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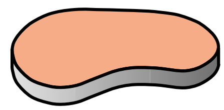
Kirchhoff Plates: Field Equations

Plate Structures

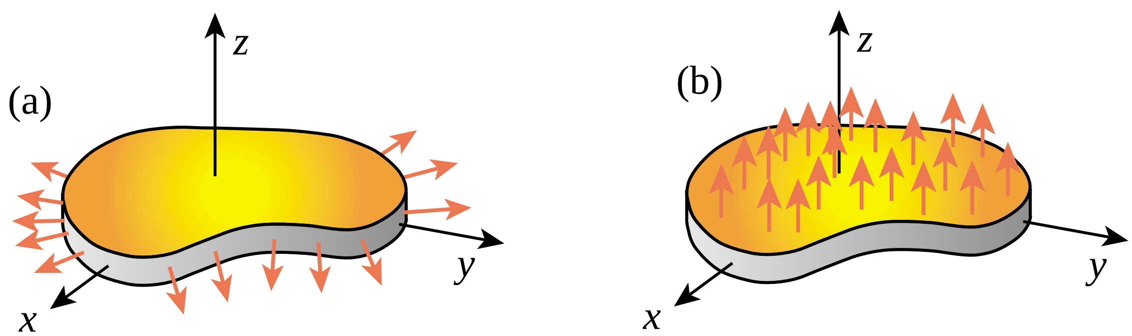
A *plate* is a three dimensional body characterized by



***Thinness*: one of the plate dimensions, the *thickness*, is much smaller than the other two**

***Flatness*: the *midsurface* of the plate is a plane**

Plate: Membrane vs Bending



Reduction to Two Dimensional Problem

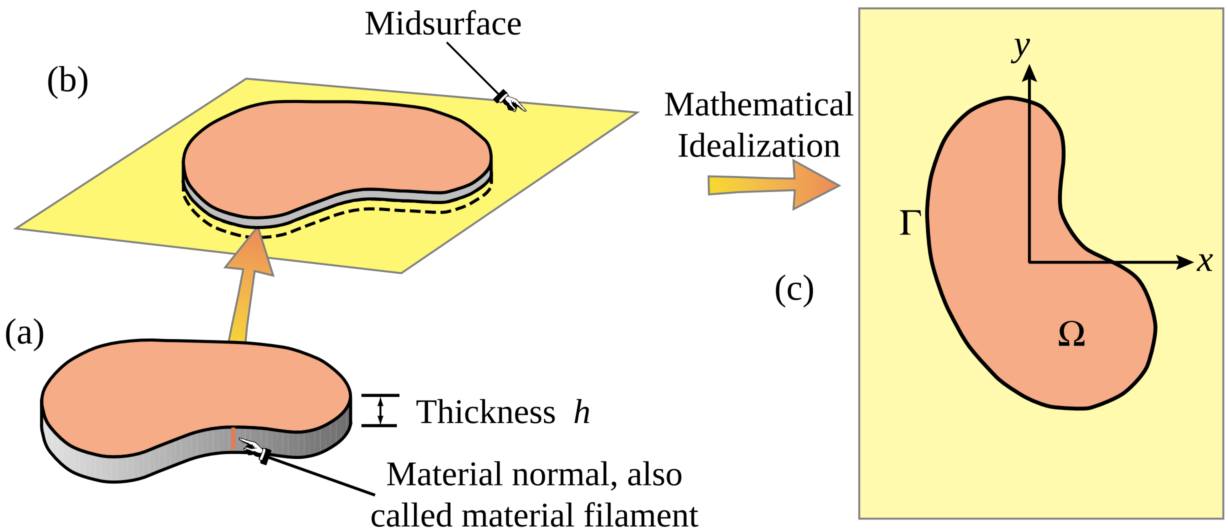


Plate Models

Bent membrane	geometrically nonlinear	global
von Karman	geometrically nonlinear	global
**** Kirchhoff***	geometrically linear	global
**** Reissner-Mindlin***	geometrically linear	global
High Order Composite	geometrically linear	local
Exact: 3D elasticity	geometrically linear	local

