

# 25

## Kirchhoff Plates: BCs and Variational Forms

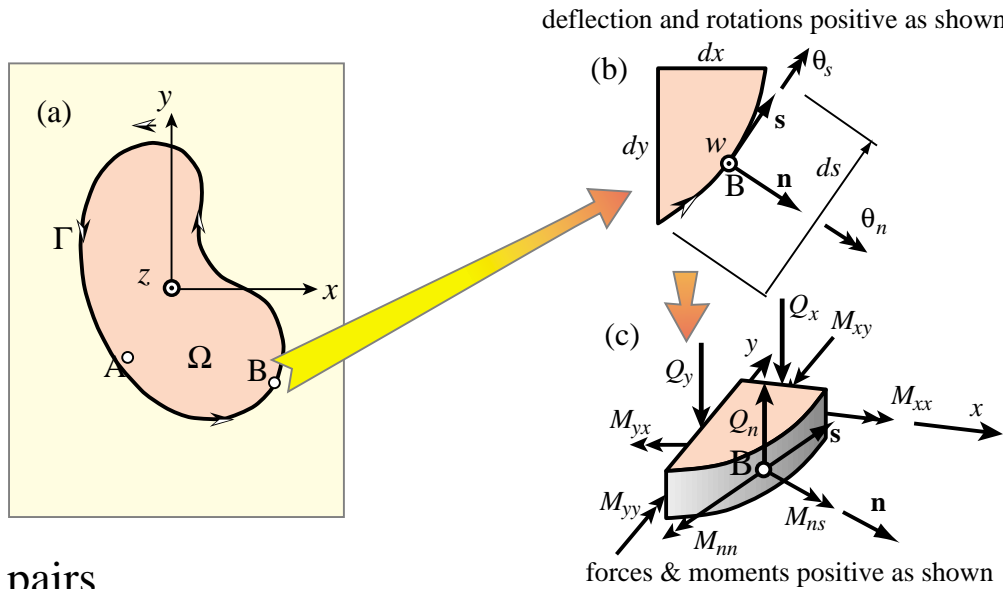
## **BCs in Kirchhoff Plates (and Thin Shells) - Mathematical Difficulties**

**Rotational freedoms introduce displacement derivatives  
(differentiation is always a source for trouble)**

**Some BCs are difficult to realize experimentally  
e.g. simple support**

**Kirchhoff model permits only two independent conditions  
at boundary points, whereas the natural number is three**

# Conjugate Quantities on Kirchhoff Plate Boundary



Work pairs

$$w, \quad \frac{\partial w}{\partial n} = -\theta_s, \quad \frac{\partial w}{\partial s} = \theta_n \quad Q_n, \quad M_{nn}, \quad M_{ns}$$

Boundary work integral

$$W_B = \int_{\Gamma} \left( Q_n w + M_{ns} \frac{\partial w}{\partial s} + M_{nn} \frac{\partial w}{\partial n} \right) ds = \int_{\Gamma} \left( Q_n w + M_{ns} \theta_n - M_{nn} \theta_s \right) ds$$

## The Modified Shear (aka "Kirchhoff Shear Force")

$$W_B|_A^B = \int_A^B \left[ \left( Q_n - \frac{\partial M_{ns}}{\partial s} \right) w + M_{nn} \frac{\partial w}{\partial n} \right] dt + M_{ns} w|_A^B.$$

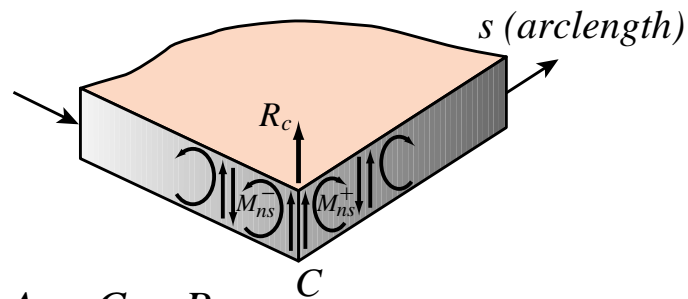
$$V_n = Q_n - \frac{\partial M_{ns}}{\partial s},$$

$$W_B|_A^B = \int_A^B \left( V_n w + M_{nn} \frac{\partial w}{\partial n} \right) dt + M_{ns} w|_A^B.$$



responsible for "corner forces"

