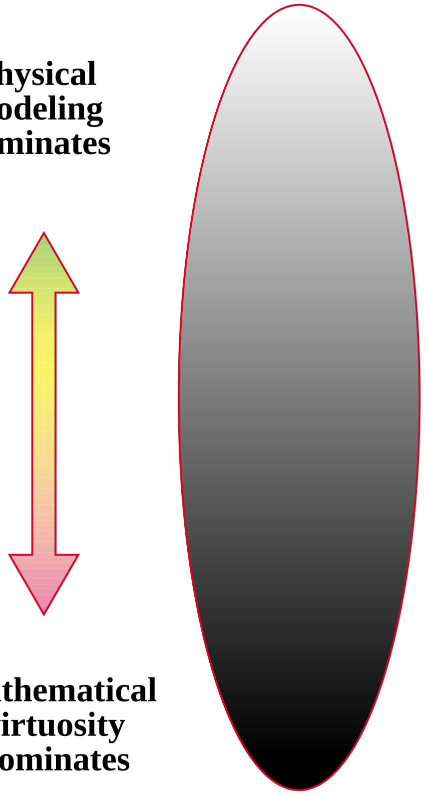
22

Element Fabrication Overview

Spectrum of Element Construction Methods



Physical modeling dominates

Mathematical virtuosity dominates

Experiment-fitted (empirical)

Application-fitted (heuristic)

Templates

G4 formulations: ANS, FF, ...

Finite volume

Ritz-Galerkin conforming

Mixed, Hybrids

Petrov-Galerkin

Spectral

Boundary elements

A Catalog of Finite Element Formulations

pre-1980 "oldies"   post-1980 "comers"

- **Isoparametric (isoP)**
- **Fixed-Up Isoparametric**
- **Subparametric (subP)**
- **Mixed Hellinger-Reissner (HR)**
- **Equilibrium Stress Hybrid**

- **Assumed Strains**
- **Free Formulation (FF)**
- **Assumed Natural Deviatoric Strains (ANDES)**
- **Templates**

Reviewed in following slides

The Isoparametric (isoP) Formulation

Source	Outgrow of Taig's quadrilateral presented in [193]. Extended by Irons to arbitrary geometries [119,122].
Master field(s)	One internal: displacements. Strains and stresses are slaves.
Var. principle	Total Potential Energy (TPE)
Description	Geometry and displacements interpolated by same shape functions. Numerical integration by Gauss quadrature.
Applicability	Problems governed by functional with variational index one. Only displacements as degrees of freedom.
Popularity	Huge despite limitations noted below.
Strengths	Well established (nearly 40 years old in 2006), widely implemented. Advantages and shortcomings well known. Systematic rules, valid regardless of dimensionality and element complexity. Naturally handles elements with curved sides/faces. Technique easily extended to non-structural elements (thermal, fluids, EMs,...) as long as variational index in the primary variable stays one. Shape functions useful to express interpolation rules for other applications, such as data fitting.
Limitations	Low order isoP models may have poor performance in terms of locking and distortion sensitivity. Some fixed-up devices are described in §20.2.2. Simplex elements of this type (linear triangles and tetrahedra) cannot be improved, however, by any device. Not applicable to problems with variational index greater than one in displacements, such as Bernoulli-Euler beams, Kirchhoff plates and thin shells. Cannot handle elements with rotational or derivative-type freedoms without substantial modifications that usually involve restriction to the subP formulation outlined in §20.2.3. "Blows up" for incompressible plane strain, axisymmetric and solid elements.

