23

The Patch Test

TABLE OF CONTENTS

		Page
§23.1.	**Introduction**	23–3
§23.2.	**Motivation**	23–3
	§23.2.1. The Black Cat	23–3
	§23.2.2. Variational Crimes	23–3
	§23.2.3. Has the Jury Reached a Veredict?	23–5
§23.3.	**Terminology**	23–5
	§23.3.1. Patches	23–6
	§23.3.2. Generalizations	23–6
§23.4.	**The Physical Patch Test**	23–7
	§23.4.1. The Displacement Test Space	23–8
	§23.4.2. The Displacement Patch Test	23–8
	§23.4.3. DPT Q&A	23–9
	§23.4.4. The Stress Test Space	23–11
	§23.4.5. The Force Patch Test	23–12
	§23.4.6. FPT Q&A	23–12
§23.5.	**One Element Tests: IET and SET**	23–13
§23.6.	***Historical Background**	23–14
§23.7.	**The Veubeke Equilibrium Triangle**	23–15
	§23.7.1. Kinematic Relations	23–16
	§23.7.2. Stiffness Matrix	23–17
	§23.7.3. Implementation	23–17
	§23.7.4. Patch Test Configuration	23–17
§23.	**Exercises**	23–19

§23.1. Introduction

Chapter 20 tangentially mentioned High Performance (HP) elements. These were defined in [72] as "simple elements that deliver engineering accuracy with arbitrary coarse meshes." A major goal of the post-1980 element formulation reviewed in Chapter 20 is to produce such elements.

A key tool used in these advanced formulations is the *patch test* and a specialization thereof called the *Individual Element Test* or IET. These tools are outlined in this Chapter. Some of this material is taken verbatim from [78].

§23.2. Motivation

The historical origins of the patch test are detailed at the end of this Chapter. Following is a summary of the motivation behind it.

§23.2.1. The Black Cat

By the end of FEM Generation 1 (1952-1962), closed by Melosh's thesis [177] it was generally recognized that the *displacement-assumed* form could be viewed as variation of the century-old Rayleigh-Ritz direct variational method. The new idea introduced by FEM was that trial functions had local support extending over element patches. To guarantee convergence as the mesh was refined, those functions ought to satisfy the conditions discussed in Chapter 19 of [86]: continuity, completeness and stability[1].

Understanding of those conditions was reached only gradually and published in fits and starts. Continuity requirements were quickly found because for displacement elements they are easily visualized: noncompliance means materials gaps or interpenetration. They were first unequivocally stated in [177,178].

Completeness was understood in stages: first rigid body modes and then constant strain (or curvature) states. Stability in terms of rank sufficiency was mentioned occasionally but often dismissed as comparatively unimportant, until the early 1980.

The rudimentary state of the mathematical theory did not impede progress. All the contrary. The years of Generation 2: 1963–1972 marked a period of exuberant productivity and advances in FEM for the Direct Stiffness Method: solid elements, new plates and shell models, as well as the advent of isoparametric elements with numerical quadrature. It was also a period of mystery: FEM was a black cat in a dark cellar at midnight. The patch test was the first device to throw some light on that cellar.

§23.2.2. Variational Crimes

The equivalence between FEM and classical Ritz as understood by 1970 is well described by Strang in a paper [226] that introduced the term "variational crimes". Here is an excerpt from the Introduction of that paper:

> "The finite element method is nearly a special case of the Rayleigh-Ritz technique. Both methods begin with a set of trial functions $\phi_1(x), \ldots, \phi_N(x)$; both work with the space of linear combinations $v^h = \sum q_i \phi_i$; and both chose the particular combination (we will call it u^h) which mininizes

[1] Stability includes rank sufficiency and positive Jacobian determinant

> a given quadratic functional. In many applications this functional corresponds to potential energy. The convergence of the Ritz approximation is governed by a single fundamental theorem: *if the trial functions ϕ_j are admissible in the original variational principle, then u^h is automatically the combination which is closest* (in the sense of strain energy), *to the true solution u*. Therefore convergence is proved by establishing that as $N \to \infty$, the trial functions fill out the space of all admissible functions. The question of *completeness* dominates the classical Ritz theory.
>
> For finite elements a new question appears. Suppose the basic rule of the Ritz method is broken, and the trial functions are not quite admissible in the true variational problem: it is still possible to prove that u^h converges to u, and to estimate the rate of convergence? This question is inescapable if we want to analyze the method as *it is actually used* [my italics]. We regard the convenience and effectiveness of the finite element technique as conclusively established; it has brought a revolution in the calculations of structural mechanics, and other applications are rapidly developing [this was written in the early 1970s]. Our goal is to examine the modification of the Ritz procedure which have been made (quite properly, in our view) in order to achieve an efficient finite element system. The hypotheses of the classical Ritz theory are violated because of computational necessity, and we want to understand the consequences."

Later in this paper Strang introduces the term *variational crimes* to call attention to the fact that many existing elements known at that time did not comply with the laws of the classical Ritz method. In decreasing order of seriousness the violations are:

The Sins of Delinquent Elements

(1) Lack of completeness
(2) Lack of invariance: element response depends on observer frame
(3) Rank deficiency
(4) Nonconformity: violation of interelement continuity required by variational index
(5) Inexact treatment of curved boundaries and essential BCs
(6) Inexact, but rank sufficient, numerical integration

(23.1)

The term *delinquent element* was coined by Bruce Irons to denote one guilty of one or more of these sins. He also used *naughty element*.[2]

In Chapter 4 ("Variational Crimes") of the Strang and Fix monograph [227] attention is focused on the last three items in (23.1). The sensational title served the intended purpose: it attracted interdisciplinary attention from both mathematicians and the FEM community. But thirty years later we take the label as too sweeping. Some "crimes", notably (4) and (6), may be actually virtues, that is, beneficial to performance when honestly acknowledged.[3] That chapter actually says little about items (1) through (3), which are the truly evil ones.

Remark 23.1. What does Strang mean by "classical Ritz theory"? A bit of history is appropriate here. Lord Rayleigh in his *Theory of Sound* masterpiece calculated approximate vibration frequencies by assuming the form of a vibration mode as a polynomial satisfying kinematic BCs and endowed with a free coefficient, say c. Inserting this into the functional of acoustics produces an algebraic function $\Pi(c)$ from which c is determined

[2] An old Middle English word: "So shines a good deed ina naughty world" (William Shakespeare, *Merchant of Venice*).