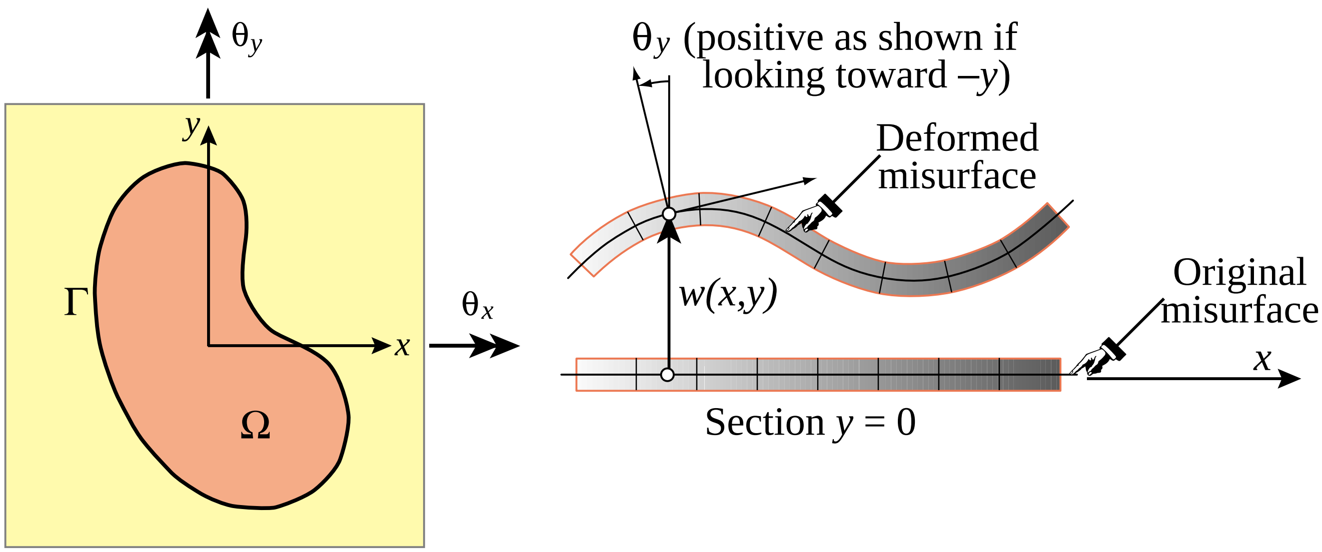
*** treated in this course**

The Kirchhoff Plate Model

Behavioral assumptions:

- **thin plate but** $w << h$
- **uniform thickness or varies slowly**
- **symmetric fabrication about midplane**
- **transverse loads distributed over areas of char dimension >** h
- **support conditions respect inextensional bending**

Main Kinematic Assumption for Kirchhoff Plate



"Material normals remain straight after deformation and normal to the deformed misurface"

Kinematic Relations

Deflection of plate midsurface along z

$$w = w(x,y)$$

Rotations of material normal about x, y

$$\theta_x = \frac{\partial w}{\partial y}, \qquad \theta_y = -\frac{\partial w}{\partial x}$$

Displacement of a material particle $P(x,y,z)$

$$u_x = -z\frac{\partial w}{\partial x} = z\theta_y, \quad u_y = -z\frac{\partial w}{\partial y} = -z\theta_x, \quad u_z = w$$

Kinematic Relations (cont'd)

