18

Hexahedron Elements

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§18.1. Introduction

The generalization of a quadrilateral three-dimensions is a *hexahedron*, also known in the finite element literature as *brick*. A hexahedron is topologically equivalent to a cube. It has eight corners, twelve edges or sides, and six faces.

Finite elements with this geometry are extensively used in modeling three-dimensional solids. Hexahedra also have been the motivating factor for the development of "Ahmad-Pawsey" shell elements through the use of the "degenerated solid" concept.

The construction of hexahedra shape functions and the computation of the stiffness matrix was greatly facilitated by three advances in finite element technology: natural coordinates, isoparametric description and numerical integration. Together these revolutionized the finite element field in the mid-1960's.

§18.1.1. Natural Coordinates

Before presenting examples of hexahedron elements, we have to introduce the appropriate *natural coordinate system* for that geometry. The natural coordinates for this geometry are called ξ, η and μ, and are called *isoparametric hexahedral coordinates* or simply *natural coordinates*.

These coordinates are illustrated in Figure 18.1. As can be seen they are very similar to the quadrilateral coordinates ξ and η used in IFEM. They vary from -1 on one face to +1 on the opposite face, taking the value zero on the "median" face. As in the quadrilateral, this particular choice of limits was made to facilitate the use of the standard Gauss integration formulas.

§18.1.2. Corner Numbering Rules

The eight corners of a hexahedron element are locally numbered 1, 2 ... 8. The corner numbering rule is similar to that given for the 4-node tetrahedron in Chapter 14. Again the purpose is to guarantee a positive volume (or, more precisely, a positive Jacobian determinant at every point). The transcription of those rules to the hexahedron element is as follows:

1. Chose one starting corner, which is given number 1, and one initial face pertaining to that corner (given a starting corner, there are three possible faces meeting at that corner that may be selected).

2. Number the other 3 corners as 2,3,4 traversing the initial face counterclockwise[1] while one looks at the initial face from the opposite one.

3. Number the corners of the opposite face directly opposite 1,2,3,4 as 5,6,7,8, respectively.

[1] "Anticlockwise" in British.

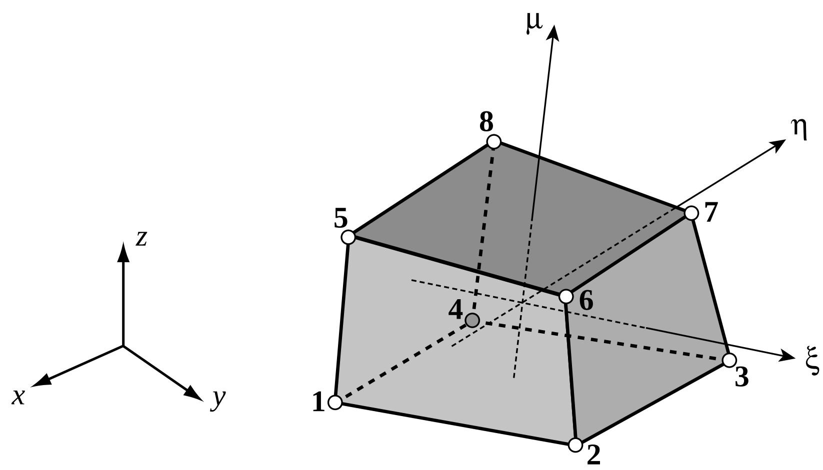

Figure 18.1. The 8-node hexahedron and the natural coordinates ξ, η, μ. The definition of the latter is the same for higher order models.

The definition of ξ, η and μ can be now be made more precise:

ξ goes from -1 from (center of) face 1485 to $+1$ on face 2376

η goes from -1 from (center of) on face 1265 to $+1$ on face 3487

μ goes from -1 from (center of) on face 1234 to $+1$ on face 5678

The center of a face is the intersection of the two medians.