[3] "To live outside the law, you must be honest." (Bob Dylan, *Absolutely Sweet Marie*).

- Multielement Tests
  - Homogeneous (Irons et al)
    - Displacement Patch Test
    - Force (aka Stress) Patch Test
    - Mixed Patch Tests
  - Heterogeneous
- One Element Tests
  - Individual Element Test (IET: Bergan-Hanssen)
  - Single Element Test (SET: Taylor et al)

covered in this Chapter

Can be done numerically or symbolically

FIGURE 23.1. Variants of the patch test.

by minimization: $\partial\Pi/\partial c = 0$. Inserting into the Rayleigh quotient gives the squared frequency. This is called a *direct variational method*.

The Swiss mathematician Ritz extended in 1908 this idea to multiple coefficients by taking a linear combination of suitable admissible functions. These are now called *trial functions*. In the original (= classical) Ritz method, trial functions are infinitely smooth (typically polynomials) over the computational domain and satisfy the essential BCs exactly. The necessary integrals were done analytically over the computational domain: no numerical quadrature was used.

§23.2.3. Has the Jury Reached a Veredict?

In the present state of knowledge only the first two items in the alleged–crimes list (23.1) are truly serious. Certainly (1) is a capital offense. Not only it precludes convergence, but may cause convergence to the solution of the wrong problem, misleading unsuspecting users. Lack of invariance (2) can also lead to serious mistakes.

The effect of rank deficiency (3) is very problem dependent (dynamics versus statics, boundary conditions, etc.). While acceptable in special codes used by experts[4] it should be avoided in elements intended for general use. Veredict: a rank-deficient element should be kept on probation.

The effect of nonconformity (4) was unpredictable before the development of the patch test. By now the consequences are well understood. In the hands of experts it has become a useful tool for fabricating high performance elements.

Inexact but rank sufficient integration (5) is often beneficial so its presence in that list is highly questionable. Finally the effect of (6) is generally minor but may require some attention when boundary layers may occur, as in the Reissner-Mindlin plate model.

[4] For example, hydrocodes used for simulating a rapid transient response.

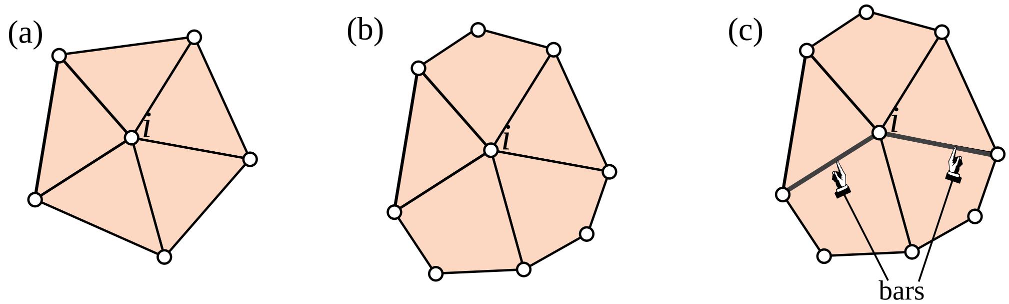

FIGURE 23.2. An element patch is the set of all elements attached to a patch node, herein labeled i. (a) illustrates a patch of triangles; (b) a mixture of triangles and quadrilaterals; (c) a mixture of triangles, quadrilaterals, and bars.

§23.3. Terminology

The original patch test form, historically outlined in §23.7, has evolved since into a great number of variants, summarized in Figure 23.1. Of these only the multielement homogeneous patch test in the displacement and force forms will be covered in detail here. Other versions will be described briefly with appropriate references if available. Before explaining the test procedure for the multielement versions, it is appropriate to introduce the concept of patch.

§23.3.1. Patches

We recall here the concept of *element patch*, or simply *patch*, introduced in Chapter 19 of IFEM. This is the set of all elements attached to a given node. This node is called the *patch node*. The definition is illustrated in Figure 23.2, which depicts three different kind of patches attached to patch node i in a plane stress problem. The patch of Figure 23.2(a) contains only one type of element: 3-node linear triangles. The patch of Figure 23.2(b) mixes two plane stress element types: 3-node linear triangles and 4-node bilinear quadrilaterals. The patch of Figure 23.2(c) combines three element types: 3-node linear triangles, 4-node bilinear quadrilaterals, and 2-node bars. Patches may also include a mixture of widely different element types, as illustrated for three dimensional space in Figure 23.2.

If all elements of the patch are of assumed-displacement type and hence defined via shape functions, we can define a *patch trial function* (PTF) the union of shape functions activated by setting a degree of freedom (DOF) at the patch node to unity, while all other freedoms are zero. If the elements are conforming, a patch trial function “propagates” only over the patch, and is zero beyond it.[5]

If one or more of the patch elements are not of displacement type, shape functions generally do not exist. Still the patch will respond in some way (strains, stresses, etc) to the activation of a patch-node DOF. That response, however, might not be so easily visualized.

§23.3.2. Generalizations

The definition of patch as “all elements attached to a node” is a matter of convenience. It simplifies some theoretical interpretations as well as the description of the physical test. But the concept can

[5] If the elements are nonconforming, the PTF is by convention truncated at the patch boundary.