Strain-displacement equations

$$
\begin{aligned}
e_{xx} &= \frac{\partial u_x}{\partial x} = -z\frac{\partial^2 w}{\partial x^2} = -z\,\kappa_{xx},\\
e_{yy} &= \frac{\partial u_y}{\partial y} = -z\frac{\partial^2 w}{\partial y^2} = -z\,\kappa_{yy},\\
e_{zz} &= \frac{\partial u_z}{\partial z} = -z\frac{\partial^2 w}{\partial z^2} = 0,\\
2e_{xy} &= \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = -2z\frac{\partial^2 w}{\partial x\partial y} = -2z\,\kappa_{xy},\\
2e_{xz} &= \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} = -\frac{\partial w}{\partial x} + \frac{\partial w}{\partial x} = 0,\\
2e_{yz} &= \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} = -\frac{\partial w}{\partial y} + \frac{\partial w}{\partial y} = 0
\end{aligned}
$$

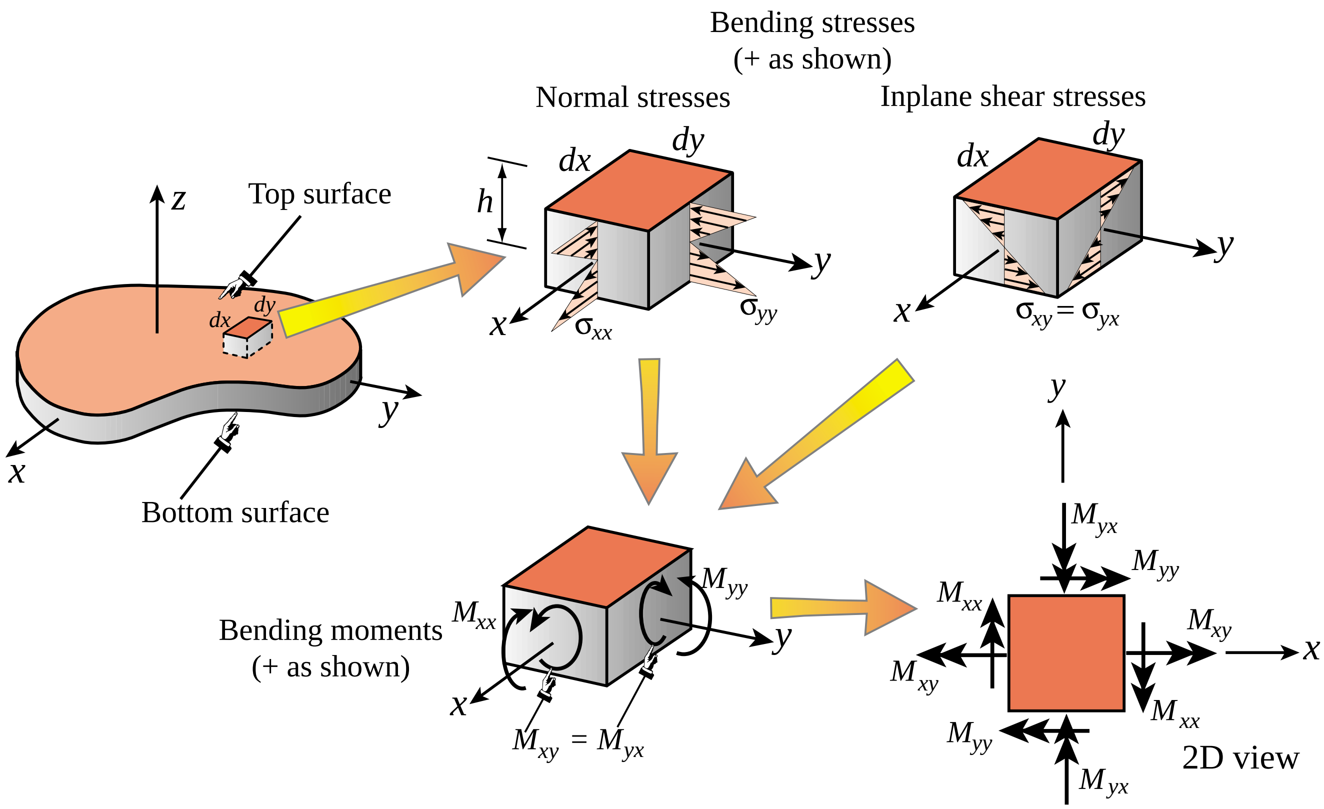
in which the κ's are the plate midsurface curvatures

$$
\kappa_{xy} = \frac{\partial^2 w}{\partial x^2},\quad \kappa_{yy} = \frac{\partial^2 w}{\partial y^2},\quad \kappa_{xy} = \frac{\partial^2 w}{\partial x\partial y}
$$