§18.2. The Eight Node (Trilinear) Hexahedron

The eight-node hexahedron shown in Figure 18.2 is the simplest member of the hexahedron family. It is defined by

$$\begin{bmatrix} 1 \\ x \\ y \\ z \\ v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 \\ z_1 & z_2 & z_3 & z_4 & z_5 & z_6 & z_7 & z_8 \\ v_{x1} & v_{x2} & v_{x3} & v_{x4} & v_{x5} & v_{x6} & v_{x7} & v_{x8} \\ v_{y1} & v_{y2} & v_{y3} & v_{y4} & v_{y5} & v_{y6} & v_{y7} & v_{y8} \\ v_{z1} & v_{z2} & v_{z3} & v_{z4} & v_{z5} & v_{z6} & v_{z7} & v_{z8} \end{bmatrix} \begin{bmatrix} N_1^{(e)} \\ N_2^{(e)} \\ \vdots \\ N_8^{(e)} \end{bmatrix} \tag{18.1}$$

The hexahedron coordinates of the corners are (see Figure 18.1)

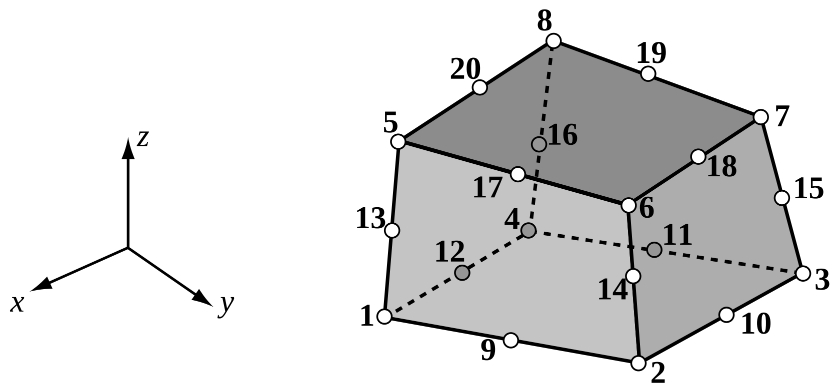

Figure 18.2. The 20-node hexahedron element — note node numbering conventions.

node	ξ	η	μ
1	−1	−1	−1
2	+1	−1	−1
3	+1	+1	−1
4	−1	+1	−1
5	−1	−1	+1
6	+1	−1	+1
7	+1	+1	+1
8	−1	+1	+1

The shape functions are

$$
\begin{aligned}
N_1^{(e)} &= \tfrac{1}{8}(1-\xi)(1-\eta)(1-\mu), & N_2^{(e)} &= \tfrac{1}{8}(1+\xi)(1-\eta)(1-\mu)\\
N_3^{(e)} &= \tfrac{1}{8}(1+\xi)(1+\eta)(1-\mu), & N_4^{(e)} &= \tfrac{1}{8}(1-\xi)(1+\eta)(1-\mu)\\
N_5^{(e)} &= \tfrac{1}{8}(1-\xi)(1-\eta)(1+\mu), & N_6^{(e)} &= \tfrac{1}{8}(1+\xi)(1-\eta)(1+\mu)\\
N_7^{(e)} &= \tfrac{1}{8}(1+\xi)(1+\eta)(1+\mu), & N_8^{(e)} &= \tfrac{1}{8}(1-\xi)(1+\eta)(1+\mu)
\end{aligned}
\tag{18.2}
$$

These eight formulas can be summarized in a single expression:

$$
N_1^{(e)} = \tfrac{1}{8}(1+\xi\xi_i)(1+\eta\eta_i)(1+\mu\mu_i) \tag{18.3}
$$

where ξ_i, η_i and μ_i denote the coordinates of the i^{th} node.

§18.3. The 20-node (Serendipity) Hexahedron

The 20-node hexahedron is the analog of the 8-node "serendipity" quadrilateral. The 8 corner nodes are augmented with 12 side nodes which are usually located at the midpoints of the sides.

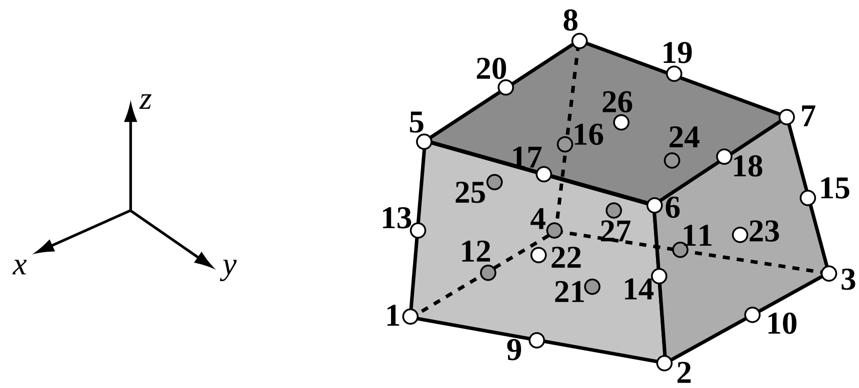

Figure 18.3. The 27-node hexahedron element — note node numbering conventions.