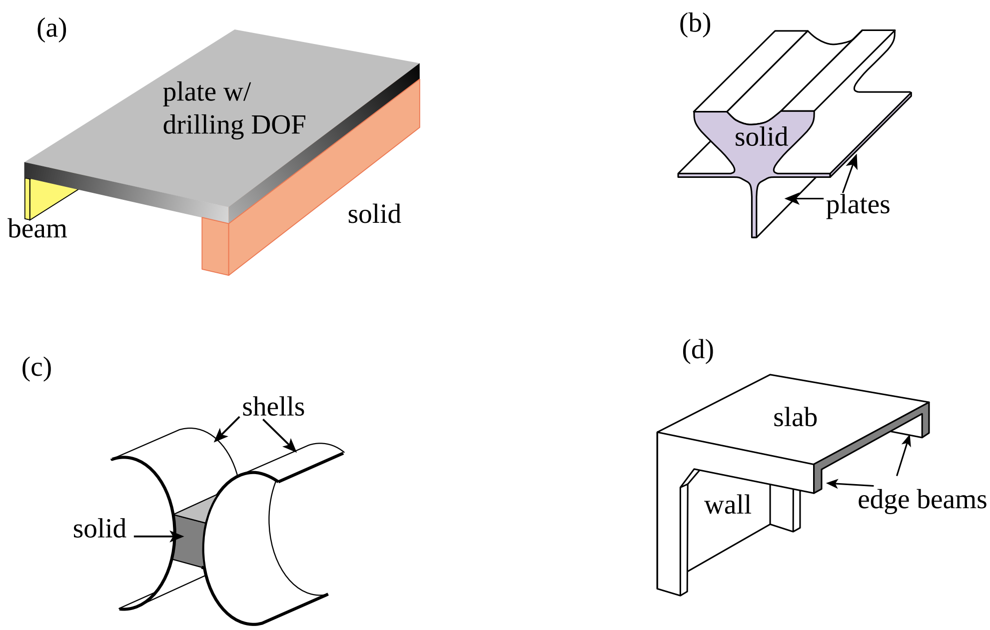

FIGURE 23.3. Heterogeneous patch test test the mixability of elements of different types, as illustrated in (a) through (d). These combinations test the *mixability* of elements.

be generalized to more general assemblies. Recall from Chapter 11 of IFEM that a superelement was defined as any grouping of elements that possesses no kinematic deficiencies other than rigid body modes. Not all superelements, however, are suitable for the test. The following definition characterizes a mild generalization:

> *Any superelement consisting of two or more elements can be used as a patch if it can be viewed as the limit of a mesh refinement process*

This "limit mesh condition" characterizes *homogeneous patches*, that is, made up of the same element and having the same material and fabrication properties. In particular, no holes or cracks are permitted. On the other hand, the number of interior nodes can be arbitrary.

The restriction to at least two elements is to have an *interface* on which to test interelement nonconformity effects. However, the test can be reduced to *one element* under some assumptions discussed in later Chapters. Those are called *individual element tests* and are briefly covered in §23.6.

Heterogeneous patches are those including different element types to test element mixability, as in Figure 23.3. Those are not considered in this Chapter.

§23.4. The Physical Patch Test

Like FEM, the patch test has both a physical and a mathematical interpretation. Both are practically useful. The physical form was originally put forward by Irons in the Appendix to a 1965 paper [18]. It was described in more detail in later publications [144,147]. More historical details are provided at the end of the Chapter.

The essential idea behind the physical patch test is: *a good element must solve simple problems exactly* whether individually, or as component of arbitrary patches. The test has two *dual* forms:

Displacement Patch Test or DPT: applies boundary displacements to patch and verifies that the patch response reproduces exactly rigid body modes and constant strain states.

Force Patch Test or FPT: applies boundary forces to patch and verifies that the patch response reproduces exactly constant stress states.

There are also *mixed patch tests* that incorporate both force and displacement BCs. These will not be treated here to keep things simple.

§23.4.1. The Displacement Test Space

What is the "simple problem" alluded to above? For the displacement version, Irons reasoned as follows. In the limit of a mesh refinement process, the state of strain inside each displacement-assumed element is sensibly constant. (For bending models, strains are replaced by curvatures.) The associated displacement states are therefore linear in elements modeling bars, plane elasticity or 3D elasticity, and quadratic in beams, plates and shells. These limit displacement states are of two types: *rigid body modes* or r-modes, and *constant strain modes* or c-modes (constant curvatures in bending models). For brevity this set will be called the *displacement test space*.

In rectangular Cartesian coordinates the limit displacement fields are represented by *polynomials* in the coordinates. If the variational index is m and the number of space dimensions n, those polynomials have degree $\leq m$ in n variables. That set of polynomials is called $\mathcal{P}_n^m$. For example if $m = 1$ in a plane stress problem ($n = 2$), the displacement test space contains

$$u_x \in \mathcal{P}_2^1,\; u_y \in \mathcal{P}_2^1, \quad \text{or} \quad \mathbf{u}(x,y) = \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} c_{x0} + c_{x1}x + c_{x2}y \\ c_{x0} + c_{x1}x + c_{x2}y \end{bmatrix} \tag{23.2}$$

This test space has obviously dimension six: $u_x = \{1, x, y\}$, $u_y = \{0, 0, 0\}$ $u_y = \{1, x, y\}$, $u_x = \{0, 0, 0\}$. Therefore it is sufficient to test six cases since for linear FEM models any combination thereof will also pass the test. It is customary, however, to test for three rigid-body and three constant-strain modes separately, using the so-called rc basis:

$$r \text{ modes: } \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} -y \\ x \end{bmatrix}, \quad c \text{ modes: } \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ y \end{bmatrix} \text{ and } \begin{bmatrix} y \\ x \end{bmatrix}, \tag{23.3}$$

This polynomial space is exactly that which appears in the definition of completeness for individual elements given in Chapter 19 of IFEM. Hence the patch test, administered in this form, can be viewed as a *completeness test on arbitrary patches*.

§23.4.2. The Displacement Patch Test

We are now in a position to apply the so called *displacement patch test* or DPT.[6] This will be illustrated by the 2D patch shown in Figures 23.3 and 23.4, which pertains to a plane stress problem.

Figure 23.4(a) shows the application of a rigid-body-mode (r-mode) test displacement field, $u_x = 1$ and $u_y = 0$. The procedure is as follows. Pick a patch. Evaluate the displacement field at the external nodes of the patch, and apply as prescribed displacements. Set forces at interior DOFs to

[6] Also known as *kinematic patch test* or *Dirichlet patch test.* The latter name should be palatable to mathematicians.