Advanced FEM

Bending Stresses and Moments

Showing Positive Sign Conventions



Moment-Curvature Relations

Wall fabrication assumptions:

- **Plate is homogeneous**
- **Each plate lamina** z **= constant is in plane stress**
- **Material obeys Hooke's law in plane stress:**

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix} = -z \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ 2\kappa_{xy} \end{bmatrix}$$

Moment-Curvature Relations (cont'd)

Bending moments are obtained by integrating the in-plane wall stresses over the thickness

$$M_{xx}\,dy = \int_{-h/2}^{h/2} -\sigma_{xx}\,z\,dy\,dz \quad \Rightarrow \quad M_{xx} = -\int_{-h/2}^{h/2} \sigma_{xx}\,z\;dz,$$
$$M_{yy}\,dx = \int_{-h/2}^{h/2} -\sigma_{yy}\,z\,dx\,dz \quad \Rightarrow \quad M_{yy} = -\int_{-h/2}^{h/2} \sigma_{yy}\,z\,dz,$$
$$M_{xy}\,dy = \int_{-h/2}^{h/2} -\sigma_{xy}\,z\,dy\,dz \quad \Rightarrow \quad M_{xy} = -\int_{-h/2}^{h/2} \sigma_{xy}\,z\,dz,$$
$$M_{yx}\,dx = \int_{-h/2}^{h/2} -\sigma_{yx}\,z\,dx\,dz \quad \Rightarrow \quad M_{yx} = -\int_{-h/2}^{h/2} \sigma_{yx}\,z\,dz.$$

Since $M_{xy} = M_{yx}$ (from rotational equilibrium) only 3 independent components need to be calculated

Moment-Curvature Relations (cont'd)

Carrying out the integration over the thickness:

$$\begin{bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{bmatrix} = \frac{h^3}{12} \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ 2\kappa_{xy} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{12} & D_{22} & D_{23} \\ D_{13} & D_{23} & D_{33} \end{bmatrix} \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ 2\kappa_{xy} \end{bmatrix}$$

For isotropic material of modulus E and Poisson's ratio ν

$$\begin{bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{bmatrix} = D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1+\nu) \end{bmatrix} \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ 2\kappa_{xy} \end{bmatrix}$$