The Fixed-Up Isoparametric Formulation

Source	Bag of tricks emerging over period 1969–75: [47,195,220,230]. Equivalence to mixed methods: [115]. Comprehensive exposition in Hughes' book [118].
Master field(s)	Same as isoP.
Var. principle	Total Potential Energy (TPE)
Description	This category embraces a number of ad-hoc devices introduced to fix or at least alleviate the poor performance of certain isoparametric models, especially low-order ones, as regards locking (extreme overstiffness) and distortion sensitivity. Most common devices include (1) reduced integration, (2) selective integration, (3) directional integration, and (4) incompatible displacement modes.
Applicability	Same as isoP formulation.
Popularity	High because fixed-up devices are easy to implement. Expected knowledge does not rise to level of variational techniques used in mixed and hybrid formulations.
Strengths	Easy to implement. Software reuse (for example, of shape function modules) is facilitated. This is particularly important in legacy and nonlinear codes.
Limitations	Backstabbing erratic effects. As more tricks are introduced, unpleasant surprises can happen, such as hourglassing and invariance loss. Repair can produce more unexpected side effects: "turn on the light, the shower goes on." Eventually legacy software adorned with these accoutrements may become untouchable for fear of additional side effects.

The Subparametric (subP) Formulation

Source	Assumed-displacement models preceding the isoP formulation, as in [6,145,146].
Master field(s)	One internal: displacements. Strains and stresses are slaves.
Var. principle	Total Potential Energy (TPE)
Description	Element geometry kept simple, typically restricted to the simplest possible shapes. Displacements interpolated by equal or higher order functions than geometry.
Applicability	Problems governed by functionals of any variational index. Handles elements with rotational or derivative-type freedoms.
Popularity	Medium. Easy to present and teach.
Strengths	Applicability wider than that of isoP elements. Can directly handle C^1 beams, plates and shells.
Limitations	Element geometry is restricted, restricting modeling flexibility. No systematic construction rules. As in the isoP case, performance of low-order elements may be poor. For some configurations full interelement compatibility may be difficult or even impossible to attain.

The Mixed Hellinger-Reissner (HR) Formulation

Source	First FEM use proposed by Herrmann [109] for incompressible and nearly-incompressible elasticity.
Master field(s)	Two internal: displacements and stresses. Two strain slaves.
Var. principle	Hellinger-Reissner (HR)
Description	Displacement master field may be interpolated with shape functions. Separate master stress interpolation (or pressure) usually done in Cartesian coordinates. Stress DOFs (or stress parameters) may be condensed out at element level if (1) stress (or pressure) may jump between adjacent elements and (2) material is compressible. Else stress DOFs must go into the assembly.
Applicability	Two major uses: (1) treating incompressible or nearly-incompressible materials, and (2) improving the performance of low-order displacement models. For case (1) pressure is assumed in addition to displacements.
Popularity	Low since it requires advanced knowledge. Hampered by the discovery of fixed-up devices for isoP elements, which under certain restrictions effectively produce the equivalent of mixed elements with simpler tricks.
Strengths	Possible way to improve element performance. Proven useful for the incompressible case. If stress DOFs can be eliminated, condensed model looks like a displacement element to the assembler. Comparing strains from displacements and stresses allows easy construction of an element-level error measure.
Limitations	Requires knowledge of mixed variational principles, not an easy subject. Prone to failure because of rank deficiency if stress (or pressure) and displacement assumptions violated certain stability conditions.[1] Even if element is stable, expected improvements in performance may not necessarily materialize: “mixed elements lead to mixed results” (G. Strang).

The Equilibrium Stress Hybrid Formulation

Source	Pian [157,158], variational basis by Pian and Tong [159].
Master field(s)	Over element: internal equilibrium stress field, interelement-compatible boundary displacements. Slave Strains. Displacement field inside element known only weakly.
Var. principle	Total Potential Complementary Energy (TCPE) augmented by dislocation potential.
Description	Ingenious motivation: facilitate interelement compatibility by assuming only boundary displacements while mainting internal equilibrium, which should result in better stress recovery.
Applicability	Applicable in principle to any linear element.
Popularity	Medium. Despite age[2] requires advanced knowledge.
Strengths	When feasible, a proven way to improve element stress-recovery performance. Especially useful when assumed-displacement interelement compatibility is hard to achieve, as in thin plates and shells.
Limitations	Requires knowledge of hybrid variational principles, a tough subject. Construction of an invariant equilibrated stress field may be difficult or even impossible for arbitrary geometries.[3] Difficult to extend to geometrically nonlinear problems.

