_b=33$
$m=3$	N/A	N/A	N/A	N/A	$N_b=33$

The first interesting solution is boxed. It corresponds to a complete quintic polynomial $n=5$ with 21 DOFs, all fundamental. Six degrees of freedom are required at each corner: w_i, w_{xi}, w_{yi}, w_{xxi}, w_{xyi}, w_{yyi}, $i=1,2,3$, plus one normal slope (or side rotation) at each midpoint. The resulting element, called CCT-21, was presented in the thesis cited in footnote 7. The element, however, is too complex for use in standard FEM codes because it does not mix easily with other elements such as beams. So it has only been used in special purpose codes for plates only.

[7] C. A. Felippa, Refined finite element analysos of linear and nonlinear two-dimensional structures, *Ph.D. Dissertation*, Department of Civil Engineering, University of California at Berkeley, 1966.

§26.4. Early Work

By the late 1950s the success of the Finite Element Method with membrane problems (for example, for wing covers and fuselage panels) led to high hopes in its application to plate bending and shell problems. The first results were published by 1960. But until 1965 only rectangular models gave satisfactory results. The construction of successful triangular elements to model plates and shells of arbitrary geometry proved more difficult than expected. Early failures, however, led to a more complete understanding of the theoretical basis of FEM and motivated several advances taken for granted today.

The major source of difficulties in plates is due to stricter continuity requirements. The objective of attaining normal slope interelement compatibility posed serious problems, documented in the form of Limitation Theorems in §26.3. By 1963 researchers were looking around "escape ways" to bypass those problems. It was recognized that completeness, in the form of exact representation of rigid body and constant curvature modes, was fundamental for convergence to the analytical solution, a criterion first enuncaite by Melosh.[8] The effect of compatibility violations was more difficult to understand until the patch test came along.

§26.4.1. Rectangular Elements

The first successful rectangular plate bending element was developed by Adini and Clough[9]. This element has 12 degrees of freedom (DOF). It used a complete third order polynomial expansion in x and y, aligned with the rectangle sides, plus two additional x^3y and xy^3 terms. The element satisfies completeness as well as transverse deflection continuity but normal slope continuity is only maintained at the four corner points. The same element results from another expansion proposed by Melosh (1963 reference cited), which erroneously states that the element satisfies C^1 continuity. The error was noted in subsquent discussion.[10] In 1961 Melosh had proposed[11] had proposed a rectangular plate element constructed with beam-like edge functions damped linearly toward the opposite side, plus a uniform twisting mode. Again C^0 continuity was achieved but not C^1 except at corners.

Both of the foregoing elements displayed good convergence characteristics when used for rectangular plates. However the search for a compatible displacement field was underway to try to achieve monotonic convergence. A fully compatible 12-DOF rectangular element was apparently first developed by Papenfuss in an obscure reference.[12] The element appears to have been rediscovered several times. The simplest derivation can be carried out with products of Hermite cubic polynomials, as noted below. Unfortunately the uniform twist state is not include in the expansion and consequently the element fails the completeness requirement, converging monotonically to a zero twist-curvature solution: the right answer for the wrong problem.

In a brief but important paper, Irons and Draper[13] stressed the importance of completeness for uniform strain modes (constant curvature modes in the case of plate bending). They proved that it is impossible to construct any polygonal-shape plate element with only 3 DOFs per corner and continuous corner curvatures that can simultaneously maintain normal slope conformity and inclusion of the uniform twist mode. This negative

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[8] R. J. Melosh, Bases for the derivation of stiffness matrices for solid continua, *AIAA J.*, bf 1, 1631–1637, 1963.

[9] A. Adini and R. W. Clough, Analysis of plate bending by the finite element method, NSF report for Grant G-7337, Dept. of Civil Engineering, University of California, Berkeley, 1960. Also A. Adini, Analysis of shell structures by the finite element method, Ph.D. Dissertation, Department of Civil Engineering, University of California, Berkeley, 1961.

[10] J. L. Tocher and K. K. Kapur, Comment on Melosh's paper, *AIAA J.*, **3**, 1215–1216, 1965.

[11] R. J. Melosh, A stiffness matrix for the analysis of thin plates in bending, *J. Aero. Sci.*, **28**, 34–42, 1961.

[12] S. W. Papenfuss, Lateral plate deflection by stiffness methods and application to a marquee, M. S. Thesis, Department of Civil Engineering, University of Washington, Seattle, WA, 1959.

[13] B. M. Irons and K. Draper, Inadequacy of nodal connections in a stiffness solution for plate bending, *AIAA J.*, **3**, 965–966, 1965.

result, presented in §26.3 as Limitation Theorem II, effectively closed the door to the construction of the analog of isoparametric elements in plate bending.