The numbering scheme is illustrated in Figure 18.2. For elasticity applications this element have $20 \times 3 = 60$ degrees of freedom.

The 8-node quadrilateral studied in IFEM cannot represent a complete biquadratic expansion in the quadrilateral coordinates ξ and η, that is, the nine terms $1, \xi, \eta, \xi^2, \ldots, \xi^2\eta^2$. One has to go to the 9-node (biquadratic) quadrilateral to achieve that.

Likewise, the 20 node hexahedron is incapable of accomodating a full triquadratic expansion in ξ, η and μ; that is $1, \xi, \eta, \mu, \eta^2, \ldots, \xi^2\eta^2\mu^2$. A 27-node hexahedron is required for that. That element is described in the next section.

The shape functions of the 20-node hexahedron can be grouped as follows. For the corner nodes $i = 1, 2, \ldots, 8$:

$$N_i^{(e)} = \tfrac{1}{8}(1+\xi\xi_i)(1+\eta\eta_i)(1+\mu\mu_i)(\xi\xi_i+\eta\eta_i+\mu\mu_i-2). \tag{18.4}$$

For the midside nodes $i = 9, 11, 17, 19$:

$$N_i^{(e)} = \tfrac{1}{4}(1-\xi^2)(1+\eta\eta_i)(1+\mu\mu_i). \tag{18.5}$$

For the midside nodes $i = 10, 12, 18, 20$:

$$N_i^{(e)} = \tfrac{1}{4}(1-\eta^2)(1+\xi\xi_i)(1+\mu\mu_i). \tag{18.6}$$

For the midside nodes $i = 13, 14, 15, 16$:

$$N_i^{(e)} = \tfrac{1}{4}(1-\mu^2)(1+\xi\xi_i)(1+\eta\eta_i). \tag{18.7}$$

§18.4. The 27-Node Hexahedron

A 27-node hexahedron can indeed be constructed by adding 7 more nodes: 6 on each face center, and 1 interior node at the hexahedron center. See Figure 18.3. In elasticity application such an element has $27 \times 3 = 81$ degrees of freedom.

(To be completed).

§18.5. Partial Derivatives

The calculation of partial derivatives of hexahedron shape functions with respect to Cartesian coordinates follows techniques similar to that discussed for two-dimensional quadrilateral elements in IFEM. Only the size of the matrices changes because of the appearance of the third dimension.

§18.5.1. The Jacobian

The derivatives of the shape functions are given by the usual chain rule formulas:

$$\begin{aligned}
\frac{\partial N_i^{(e)}}{\partial x} &= \frac{\partial N_i^{(e)}}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial N_i^{(e)}}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial N_i^{(e)}}{\partial \mu}\frac{\partial \mu}{\partial x},\\
\frac{\partial N_i^{(e)}}{\partial y} &= \frac{\partial N_i^{(e)}}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial N_i^{(e)}}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial N_i^{(e)}}{\partial \mu}\frac{\partial \mu}{\partial y},\\
\frac{\partial N_i^{(e)}}{\partial z} &= \frac{\partial N_i^{(e)}}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial N_i^{(e)}}{\partial \eta}\frac{\partial \eta}{\partial z} + \frac{\partial N_i^{(e)}}{\partial \mu}\frac{\partial \mu}{\partial z}.
\end{aligned} \tag{18.8}$$

In matrix form

$$\begin{bmatrix} \dfrac{\partial N_i^{(e)}}{\partial x} \\ \dfrac{\partial N_i^{(e)}}{\partial y} \\ \dfrac{\partial N_i^{(e)}}{\partial z} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \eta}{\partial x} & \dfrac{\partial \mu}{\partial x} \\ \dfrac{\partial \xi}{\partial y} & \dfrac{\partial \eta}{\partial y} & \dfrac{\partial \mu}{\partial y} \\ \dfrac{\partial \xi}{\partial z} & \dfrac{\partial \eta}{\partial z} & \dfrac{\partial \mu}{\partial z} \end{bmatrix} \begin{bmatrix} \dfrac{\partial N_i^{(e)}}{\partial \xi} \\ \dfrac{\partial N_i^{(e)}}{\partial \eta} \\ \dfrac{\partial N_i^{(e)}}{\partial \mu} \end{bmatrix}. \tag{18.9}$$