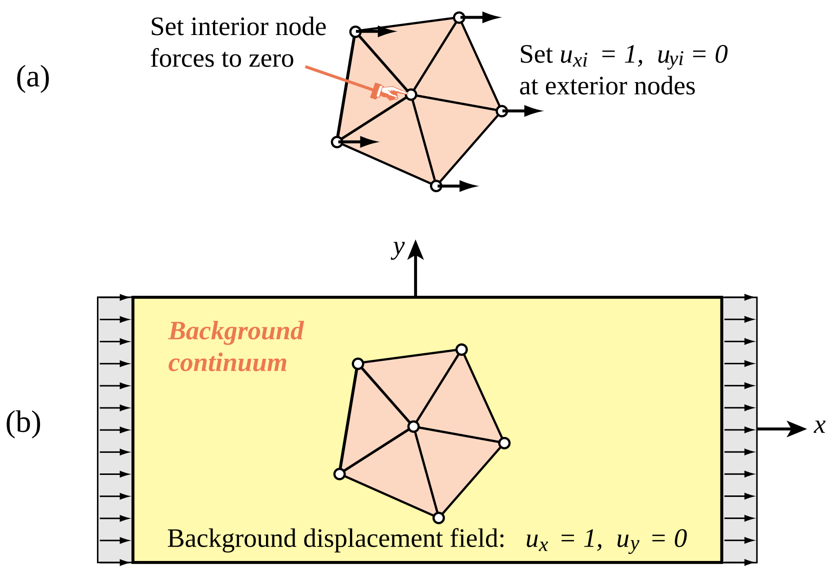

FIGURE 23.4. A two-dimensional rigid-body-mode displacement patch test (DPT). Prescribed displacement field is $u_x = 1$, $u_y = 0$, which represents a rigid x-motion; (a) shows a patch with external nodes given prescribed forces; (b) interpretation as replacing part of a "background continuum" with the patch.

zero. Solve for the displacement components of the interior nodes (two in the example). These should agree with the value of the displacement field at that node. Recover the strain field over the elements: all components should vanish identically at any point.

Figure 23.5(a) illustrates the application of a constant-strain-mode (c-mode) test displacement field $u_x = x$ and $u_y = 0$. This gives unit x strain $e_{xx} = \partial u_x/\partial x = 1$, others zero. Take the same patch. Evaluate the displacement field at the external nodes of the patch, and apply as prescribed displacements. Set forces at interior DOFs to zero. Solve for the displacement components of the interior nodes (two in the example). These should agree with the value of the displacement field at that node. Recover the strain field over the elements: all components should vanish except $e_{xx} = 1$ at any point.

If all test displacement/strain states are reproduced correctly, the DPT is passed for the selected patch, and failed otherwise. A shortcut possible in symbolic computation is to insert directly the polynomial basis (23.2), keeping the coefficients as variables.

§23.4.3. DPT Q&A

The foregoing subsection gives the recipe for running DPTs. Now for the questions.[7]

To answer most of the questions a thought experiment helps: think of the outside of the patch as a *homogeneous continuum* subject to the given displacement field. You may view the continuum as infinite if that helps. This is called the *background continuum*. This continuum must have exactly

[7] Some are answered here for the first time. Undoubtedly Irons knew the answers but never bothered to state them, possibly viewing them as too obvious.

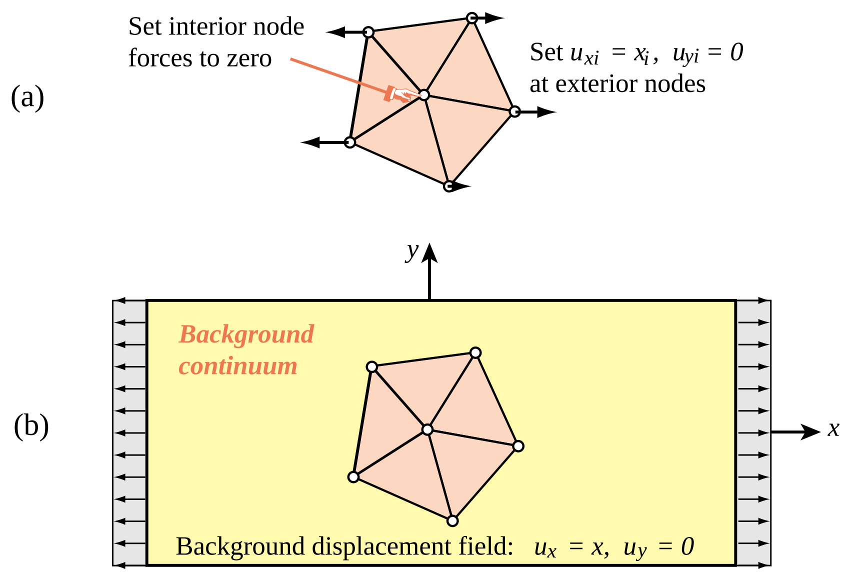

FIGURE 23.5. A two-dimensional constant-strain-mode displacement patch test (DPT). Prescribed displacement field is $u_x = x$, $u_y = 0$, which gives $e_{xx} = 1$, others zero; (a) shows a patch with external nodes given prescribed forces; (b) interpretation as replacing part of a "background continuum" with the patch.

the same material and fabrication properties as the patch. (As shown below, all elements of the patch must have the same properties).

Q1. How arbitrary can the patch be? Cut out a portion of the background continuum as illustrated in Figures 23.4(b) and 23.5(b). Replace by the patch. *Nothing should happen*. If the patch disturbs the given field, it is not admissible. For specific constraints see Q2.

Q2. Can one have elements of different material or fabrication properties in the patch? No. Think of the background continuum under constant strain. Replacing a portion by that kind of patch by a non-homogeneous medium will produce non-uniform strain. (For rigid body mode tests, however, those differences will not have any effect.)

Q3. Why are the displacements applied to exterior nodes only? Because those are the one in contact with the background continuum, which provides the displacement (Dirichlet) boundary conditions for the patch problem. The interior nodes are solved from the FEM equilibrium conditions.

Q4. What happens if all patch nodes and DOF are given displacements? Nothing very useful. The patch will behave as if the elements were disconnected, so there is no difference from testing single elements. The results only would check if completeness is verified in an individual element. This is useful as a strain computation check and nothing more.

Q5. Can elements of different geometry be included in the test? Yes. For example, triangles and quadrilaters may be mixed in a 2D test as long as material and fabrication properties are the same. Thus the patch of Figure 23.2(b) is acceptable. But that of Figure 23.2(c) is not, because the two bars are a different structural type, which would alter constant strain states.