## Physical Interpretation of Modified Shear at Plate Corner

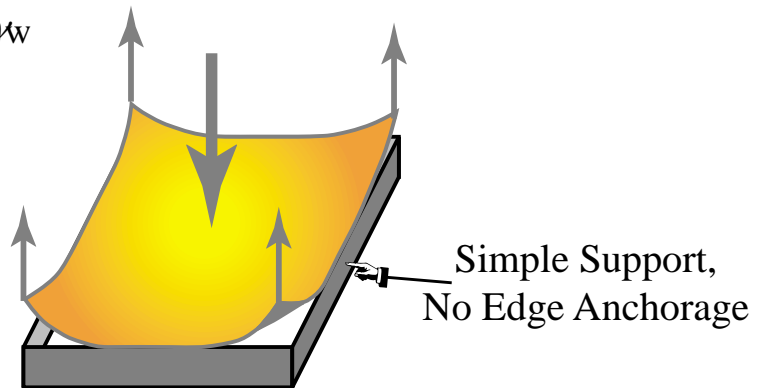


At a corner point  $A \rightarrow C \leftarrow B$ :

$$M_{ns}|_A^B = R_c w = (M_{ns}^+ - M_{ns}^-) w$$

$$R_c = M_{ns}^+ - M_{ns}^-$$

This is a "corner lifting" force



## Common Plate Boundary Conditions

**Clamped or Fixed Edge** (with  $s$  along edge)

$$w = 0, \quad \frac{\partial w}{\partial n} = -\theta_s = 0$$

**Simply Supported Edge** (with  $s$  along edge)

$$w = 0, \quad M_{nn} = 0$$

**Free Edge** (with  $s$  along edge)

$$V_n = 0, \quad M_{nn} = 0$$

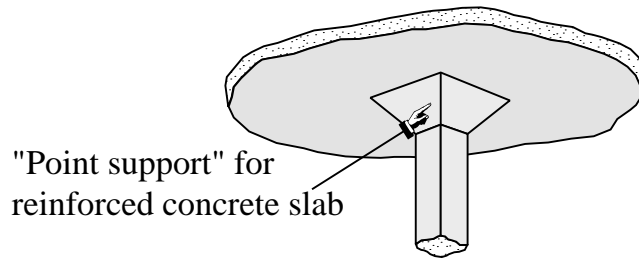
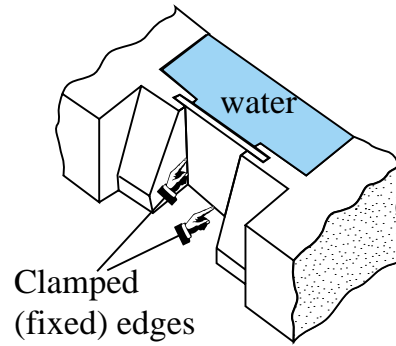
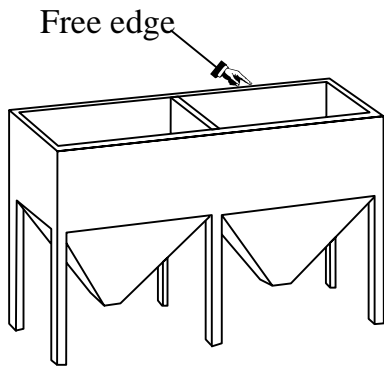
**Symmetry Line** (with  $s$  along line)