The 3×3 matrix that appears in (18.9) is $\mathbf{J}^{-1}$, the inverse of:

$$\mathbf{J} = \frac{\partial(x, y, z)}{\partial(\xi, \eta, \mu)} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} & \dfrac{\partial z}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} & \dfrac{\partial z}{\partial \eta} \\ \dfrac{\partial x}{\partial \mu} & \dfrac{\partial y}{\partial \mu} & \dfrac{\partial z}{\partial \mu} \end{bmatrix}. \tag{18.10}$$

Matrix $\mathbf{J}$ is called the *Jacobian matrix* of (x, y, z) with respect to (ξ, η, μ). In the finite element literature, matrices $\mathbf{J}$ and $\mathbf{J}^{-1}$ are called simply the *Jacobian* and *inverse Jacobian*, respectively, although such a short name is sometimes ambiguous. The notation

$$\mathbf{J} = \frac{\partial(x, y, z)}{\partial(\xi, \eta, \mu)}, \qquad \mathbf{J}^{-1} = \frac{\partial(\xi, \eta, \mu)}{\partial(x, y, z)}. \tag{18.11}$$

is standard in multivariable calculus and suggests that the Jacobian may be viewed as a generalization of the ordinary derivative, to which it reduces for a scalar function $\mathbf{x} = x(\xi)$.

§18.5.2. Computing the Jacobian Matrix

The isoparametric definition of hexahedron element geometry is

$$x = x_i\, N_i^{(e)}, \qquad y = y_i\, N_i^{(e)}, \qquad z = z_i\, N_i^{(e)}, \tag{18.12}$$

where the summation convention is understood to apply over $i = 1, 2, \ldots n$, in which n denotes the number of element nodes.

Differentiating these relations with respect to the hexahedron coordinates we construct the matrix $\mathbf{J}$ as follows:

$$\mathbf{J} = \begin{bmatrix} x_i \frac{\partial N_i^{(e)}}{\partial \xi} & y_i \frac{\partial N_i^{(e)}}{\partial \xi} & z_i \frac{\partial N_i^{(e)}}{\partial \xi} \\ x_i \frac{\partial N_i^{(e)}}{\partial \eta} & y_i \frac{\partial N_i^{(e)}}{\partial \eta} & z_i \frac{\partial N_i^{(e)}}{\partial \eta} \\ x_i \frac{\partial N_i^{(e)}}{\partial \mu} & y_i \frac{\partial N_i^{(e)}}{\partial \mu} & z_i \frac{\partial N_i^{(e)}}{\partial \mu} \end{bmatrix}. \tag{18.13}$$

Given a point of hexahedron coordinates (ξ, η, μ) the Jacobian $\mathbf{J}$ can be easily formed using the above formula, and numerically inverted to form $\mathbf{J}^{-1}$.

Remark 18.1. The inversion formula for a matrix of order 3 is

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \qquad \mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, \tag{18.14}$$

where

$$\begin{aligned} A_{11} &= a_{22}a_{33} - a_{23}a_{32}, & A_{22} &= a_{33}a_{11} - a_{31}a_{13}, \\ A_{33} &= a_{11}a_{22} - a_{12}a_{21}, & A_{12} &= a_{23}a_{31} - a_{21}a_{33}, \\ A_{23} &= a_{31}a_{12} - a_{32}a_{11}, & A_{31} &= a_{12}a_{23} - a_{13}a_{22}, \\ A_{21} &= a_{32}a_{13} - a_{12}a_{33}, & A_{32} &= a_{13}a_{21} - a_{23}a_{11}, \\ A_{13} &= a_{21}a_{22} - a_{31}a_{22}, & |\mathbf{A}| &= a_{11}A_{11} + a_{12}A_{21} + a_{13}A_{31}. \end{aligned} \tag{18.15}$$

(The determinant can in fact be computed in 9 different ways.)