Q6. Can patch tests be done by hand? Only for some very simple one-dimensional configurations. Otherwise computer use is mandatory. A strong case for the use of computers anyway is that patch

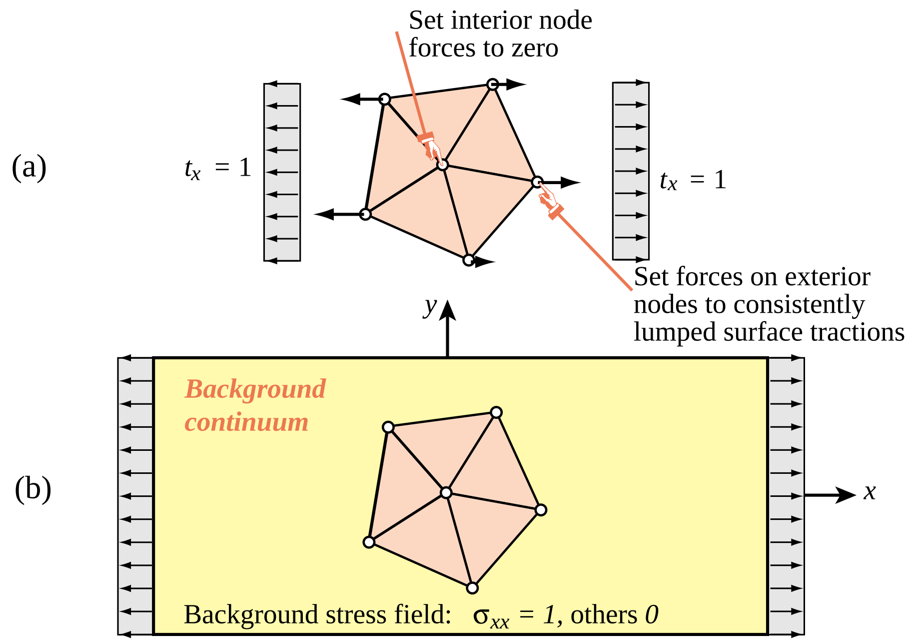

FIGURE 23.6. A two-dimensional force patch test: $\sigma_{xx} = 1$, (a) shows a patch with external patch nodes given prescribed forces; (b) interpretation as replacing part of a stressed "background continuum" with the patch.

tests are often used to verify actual code.

Q7. Are computer patch tests necessarily numeric? Not necessarily. Numerics is of course mandatory is one is verifying element code written in Fortran or C. But the use of computer algebra systems is growing as a tool to *design* elements. In element design it is very convenient to be able to leave geometric, material and fabrication properties as variables, In addition, one can parametrize patch geometries so that a symbolic test is equivalent to an infinite number of numeric tests.

Q8. The applied strain $e_{xx} = 1 = 100\%$ is huge. Finite elements with linearized kinematics are valid only for very small strains. Why are the DPT results correct? This is posed as a Homework exercise.

§23.4.4. The Stress Test Space

For the force version of the physical patch test one needs to reword part of the discussion. Strains become stresses, and boundary displacements become surface tractions.

To create "simple problems" again one thinks of a mesh refinement process. In the limit the state of stress inside each element is sensibly constant. The test space is that of *constant stress modes*. (For elements that model bending, stresses are replaced by moments.) For a two-dimensional plane stress problem, the test space is spanned by two axial stresses and a shear stress

$$\sigma_{xx},\ \sigma_{yy},\ \sigma_{xy} \tag{23.4}$$

since the stress in any other direction is a linear combination of these. Alternatively one could run three axial stresses in three independent directions:

$$\sigma_1,\ \sigma_2,\ \sigma_3 \tag{23.5}$$

for example at directions 0°, 90° and 45° from x. This avoids the shear patch test, which is more difficult to program.

In 1D, or 2D models integrated through the thickness, one applies stress resultants, such as membrane or axial forces and moments.

For the DPT we distinguished between rigid-body and constant strain modes. The distinction is not necessary for stress modes. This renders this test form simpler in some ways but more complicated in others, in that rigid-body motions must be precluded to avoid singularities.

§23.4.5. The Force Patch Test

The procedure for the force *force patch test* or FPT[8] will be illustrated by the 2D patch shown in Figure 23.6, which pertains to a plane stress problem.

Figure 23.6(a) shows the FPT administered for uniform stress $\sigma_{xx} = 1$, others zero. On the patch boundary of the patch apply a uniform surface traction $t_x = \sigma_x x$. Convert this to nodal forces using a consistent force lumping approach. Forces at interior DOFs should be zero. Apply a minimal number of displacement BC to eliminate rigid body motions (more about this below). Solve for the displacement components, and recover strains and stresses over the elements. The computed stresses should recover exactly the test state.

If all test states are correctly reproduced, the FPT is passed, and failed otherwise. Note that there is no such thing as "PT passed 90%". In this form the test is always pass-or-fail (ex

§23.4.6. FPT Q&A

Q1. How arbitrary can the patch be? The "background check" also applies. Nothing should happen to the continuum upon inserting the patch.

Q2. Can one have elements of different material or fabrication properties in the patch? There is a bit more leeway here than in the DPT. The flux of stress, or of stress resultants, should be preserved. This is easier under statically determinate conditions. For example, in the case of a plane-beam test under pure moment, a patch of two beams with different cross sections should work. There are more restrictions in 2D and 3D because of statically indeterminacy cross-effects, such as Poisson's ratio. In practice most tests of this type are done under the same restrictions as DPT.

Q3. Can singular-stiffness problems arise? Certainly. The test is supposed to be conducted on a free-free patch under equilibrium conditions, but the elements to be checked are likely to be of displacement type. Unsuppressed rigid-body modes (RBMs) will result in a singular patch stiffness, impeding the computation of node displacements. There are basically four ways around this:

1. Apply the *minimum* number of independent displacement BC to suppress the RBMs. It is important not to overconstrain the system, as that may cause force flux perturbations. The way to check that you are doing the right thing is to compute the reaction forces: *they must be*

[8] Also known as *static patch test* or *flux patch test.*

identically zero since the stress states are in self-equilibrium. The test now becomes of *mixed force-displacement type* since some nodal displacements must be prescribed.

2. Use an equation solver that automatically removes the RBMs. This can be tricky in floating-point work. Unfeasible if the solver comes from a standard library, as in commercial packages.
3. Project the patch stiffness matrix onto the space of deformation modes. This technique is tricky and difficult to use.
4. Use the free-free flexibility technique of [89]. Unlike the preceding one this is foolproof but since the technique is very recent, it has not been used for this purpose.

Q4. When should this test be applied? Ideally both DPT and FPT should be administered when one is checking element code. For example, DPT directly checks rigid body motions but FPT does not. For single-master displacement-assumed elements there is overlap in the strain and stress checks if the stresses are recovered from strains.