The construction of fully compatible polynomials expansions of various orders for rectangular shapes was solved by Bogner et al in 1965[14] through Hermitian interpolation functions. In their paper they rederived Papenfuss' element, but in an Addendum[15] they recognised the lack of the twist mode and an additional degree of freedom: the twist curvature, was added at each corner. The 16-DOF element is complete and compatible, and produced excellent results. More refined rectangular elements with 36 DOFs have been also developed using fifth order Hermite polynomials.

§**26.4.2. Triangular Elements**

Flat triangular plate elements have a wider range of application than rectangular elements since they naturally conform to the analysis of plates and shells of arbitrary geometry for small and large deflections. But as previously noted the development of adequate kinematic expansions was not an easy problem and has kept researchers busy for decades.

The success of incompatible rectangular elements is due to the fact that the assumed polynonial expansions for w can be considered as "natural" deformation modes, after a trivial reduction to nondimensional form. They are intrisically related to the geometry of the element because the local system is chosen along two preferred directions. Lack of C^1 continuity between corners disppears in the limit of a mesh refinement.

Early attempts to construct triangular elements tried to mimic that scheme, using a RCC system arbitrarily oriented with respect to the element. This lead to an unpleasant lack of invariance whenever an *incomplete* polynomial was selected, since kinematic constraints were artificially imposed. Furthermore the role of completeness was not understood. Thus the first suggested expansion for a triangular element with 9 DOFs[16]

$$w = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 y^2 \alpha_6 x^3 + \alpha_7 x^2 y + \alpha_8 x y^2 + \alpha_9 y^3 \tag{26.7}$$

in which the xy term is missing, violates compability, completeness and invariance requirements. The element converges, but to the wrong solution with zero twist curvature.

Tocher in his thesis cited above tried two variants of the cubic expansion:

1 Combining the two cubic terms: $x^2 y + x y^2$.

2 Using a complete 10-term cubic polynomial The first choice satisfies completeness but violates compatibility and invariance. The second assumption satisfies completeness and invariance but violates compatibility and poses the problem: what to do with the extra DOF? Tocher decided to eliminate it by a generalized inversion process, which unfortunately leads to discarding a fundamental degree of freedom. This led to an extremely flexible (and non convergent) element. The elimination technique of Bazeley et. al. discussed in Chapter 25 was more successful and produced an element which is still in use today.

The first fully compatible 9-DOF cubic triangle was finally constructed by the macroelement technique.[17]. The triangle was divided into three subtriangles, over each of which a cubic expansion with linear variation along

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[14] F. K. Bogner, R. L. Fox and L. A. Schmidt Jr., The generation of interelement compatible stiffness and mass matrices by the use of interpolation formulas, *Proc. Conf. on Matrix Methods in Structural Mechanics*, WPAFB, Ohio, 1965, in *AFFDL TR 66-80*, pp. 397–444, 1966.

[15] Addendum to aforementioned paper, 411–413 in *AFFDL TR 66-80*.

[16] J. L. Tocher, Analysis of plate bending using triangular elements, *Ph. D. Dissertation*, Dept. of Civil Engineering, University of California, Berkeley, California, 1963.

[17] R. W. Clough and J. L. Tocher, Finite element stiffness matrices for analysis of plate bending, Proc. Conf. on Matrix Methods in Structural Mechanics, WPAFB, Ohio, 1965, in *AFFDL TR 66-80*, 515–545, 1966.

the exterior side was assumed. A similar element with quadratic slope variation and 12 DOF was constructed by the writer.[18] The original derivations, carried out in x, y coordinates were considerably simplified later by using triangular coordinates.

The 1965 paper by Bazeley et al.[19] was an important milestone. In it three plate bending triangles were developed. Two compatible elements were developed using rational functions. Experiments showed them to be quite stiff and have no interest today. An incompatible element called the BCIZ triangle since was obtained by eliminating the 10th DOF from a complete cubic in such as way that completeness was maintained. This element is incompatible. Numerical experiments showed that it converged for some mesh patterns but not for others. This puzzling behavior led to the invention of the patch test.[20] The patch test was further developed by Irons and coworkers in the 1970s.[21] A mathematical version is presented in the Strang-Fix monograph.[22]

§26.4.3. Quadrilateral Elements

Arbitrary quadrilaterals can be constructed by assembling several triangles, and eliminating internal DOFs, if any by static condensation. This represents an efficient procedure to take into account that the four corners need not be on a plane. The article by Clough and Felippa cited above presents the first quadrilateral element constructed this way. That element was included in the open-source SAP family of FEM codes and used for shell analysis since 1968.