where

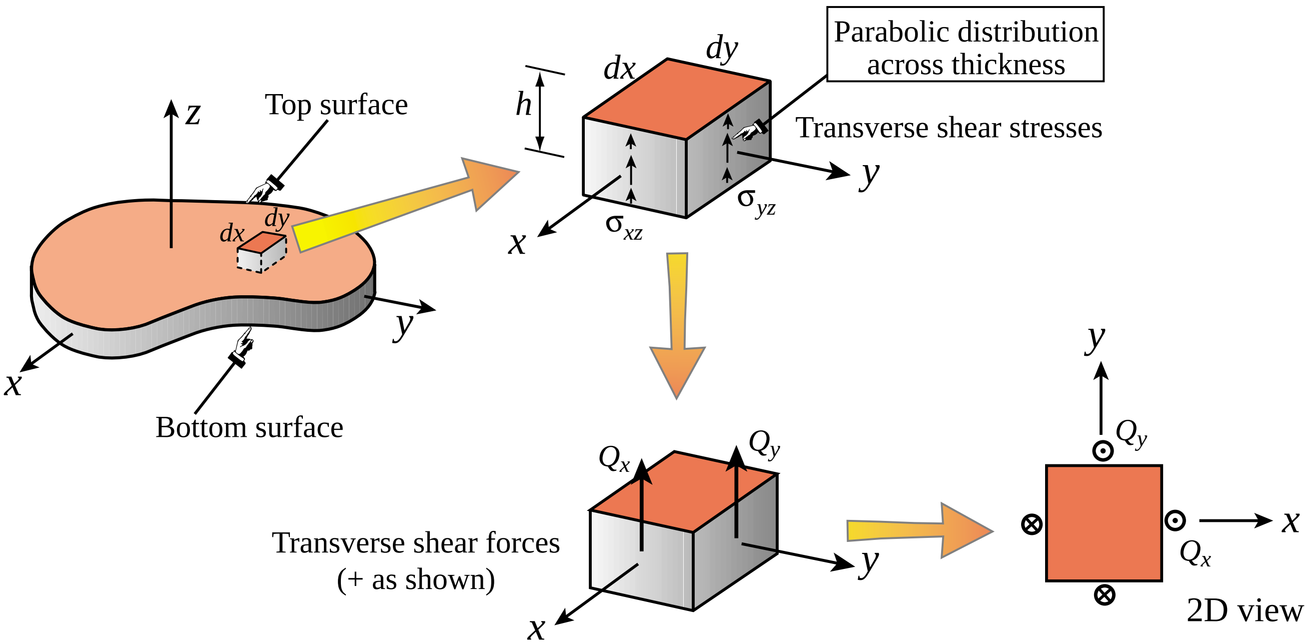
$$D = \frac{E h^3}{12(1-\nu^2)}$$

is the plate rigidity

Max/min stress computation given the moments:

$$\sigma_{xx}^{max,min} = \pm\frac{6M_{xx}}{h^2}, \quad \sigma_{yy}^{max,min} = \pm\frac{6M_{yy}}{h^2}, \quad \sigma_{xy}^{max,min} = \pm\frac{6M_{xy}}{h^2} = \sigma_{yx}^{max,min}$$

Transverse Shear Stresses and Forces



Transverse Shear Stresses and Forces (cont'd)

Wall distribution in a homogeneous plate

$$\sigma_{xz} = \sigma_{xz}^{max}\left(1 - \frac{4z^2}{h^2}\right), \quad \sigma_{yz} = \sigma_{yz}^{max}\left(1 - \frac{4z^2}{h^2}\right).$$

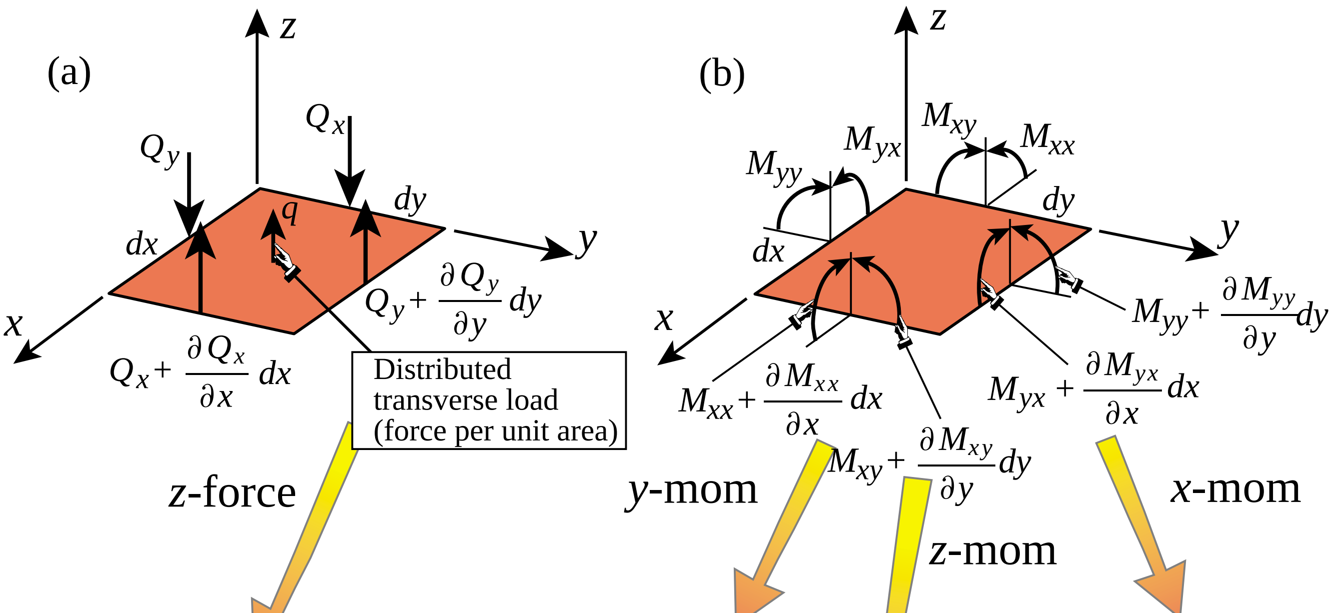
Integrating over the thickness provides the transverse shear forces