§23.5. One Element Tests: IET and SET

Because of practical difficulties incurred in testing all possible patches there have been efforts directed toward translating the original test into statements involving a single element. These will be collectively called *one-element tests*.

The first step along this path was taken by Strang [227], who using integration by parts recast the original test in terms of "jump" contour integrals over element interfaces. An updated account is given by Griffiths and Mitchell [116], who observe that Strang's test can be passed in three different ways.

JCS: Jump integrals cancel over common sides of adjacent elements. Examples: Fraeijs de Veubeke's 3-midside-node triangle [101], Morley's constant-moment plate element [184].

JOS: Jump integrals cancel over opposite element sides. Example: Wilson's incompatible plane rectangle [268].

JEC: Jump integrals cancel over the element contour. Examples are given in the aforementioned article by Griffiths and Mitchell [116].

Another consequential development, not so well publicized as Strang's, was undertaken by Bergan and coworkers at Trondheim over the decade 1974–1984. The so called *individual element test*, or IET, was proposed by Bergan and Hanssen [25]. The underlying goal was to establish a test that could be directly carried out on the stiffness equations of a single element — an obvious improvement over the multielement form. In addition the test was to be *constructive*, that is, used as a guide during element formulation, rather than as *a posteriori* check.

The IET has a simple physical motivation: to demand pairwise cancellation of tractions among adjacent elements that are subjected to a common uniform stress state. This is in fact the 'JCS' case of the Strang test noted above. Because of this inclusion, the IET is said to be a *strong* version of the patch test in the following sense: any element passing the IET also verifies the conventional multielement form of the patch test, but the converse is not necessarily true.

The IET goes beyond Strang's test, however, in that it provides *a priori* rules for constructing finite elements. These rules have formed the basis of the Free Formulation (FF) developed by Bergan and Nygård [27]. The test has also played a key role in the development of high performance finite elements undertaken by the writer.

In an important paper written in response to Stummel's counterexample, Taylor et al [240] defined multielement patch tests in more precise terms, introducing the so-called A, B and C versions. They also discussed a one-element test called the "single element test," herein abbreviated to SET. They used the BCIZ plate bending element [18] to show that an element may pass the SET but fail multielement versions, and consequently that tests involving single elements are to be viewed with caution.

§**23.6. *Historical Background**

The patch test for convergence is a fascinating area in the development of nonconforming finite element methods. It grew up of the brilliant intuition of Bruce Irons. Initially developed in the mid-1960s at Rolls Royce and then at the Swansea group headed by O. C. Zienkiewicz, by the early 1970s the test had became a powerful and practical tool for evaluating and checking nonconforming elements. And yet today it remains a controversial issue: accepted by many finite element developers while ignored by others, welcomed by element programmers, distrusted by mathematicians. For tracing down the origins of the test there is no better source than a 1973 survey article by the Irons and Razzaque [144]. Annotations to the quoted material are inserted in square brackets, and reference numbers have been altered to match those of the present Chapter.

"*Origins of the Patch Test*

In 1965 even engineering intuition dared not predict the behavior of certain finite elements. Experience forced those engineers who doubted it to admit that interelement continuity was important: the senior author [Bruce Irons] believed that it was necessary for convergence. It is not known which ideas inspired a numerical experiment by Tocher and Kapur [248] which demonstrated convergence within 0.3% in a biharmonic problem of plate bending, using equal rectangular elements with 1, x, y, x^2, xy, y^2, y^2, x^3, x^2y, xy^2, y^3, x^3y and xy^3, as functional basis. The nodal variables of this Ari Adini rectangle [3] are w, $\partial w/\partial x$ and $\partial w/\partial y$ at the four corners, and this element guarantees only C^0 conformity.

Some months later, research at Rolls-Royce on the Zienkiewicz nonconforming triangle [18] — a similar plate-bending element — clarified the situation. [This element is that identified by 'BCIZ' in the sequel.] Three elements with C^1 continuity were simultaneously available, and, because the shape function subroutine used for numerical integration had been exhaustively tested, the results were trustworthy. It was observed: (a) that every problem giving constant curvature over the whole domain was accurately solved by the conforming elements whatever the mesh pattern, as was expected, and (b) that the nonconforming element was also successful, but only for one particular mesh pattern. [The bending element test referred to in this passage is described in the Addendum to [18]. This Addendum was not part of the original paper presented at the First Wright-Patterson Conference held in September 1965; it was added to the Proceedings that appeared in 1966. The name "patch test" will not be found there; see the Appendix of the Strang-Fix monograph [227] for further historical details.]

Thus the patch test was born. For if the external nodes of any sub-assembly of a successful assembly of elements are given prescribed values corresponding to an arbitrary state of constant curvature, then the internal nodes must obediently take their correct values. (An internal node is defined as one completely surrounded by elements.) Conversely, if two overlapping patches can reproduce any given state of constant curvature, they should combine into a larger successful patch, provided that every external node lost is internal to one of the original patches. For such nodes are in equilibrium at their correct values, and should behave correctly as internal nodes of the extended patch. In an unsuccessful patch test, the internal nodes take unsuitable values, which introduce interelement discontinuities. The errors in deflection may be slight, but the errors in curvature may be ±20%. We must recognize two distinct types of errors:

(i) The finite element equations would not be exactly satisfied by the correct values at the internal nodes — in structural terms, we have disequilibrium;

(ii) The answers are nonunique because the matrix of coefficients **K** is semidefinite.

Role of the Patch Test

Clearly the patch test provides a *necessary* condition for convergence with fine mesh. We are less confident that it provides a *sufficient* condition. The argument is that if the mesh is fine, the patches are also small. Over any patch the correct solution gives almost uniform conditions to which the patch is known to respond correctly — provided that the small perturbations from uniform conditions do not cause a disproportionate response in the patch: we hope to prevent this by insisting that **K** is positive definite. [Given later developments, this was an inspired guess.]

The patch test is invaluable to the research worker. Already, it has made respectable

(i) Elements that do not conform,

(ii) Elements that contain singularities,

(iii) Elements that are approximately integrated,

(iv) Elements that have no clear physical basis.

In short, the patch test will help a research worker to exploit and justify his wildest ideas. It largely restores the freedom enjoyed by the early unsophisticated experimenters."