A direct construction of an arbitrary quadrilateral with 16 DOFs was presented by de Veubeke.[23] The quadrilateral is formed by a macroassembly of four triangles by the two diagonals, which are selected as a skew Cartesian coordinate system to develop the finite element fields.

§26.5. More Recent Work

The fully conforming elements developed in the mid 1960s proved “safe” for FEM program users in that convergence could be guaranteed. Performance was another matter. Triangular elements proved to be excessively stiff, particularly for high aspect ratios. A significant improvement in performance was achieved by Razzaque[24] who replaced the shape function curvatures with least-square-fitted smooth functions. This technique was later shown to be equivalent to the stress-hybrid formulation.

The first application of mixed functionals to finite elements was actually to the plate bending problem. Herrman[25] developed a mixed triangular model in which transverse displacements and bending moments are

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[18] R. W. Clough and C. A. Felippa, A refined quadrilateral element for analysis of plate bending, Proc. 2nd Conf. on Matrix Methods in Structural Mechanics, WPAFB, Ohio, 1965, in *AFFDL TR 69-23*, 1969

[19] G. P. Bazeley, Y, K. Cheung, B. M. Irons and O. C. Zienkiewicz, Triangular elements in plate bending — conforming and nonconforming solutions, Proc. Conf. on Matrix Methods in Structural Mechanics, WPAFB, Ohio, 1965, in *AFFDL TR 66-80*, pp. 547–576, 1966.

[20] Addendum to Bazeley et. al. paper cited above, pp. 573–576 in *AFFDL TR 66-80*.

[21] B. M. Irons and A. Razzaque, Experiences with the patch test for convergence of finite elements, in *Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations*, ed. by K. Aziz, Academic Press, New York, 1972.
B. M. Irons and S. Ahmad, *Techniques of Finite Elements*, Ellis Horwood Ltd, Chichester, England, 1980.

[22] G. Strang and G. Fix, *An Analysis of the Finite Element Method*, Prentice-Hall, Englewood Cliffs, N.J., 1973.

[23] B. Fraeijs de Veubeke, A conforming finite element for plate bending, *Int. J. Solids Struct.*, **4**, 95–108, 1968

[24] A. Razzaque, Program for triangular bending elements with derivative smoothing, *Int. J. Numer. Meth. Engrg.*, **6**, 333–343, 1973.

[25] L. R. Herrmann, A bending analysis for plates, in *Proceedings 1st Conference on Matrix Methods in Structural Mechanics*, AFFDL-TR-66-80, Air Force Institute of Technology, Dayton, Ohio, 577–604, 1966.

selected as master variables. A linear variation was assumed for both variables. This work was based on the HR variational principle and included the transversal shear energy. The element did not perform well in practice.

Successful plate bending elements have also been been constructed by Pian's assumed-stress hybrid method [26] The 9-dof triangles in this class are normally derived by assuming cubic deflection and linear slope variations along the element sides, and a linear variation of the internal moment field. Efficient formulations of such elements have been published.[27] Hybrid elements generally give better moment accuracy than conforming displacement elements. The derivation of these elements, however, is more involved in that it depends on finding equilibrium moments fields within the element, which is not a straightforward matter if the moments vary within the element or large deflections are considered.

Much of the recent research on displacement-assumed models has focused on relaxing or abandoning the assumptions of Kirchhoff thin-plate theory. Relaxing these assumptions has produced elements based on the so-called discrete Kirchhoff theory.[28] In this method the primary expansion is made for the plate rotations. The rotations are linked to the nodal freedoms by introduction of thin-plate normality conditions at selected boundary points, and then interpolating displacements and rotations along the boundary. The initial applications of this method appear unduly complicated. A clear and relatively simple account is given by Batoz, Bathe and Ho.[29] The most successful of these elements to date is the DKT (Discrete Kirchhoff Triangle), an explicit formulation of which has been presented by Batoz.[30]

A more drastic step consists of abandoning the Kirchhoff theory in favor of the Reissner-Mindlin theory of moderately thick plates. The continuity requirements for the displacement assumption are lowered to C^0 (hence the name "C^0 bending elements"), but the transverse shear becomes an integral part of the formulation.

Historically the first fully conforming triangular plate elements were not Clough-Tocher's but C^0 elements called "facet" elements that were derived in the late 1950s, although an account of their formulation was not published until 1965.[31] Facet elements, however, suffer from severe numerical problems for thin-plate and obtuse-angle conditions. The approach was revived later by Argyris *et. al.*[32] within the context of degenerated "brick" elements.