$$V_n = 0, \quad \frac{\partial w}{\partial n} = -\theta_s = 0$$

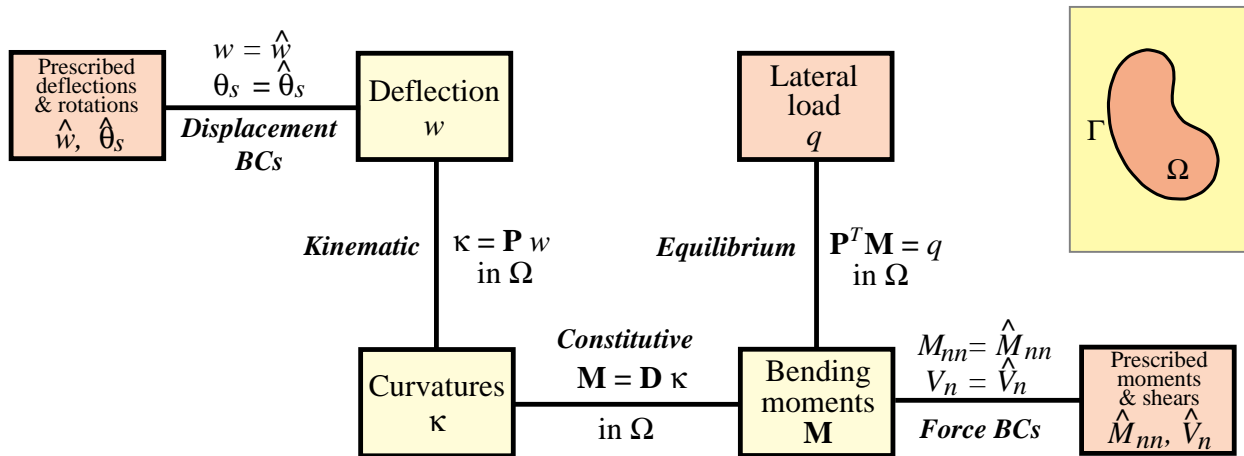
**Point Support**

$$w = 0$$

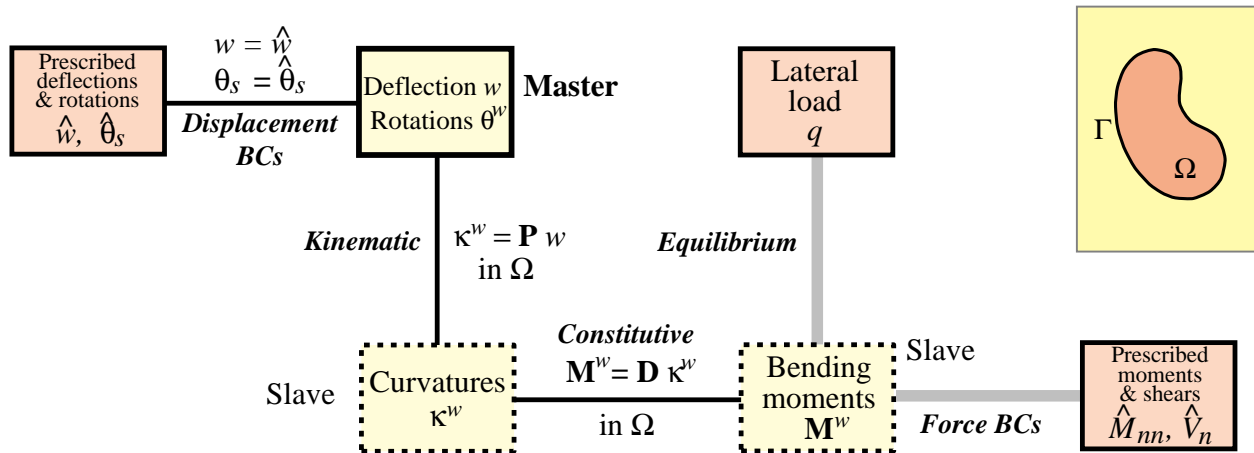
# Examples



# Strong Form Tonti Diagram



# Total Potential Energy Principle: Departure Weak Form



## Total Potential Energy Functional of Kirchhoff Plate

$$\Pi_{\text{TPE}}[w] = U_{\text{TPE}}[w] - W_{\text{TPE}}[w]$$

**internal +  
external**

**Internal energy:**

$$U_{\text{TPE}}[w] = \frac{1}{2} \int_{\Omega} (\mathbf{M}^w)^T \boldsymbol{\kappa}^w = \frac{1}{2} \int_{\Omega} (\boldsymbol{\kappa}^w)^T \mathbf{D} \boldsymbol{\kappa}^w d\Omega = \frac{1}{2} \int_{\Omega} (w \mathbf{P}^T) \mathbf{D} (\mathbf{P} w) d\Omega$$

where  $\mathbf{P} = [\partial^2/\partial x^2 \quad \partial^2/\partial y^2 \quad 2\partial^2/\partial x\partial y]^T$