The late 1960s and early 1970s were a period of unquestionable success for the test. That optimism is evident in the article quoted above, and prompted Gilbert Strang to develop a mathematical version popularized in the Strang-Fix monograph [227].

Confidence was shaken in the late 1970s by several developments. Numerical experiments, for example, those of Sander and Beckers [215] suggested that the test is not necessary for convergence, thus disproving Irons' belief stated above. Then a counterexample by Stummel [233] purported to show that the test is not even sufficient. This motivated defensive responses by Irons [148] shortly before his untimely death and by Taylor et al. [240] These papers tried to set out the engineering version of the test on a more precise basis.

Despite these ruminations many questions persist, as noted in the lucid review article by Griffiths and Mitchell [116]. The most important ones are listed below.

Q1. What is a patch? Is it the ensemble of all possible meshes? Are some meshes excluded? Can these meshes contain different types of elements?

Q2. The test was originally developed for harmonic and biharmonic problems of compressible-elasticity, for which the concept of "constant strains" or "constant curvatures" is unambiguous. But what is the equivalent concept for arches and shells, if one is unwilling to undergo a limiting process?

Q3. What are the modifications required for incompressible media? Is the test applicable to dynamic or nonlinear problems?

Q4. Are single-element versions of the test equivalent to the conventional, multielement versions?

Q5. Is the test restricted to nonconforming assumed-displacement elements? Can it be extended to encompass assumed-stress or assumed-strain mixed and hybrid elements? Initial attempts in this direction were made by Fraeijs de Veubeke in 1974.

Remark 23.2. Stummel has constructed [234] a *generalized patch test*, which is mathematically impeccable in that it provides necessary and sufficient conditions for convergence. Unfortunately such test lacks practical side benefits of Irons' patch test, such as element checkout by computer (either numerically or symbolically), because it is administered as a mathematically limiting process in function spaces. Furthermore, it does not apply to a mixture of different element types, which is of crucial importance in complex physical models.

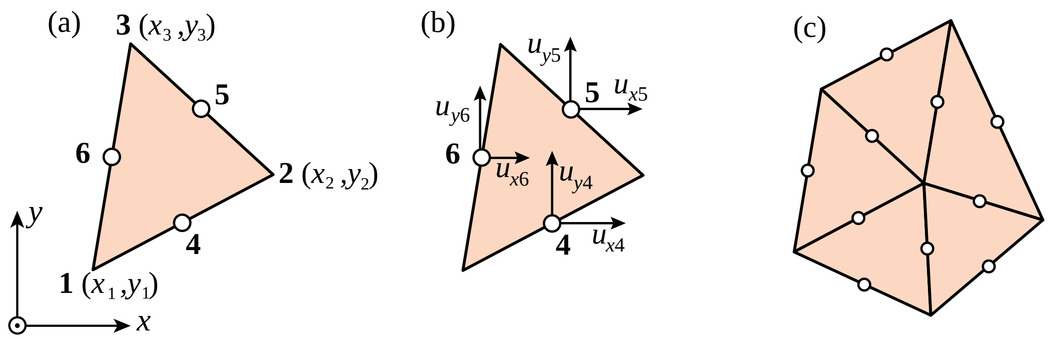

FIGURE 23.7. The Veubeke equilibrium triangle: (a) geometric definition; (b) degree-of-freedom configuration; (c) element patch showing how triangles are connected at the midpoints.

§23.7. The Veubeke Equilibrium Triangle

The Veubeke equilibrium triangle is briefly described below so as to provide background material for applying the multielement patch test in Exercise E23.2. This element differs from the conventional 3-node plane stress triangle, also known as *Turner triangle*, in the degree-of-freedom configuration. As illustrated in Figure 23.7, freedoms are moved to the midpoints {4, 5, 6} while the corner nodes {1, 2, 3} still define the geometry of the element. In the FEM terminology introduced in Chapter 6, the geometric nodes {1, 2, 3} and the connection nodes {4, 5, 6} no longer coincide. The node displacement vector collects the freedoms shown in Figure 23.7(b):

$$\mathbf{u}^e = [\, u_{x4} \quad u_{y4} \quad u_{x5} \quad u_{y5} \quad u_{x6} \quad u_{y6} \,]^T . \tag{23.6}$$

The quickest way to formulate the stiffness matrix of this element is to relate (23.6) to the node displacements of the Turner triangle, identified as

$$\mathbf{u}^e_T = [\, u_{x1} \quad u_{y1} \quad u_{x2} \quad u_{y2} \quad u_{x3} \quad u_{y3} \,]^T . \tag{23.7}$$

§23.7.1. Kinematic Relations

The node freedom vectors (23.6) and (23.7) are easily related since by linear interpolation along the sides one obviously has $u_{x4} = \frac{1}{2}(u_{x1} + u_{x2})$, $u_{y4} = \frac{1}{2}(u_{y1} + u_{y2})$, etc. Expressing those links in matrix form gives

$$\begin{bmatrix} u_{x4} \\ u_{y4} \\ u_{x5} \\ u_{y5} \\ u_{x6} \\ u_{y6} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix}, \qquad \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{x4} \\ u_{y4} \\ u_{x5} \\ u_{y5} \\ u_{x6} \\ u_{y6} \end{bmatrix}. \tag{23.8}$$

In compact form: $\mathbf{u}^e = \mathbf{T}_{VT}\,\mathbf{u}^e_T$ and $\mathbf{u}^e_T = \mathbf{T}_{TV}\,\mathbf{u}^e$, with $\mathbf{T}_{VT} = \mathbf{T}^{-1}_{TV}$. The shape functions are

$$N_4 = \zeta_1 + \zeta_2 - \zeta_3, \quad N_5 = -\zeta_1 + \zeta_2 + \zeta_3, \quad N_6 = \zeta_1 - \zeta_2 + \zeta_3. \tag{23.9}$$

Calling the Turner triangle strain-displacement matrix of as $\mathbf{B}_T$, the corresponding matrix that relates $\mathbf{e} = \mathbf{B}\,\mathbf{u}^e$ in the Veubeke equilibrium triangle becomes

$$\mathbf{B} = \mathbf{B}_T\,\mathbf{T}_{TV} = \frac{1}{A}\begin{bmatrix} y_{21} & 0 & y_{32} & 0 & y_{13} & 0 \\ 0 & x_{12} & 0 & x_{23} & 0 & x_{31} \\ x_{12} & y_{21} & x_{23} & y_{32} & x_{31} & y_{13} \end{bmatrix} \tag{23.10}$$