Successful quadrilateral C^0 elements have been developed by Hughes, Taylor and Kanolkulchai,[33] Pugh,

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[26] T. H. H. Pian, Derivation of element stiffness matrices by assumed stress distributions, *AIAA J.*, **2**, 1333–1336, 1964. T. H. H. Pian and P. Tong, Basis of finite element methods for solid continua, *Int. J. Numer. Meth. Engrg.*, **1**, 3–29, 1969.

[27] O. C. Zienkiewicz, *The Finite Element Method in Engineering Science*, McGraw-Hill, New York, 3rd edn., 1977. D. J. Allman, Triangular finite elements for plate bending with constant and linearly varying bending moments, *Proc. IUTAM Conf. on High Speed Computing of Elastic Structures*, Liège, Belgium, 105–136, 1970.

[28] J. Stricklin, W. Haisler, P. Tisdale and R. Gunderson, A rapidly converging triangular plate bending element, *AIAA J.*, **7**, 180–181, 1969. Also G. Dhatt, An efficient triangular shell element, *AIAA J.*, **8**, No. 11, 2100–2102, 1970.

[29] J. L. Batoz, K.-J. Bathe and Lee-Wing Ho, A study of three-node triangular plate bending elements, *Int. J. Numer. Meth. Engrg.*, **15**, 1771–1812, 1980.

[30] J. L. Batoz, An explicit formulation for an efficient triangular plate-bending element, *Int. J. Numer. Meth. Engrg.*, **18**, 1077–1089, 1982.

[31] R. J. Melosh, A flat triangular shell element stiffness matrix, Proc. Conf. on Matrix Methods in Structural Mechanics, WPAFB, Ohio, 1965, in *AFFDL TR 66-80*, 503–509, 1966.

[32] J. H. Argyris, P. C. Dunne, G. A. Malejannakis and E. Schelkle, A simple triangular facet shell element with applications to linear and nonlinear equilibrium and elastic stability problems, *Comp. Meths. Appl. Mech. Engrg.*, **11**, 215–247, 1977.

[33] T. J. R. Hughes, R. Taylor and W. Kanolkulchai, A simple and efficient finite element for plate bending, *Int. J. Numer. Meth. Engrg.*, **11**, 1529–1543, 1977.

Hinton and Zienkiewicz,[34] MacNeal,[35] Crisfield,[36] Tessler and Hughes,[37] Dvorkin and Bathe,[38] and Park and Stanley.[39]

Triangular elements in this class have been presented by Belytschko, Stolarski and Carpenter.[40] The construction of robust C^0 bending elements is delicate, as they are susceptible to 'shear locking' effects in the thin-plate regime if fully integrated, and to kinematic deficiencies (spurious modes) if they are not. When the proper care is exercised good results have been reported for quadrilateral elements and, more recently, for triangular elements.[41]

A different path has been taken by Bergan and coworkers, who retained the classical Kirchhoff formulation but in conjunction with the use of highly nonconforming (C^{-1}) shape functions. They have shown that interelement continuity is not an obstacle to convergence provided the shape functions satisfy certain energy and force orthogonality conditions[42] or the stiffness matrix is constructed using the *free formulation*[43] rather than the standard potential energy formulation. A characteristic feature of these formulations is the careful separation between basic and higher order assumed displacement functions or "modes". Results for triangular bending elements derived through this approach have reported satisfactory performance.[44] One of these elements, which is based on force-orthogonal higher order functions, was rated in 1983 as the best performer in its class.[45]

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[34] E. D. Pugh, E. Hinton and O. C. Zienkiewicz, A study of quadrilateral plate bending elements with reduced integration, *Int. J. Numer. Meth. Engrg.*, **12**, 1059–1078, 1978.

[35] R. H. MacNeal, A simple quadrilateral shell element, *Computers & Structures*, **8**, 175–183, 1978.
R. H. MacNeal, Derivation of stiffness matrices by assumed strain distributions, *Nucl. Engrg. Design*, **70**, 3–12, 1982.

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[37] A. Tessler and T. J. R. Hughes, A three-node Mindlin plate element with improved transverse shear, *Comp. Meths. Appl. Mech. Engrg.*, **50**, 71–101, 1985.

[38] E. N. Dvorkin and K. J. Bathe, A continuum mechanics based four-node shell element for general nonlinear analysis, *Engrg. Comp.*, **1**, 77–88, 1984.

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K. C. Park and G. M. Stanley, A Curved C^0 shell element based on assumed natural-coordinate strains, *J. Appl. Mech.*, **108**, 278–286, 1986.

[40] T. Belytschko, H. Stolarski and N. Carpenter, A C^0 triangular plate element with one-point quadrature, *Int. J. Numer. Meth. Engrg.*, **20**, 787–802, 1984.

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