**External work has three components:**

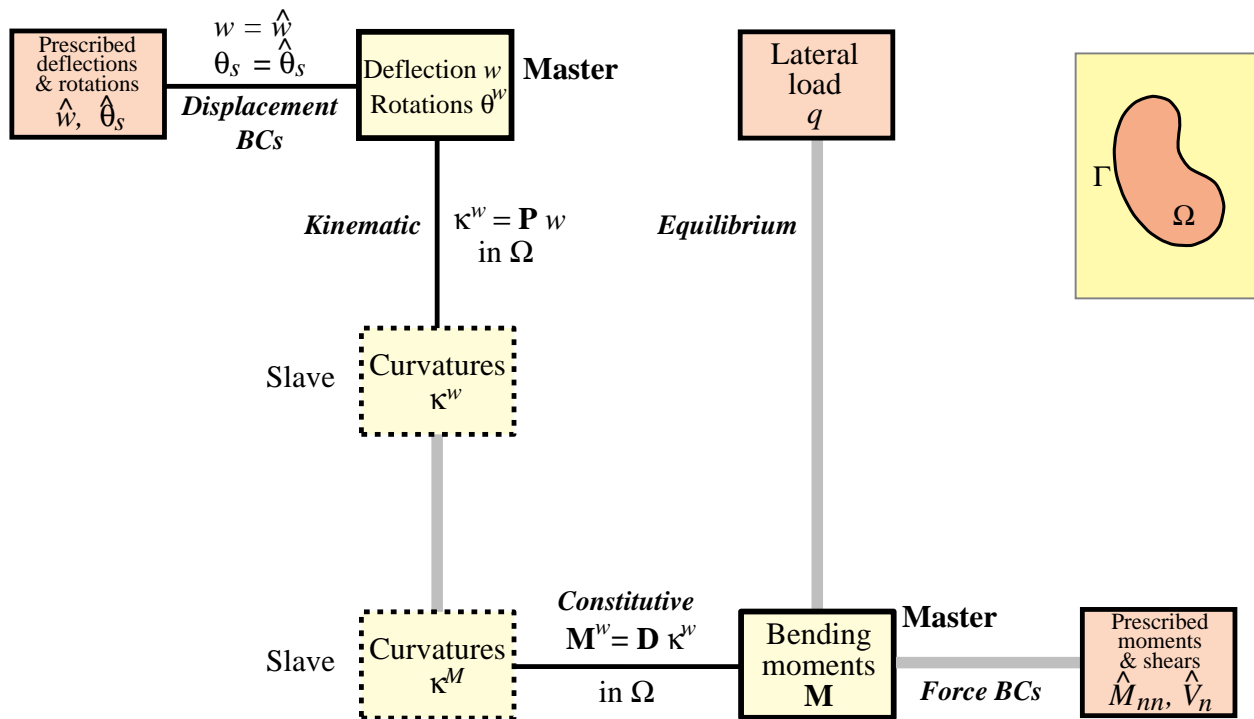
$$W_{\text{TPE}}[w] = W_q[w] + W_B[w] + W_C[w]$$

**transverse load +  
boundary forces +  
corner forces**

$$W_q[w] = \int_{\Omega} q w d\Omega, \quad W_B[w] = \int_{\Gamma_{VM}} (\hat{V}_n w - \hat{M}_{nn} \theta_s^w) d\Gamma$$

$$W_C = \sum_{j=1}^{n_c} R_{cj} w_j = \sum_{j=1}^{n_c} (\hat{M}_{ns}^+ - \hat{M}_{ns}^-) w$$