$$Q_x = \int_{-h/2}^{h/2} \sigma_{xz}\, dz = \tfrac{2}{3}\sigma_{xz}^{max}\, h, \qquad Q_y = \int_{-h/2}^{h/2} \sigma_{yz}\, dz = \tfrac{2}{3}\sigma_{yz}^{max}\, h,$$

If transverse shear forces given, maximum shear stresses are

$$\sigma_{xz}^{max} = \tfrac{3}{2}\frac{Q_x}{h}, \qquad \sigma_{yz}^{max} = \tfrac{3}{2}\frac{Q_y}{h}.$$

Internal Equilibrium Equations



$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = -q$$

$$\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} = -Q_x \qquad \frac{\partial M_{yx}}{\partial x} + \frac{\partial M_{yy}}{\partial y} = -Q_y$$

$$M_{xy} = M_{yx}$$

Internal Equilibrium Equations (cont'd)

Repeating for convenience:

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = -q \qquad \frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} = -Q_x \qquad \frac{\partial M_{yx}}{\partial x} + \frac{\partial M_{yy}}{\partial y} = -Q_y$$

$$M_{xy} = M_{yx}$$

Eliminating the shear forces and one of the twist moments gives the moment equilibrium equation

$$\frac{\partial^2 M_{xx}}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} = q$$

Matrix and Indicial Form of Field Equations

Field eqn	***Matrix form***	***Indicial form***	***Equation name for plate problem***
KE	$\boldsymbol{\kappa} = \mathbf{P}\, w$	$\kappa_{\alpha\beta} = w_{,\alpha\beta}$	**Kinematic equation**
CE	$\mathbf{M} = \mathbf{D}\, \boldsymbol{\kappa}$	$M_{\alpha\beta} = D_{\alpha\beta\gamma\delta}\, \kappa_{\gamma\delta}$	**Moment-curvature equation**
BE	$\mathbf{P}^T \mathbf{M} = q$	$M_{\alpha\beta,\alpha\beta} = q$	**Internal equilibrium equation**

Here $\mathbf{P}^T = [\, \partial^2/\partial x^2 \quad \partial^2/\partial y^2 \quad 2\,\partial^2/\partial x \partial y \,] = [\, \partial^2/\partial x_1 \partial x_1 \quad \partial^2/\partial x_2 \partial x_2 \quad 2\,\partial^2/\partial x_1 \partial x_2 \,]$,
$\mathbf{M}^T = [\, M_{xx} \quad M_{yy} \quad M_{xy} \,] = [\, M_{11} \quad M_{22} \quad M_{12} \,]$,
$\boldsymbol{\kappa}^T = [\, \kappa_{xx} \quad \kappa_{yy} \quad 2\kappa_{xy} \,] = [\, \kappa_{11} \quad \kappa_{22} \quad 2\kappa_{12} \,]$.

Greek indices, such as α, run over 1,2 only.

Strong Form Diagram of Field Equations for Kirchhoff Plate Model