The Assumed Strain Formulations

Source	For assumed Cartesian strains: MacNeal [132]. For assumed natural strains (ANS): Bathe-Dvorkin [13] for plates, Park-Stanley [153] and Huang-Hinton [265] for shells.
Master field(s)	Varies with author. Common feature is that element strain variation is assumed in some form.
Var. principle	Varies with author. Most use Total Potential Energy (TPE) with various forms of strain releases for certain energy components.
Description	The slave connection between displacements and stresses is selectively broken to alleviate locking problems. In the Assumed Natural Strain variant (the most powerful variant) the strain field is expressed in natural coordinate directions, but not necessarily in tensorial form.
Applicability	Primarily useful for plates and shells fabricated from degenerated solid elements.
Popularity	Medium, since applicability is restricted.
Strengths	When done correctly it produces high performance plates and shell elements. Main advantages arise in geometrically nonlinear analysis of shells.
Limitations	To make it work correctly requires substantial physical insight, a dab of black magic incantations and plenty of luck.

The Free Formulation (FF)

Source	Bergan and coworkers [21,22].
Master field(s)	Element response behavior split into basic and higher order. For basic response, same masters as constant-stress equilibrium hybrids. For higher order response, assumed displacements in generalized coordinates. These are not usually interelement compatible.[4]
Var. principle	Not provided in original formulation. Shown to be a mixture of hybrid and TPE functionals in [261].
Description	Fundamental idea is splitting of element response. Each component is assigned different but complementary roles. Basic component takes care of convergence and mixability. Higher order component provides stability (rank sufficiency) and accuracy. Orthogonality conditions between those components insure *a priori* satisfaction of the Individual Patch Test (IPT) of [20].
Applicability	Any mechanical element.
Popularity	Low. Splitting concept is at odds with historical tradition, variational formulation is highly unconventional, and use of generalized coordinates is unfamiliar to many developers.
Strengths	Provides high performance elements. Basing the higher order component on assumed displacements taps a well studied subject.
Limitations	Requires substantial physical insight on the part of a developer. Construction of the higher order component not easy, involving many trials to balance orthogonality conditions with rank sufficiency.

The Assumed Natural Deviatoric Strain (ANDES) Formulation

Source	Felippa and Militello [60,239].
Master field(s)	Element response behavior split into basic and higher order. For basic response, same masters as constant-stress equilibrium hybrid. For higher order response, assumed natural strains.
Var. principle	Falls in the context of parametrized variational principles [62].
Description	ANDES incorporates ideas from the Free Formulation (FF) and the Assumed Natural strain (ANS) formulation. Same response splitting as in FF, and the same basic component. Higher order component constructed with ANS ideas using assumed deviatoric strains (strains that deviate from a constant state).
Applicability	Any mechanical element.
Popularity	Low, as in the case of the FF. Deviates too much from tradition. Superseded by templates.
Strengths	Similar to FF, but construction of the higher order part allows more easy parametrization on the way to templates.
Limitations	Requires substantial expertise and a dose of luck since methodology is very new. Trial and error mandatory.

Templates

Source	First mentioned in [62]. Developed in [67,266]. Recent tutorial: [74].
Master field(s)	Element response split as in case of FF and ANDES. Basic part is constant equilibrium-stress hybrid. For higher order component, no predefined masters.
Var. principle	For basic component, equilibrium stress hybrid principle. For higher order component, no predefined principle.
Description	A template is a parametrized algebraic form of element operators (stiffness, mass, etc). This produces an infinity of possible elements that *a priori* satisfy the Individual Element Test (IET). Specific element instances obtained by assigning numerical values to parameters. A *universal template* is one that includes all possible elements that pass the IET.
Applicability	In principle any element. But see *Limitations* below.
Popularity	None, as it is a recent idea barely off the ground: “new kid on the block”
Strengths	Includes an infinite number of possible elements. Instances can be customized to produce optimal results for envisioned use.
Limitations	Development impossible by hand since one must carry along element properties (geometry, constitutive, fabrication, ...) in symbolic form. Use of a computer algebra system (CAS) mandatory. Limits in current CAS power, however, has restricted the idea to 1D and 2D elements of simple geometry.