# Hellinger-Reissner Principle: Departure Weak Form



## Hellinger-Reissner Functional for Kirchhoff Plate

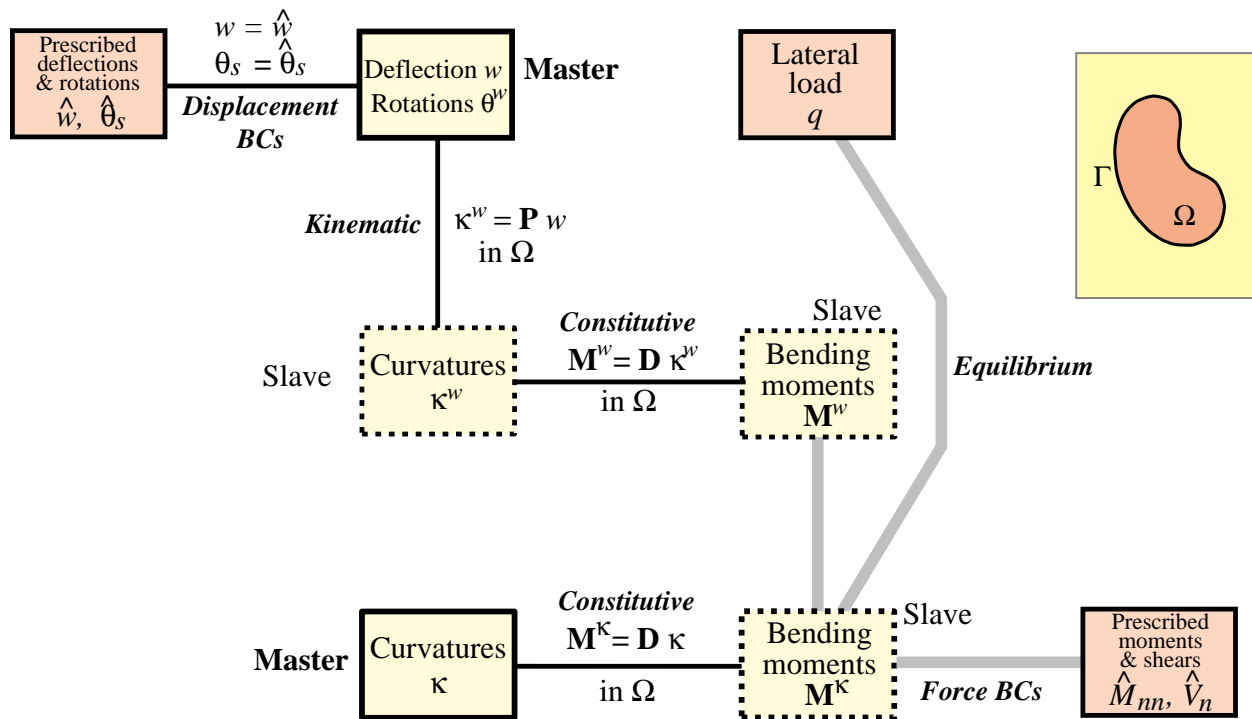
$$\Pi_{\text{HR}}[w, \mathbf{M}] = U_{\text{HR}}[w, \mathbf{M}] - W_{\text{HR}}[w, \mathbf{M}]$$

$$U_{\text{HR}}[w, \mathbf{M}] = \int_{\Omega} (\mathbf{M}^T \boldsymbol{\kappa}^w - \frac{1}{2} \mathbf{M}^T \mathbf{D}^{-1} \mathbf{M}) d\Omega = \int_{\Omega} (\mathbf{M}^T \boldsymbol{\kappa}^w - \mathcal{U}^*) d\Omega$$

$$\mathcal{U}^* = \frac{1}{2} \mathbf{M}^T \mathbf{D}^{-1} \mathbf{M} : \text{complementary energy density (per unit of plate area) in terms of moments}$$

**external work term similar to TPE**

# Fraeijs de Veubeke Curvature-Displacement Principle: Departure Weak Form



## **Fraeijs de Veubeke Curvature-Displacement Functional**

$$\Pi_{\text{FdV}}[w, \boldsymbol{\kappa}] = U_{\text{FdV}}[w, \boldsymbol{\kappa}] - W_{\text{FdV}}[w, \boldsymbol{\kappa}]$$

$$\Pi_{\text{FdV}}[w, \boldsymbol{\kappa}] = \int_{\Omega} (\boldsymbol{\kappa}^T \mathbf{M}^w - \frac{1}{2} \boldsymbol{\kappa}^T \mathbf{D} \boldsymbol{\kappa}) d\Omega$$

**external work term similar to TPE**

## Variational Index of Master Fields

Functional	transverse displacements	bending moments	plate curvatures
<b>TPE</b>	<b>2</b>		
<b>HR</b>	<b>2</b>	<b>0</b>	
<b>FdV</b>	<b>2</b>		<b>0</b>

Index  $m_w = 2$  on transverse displacements  $w$  is a bummer

Main difficulty: classical Ritz requires  $C^1$  interelement continuity, not easy to achieve with polynomials

Much research since early 1960s has been devoted to alleviate or eliminate this tough requirement