§23.7.2. Stiffness Matrix

As usual the element stiffness matrix is given by $\mathbf{K}^e = \int_{\Omega^e} h\mathbf{B}^T\,\mathbf{E}\,\mathbf{B}\,d\Omega$. For constant plate thickness h one obtains the closed form

$$\mathbf{K}^e = A\,h\,\mathbf{B}^T\,\mathbf{E}\,\mathbf{B} = \frac{h}{A}\begin{bmatrix} y_{21} & 0 & x_{12} \\ 0 & x_{12} & y_{21} \\ y_{32} & 0 & x_{23} \\ 0 & x_{23} & y_{32} \\ y_{13} & 0 & x_{31} \\ 0 & x_{31} & y_{13} \end{bmatrix}\begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix}\begin{bmatrix} y_{21} & 0 & y_{32} & 0 & y_{13} & 0 \\ 0 & x_{12} & 0 & x_{23} & 0 & x_{31} \\ x_{12} & y_{21} & x_{23} & y_{32} & x_{31} & y_{13} \end{bmatrix}. \tag{23.11}$$

§23.7.3. Implementation

The implementation of the Veubeke equilibrium triangle as a *Mathematica* module that returns $\mathbf{K}^e$ is shown in Figure 23.8. It needs only 8 lines of code. It is invoked as

`Ke=Trig3VeubekeMembraneStiffness[ncoor,Emat,h,numer];` (23.12)

where the arguments are

- `ncoor` Element corner coordinates, arranged as a list: `{{x1,y1},{x2,y2},{x3,y3}}`.
- `Emat` A two-dimensional list storing the 3 × 3 plane stress matrix of elastic moduli as `{{E11,E12,E13},{E12,E22,E23},{E13,E23,E33}}`.
- `h` Plate thickness, assumed uniform over the triangle.
- `numer` A logical flag: `True` to request floating-point computation, else `False`.

This module is exercised by the statements listed at the top of Figure 23.9, which forms a triangle with corner coordinates `{{0,0},{3,1},{2,2}}`, isotropic material matrix with $E_{11} = E_{22} = 32$, $E_{12} = 8$, $E_{33} = 16$, others zero, and unit thickness. The results are shown at the bottom of Figure 23.9. Note that the element is rank sufficient.

```
Trig3VeubekeMembraneStiffness[ncoor_,Emat_,h_,numer_]:=Module[{
  x1,x2,x3,y1,y2,y3,x12,x23,x31,y21,y32,y13,A,Be,Te,Ke},
  {{x1,y1},{x2,y2},{x3,y3}}=ncoor;
  A=Simplify[(x2*y3-x3*y2+(x3*y1-x1*y3)+(x1*y2-x2*y1))/2];
  {x12,x23,x31,y21,y32,y13}={x1-x2,x2-x3,x3-x1,y2-y1,y3-y2,y1-y3};
  Be={{y21,0,y32,0,y13,0}, {0,x12,0,x23,0,x31},
      {x12,y21,x23,y32,x31,y13}}/A;
  If [numer,Be=N[Be]]; Ke=A*h*Transpose[Be].Emat.Be;
  Return[Ke]];
```

FIGURE 23.8. Implementation of Veubeke equilibrium triangle stiffness matrix as a *Mathematica* module.

```
ncoor={{0,0},{3,1},{2,2}}; Emat=8*{{8,2,0},{2,8,0},{0,0,3}};
Ke=Trig3VeubekeMembraneStiffness[ncoor,Emat,1,False];
Print["Ke=",Ke//MatrixForm];
Print["eigs of Ke=",Chop[Eigenvalues[N[Ke]]]];
```

$$\text{Ke=}\begin{pmatrix} 140 & -60 & -4 & -28 & -136 & 88 \\ -60 & 300 & -12 & -84 & 72 & -216 \\ -4 & -12 & 44 & 20 & -40 & -8 \\ -28 & -84 & 20 & 44 & 8 & 40 \\ -136 & 72 & -40 & 8 & 176 & -80 \\ 88 & -216 & -8 & 40 & -80 & 176 \end{pmatrix}$$

```
eigs of Ke={557.318, 240., 82.6816, 0, 0, 0}
```

FIGURE 23.9. Test statements to exercise the module of Figure 23.8, and outputs.

§23.7.4. Patch Test Configuration

The Veubeke equilibrium triangle is complete since the sum of shape functions of (23.9) is unity: $N_4+N_5+N_6 = \zeta_1+\zeta_2-\zeta_3-\zeta_1+\zeta_2+\zeta_3+\zeta_1-\zeta_2+\zeta_3 = \zeta_1+\zeta_2+\zeta_3 = 1$. But it is nonconforming (interelement discontinuous) because the variation of displacements along each side is linear, but there is only one connection node at the midpoint. Thus a patch test is a good idea.

The multielement patch test is to be carried out for the two-triangle patch shown in Figure 23.10. The patch geometry is rectangular for simplicity. The patch exterior connector nodes are 5,6,7,8 and the interior node is 9. Test details are specified in Exercise E23.2.

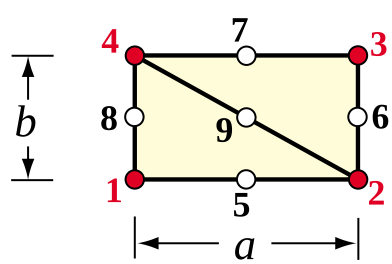

FIGURE 23.10. Two-element patch for testing Veubeke equilibrium triangle. Red filled circles and white filled circles denote geometric and connection nodes, respectively.

Homework Exercises for Chapter 23

The Patch Test

EXERCISE 23.1 [A/D:15] Answer Q8 of §23.3.3.

EXERCISE 23.2 [C:25] Carry out both displacement and force patch tests for the patch of Figure 23.10, which is fabricated with two Veubeke equilibrium triangles. Take $a = 2$, $b = 1$, $E = 10$, $\nu = 0$ and unit thickness. If a patch fails, do not procede further, and try to explain what went wrong.

EXERCISE 23.3 [A:50] (Possibly worth a Fields Medal Prize, worth $500,000). Develop a practical patch test executable on the computer, that is necessary and sufficient for convergence.