§18.6. The Strain Displacement Matrix

Having obtained the shape function derivatives, the matrix $\mathbf{B}$ for a hexahedron element displays the usual structure for 3D elements:

$$\mathbf{B} = \mathbf{D}\boldsymbol{\Phi} = \begin{bmatrix} \partial/\partial x & 0 & 0 \\ 0 & \partial/\partial y & 0 \\ 0 & 0 & \partial/\partial z \\ \partial/\partial y & \partial/\partial x & 0 \\ 0 & \partial/\partial z & \partial/\partial y \\ \partial/\partial z & 0 & \partial/\partial x \end{bmatrix} \begin{bmatrix} \mathbf{q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{q} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{q} \end{bmatrix} = \begin{bmatrix} \mathbf{q}_x & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{q}_y & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{q}_z \\ \mathbf{q}_y & \mathbf{q}_x & \mathbf{0} \\ \mathbf{0} & \mathbf{q}_z & \mathbf{q}_y \\ \mathbf{q}_z & \mathbf{0} & \mathbf{q}_x \end{bmatrix} \tag{18.16}$$

where

$$\mathbf{q} = [\, N_1^{(e)} \quad \cdots \quad N_n^{(e)} \,]$$
$$\mathbf{q}_x = \left[\frac{\partial N_1^{(e)}}{\partial x} \quad \ldots \quad \frac{\partial N_n^{(e)}}{\partial x} \right]$$
$$\mathbf{q}_y = \left[\frac{\partial N_1^{(e)}}{\partial y} \quad \ldots \quad \frac{\partial N_n^{(e)}}{\partial y} \right]$$
$$\mathbf{q}_z = \left[\frac{\partial N_1^{(e)}}{\partial z} \quad \ldots \quad \frac{\partial N_n^{(e)}}{\partial z} \right]$$

are row vectors of length n, n being the number of nodes in the element.

§18.7. Stiffness Matrix Evaluation

The element stiffness matrix is given by

$$\mathbf{K}^{(e)} = \int_{V^{(e)}} \mathbf{B}^T \mathbf{E} \mathbf{B} \, dV^{(e)}. \tag{18.17}$$

As in the two-dimensional case, this is replaced by a numerical integration formula which now involves a triple loop over conventional Gauss quadrature rules. Assuming that the stress-strain matrix $\mathbf{E}$ is constant over the element,

$$\mathbf{K}^{(e)} = \sum_{i=1}^{p_1} \sum_{j=1}^{p_2} \sum_{k=1}^{p_3} w_i w_j w_k \mathbf{B}_{ijk}^T \mathbf{E} \mathbf{B}_{ijk} \, J_{ik\ell}. \tag{18.18}$$

Here p_1, p_2 and p_3 are the number of Gauss points in the ξ, η and μ direction, respectively, while $\mathbf{B}_{ijk}$ and $\mathbf{J}_{ij\ell}$ are abbreviations for

$$\mathbf{B}_{ijk} \equiv \mathbf{B}(\xi_i, \eta_j, \mu_k), \qquad J_{ik\ell} \equiv \det\mathbf{J}(\xi_i, \eta_j, \mu_k). \tag{18.19}$$

Usually the number of integration points is taken the same in all directions: $p = p_1 = p_2 = p_3$. The total number of Gauss points is thus p^3. Each point adds at most 6 to the stiffness matrix rank. The minimum rank-sufficient rules for the 8-node and 20-node hexahedra are $p = 2$ and $p = 3$, respectively.

Remark 18.2. The computation of consistent node forces corresponding to body forces is straightforward. The treatment of prescribed surface tractions such as pressure, presents, however, some computational difficulties because hexahedron faces are not generally plane.

§18.7.1. Selecting the Integration Rule

Usually the number of integration points is taken the same in all directions: $p = p_1 = p_2 = p_3$. The total number of Gauss points is thus p^3. Each point adds at most 6 to the stiffness matrix rank. For the 8-node hexahedron this rule gives $p = 2$ because $2^3 \times 6 = 48 > 24 - 6$. For other configurations see Exercise 18.3.

Homework Exercises for Chapter 18
Hexahedron Elements

EXERCISE 18.1 [A:20] Find the shape functions associated with the 16-node hexahedron depicted in Figure E18.1(a) for node points 1 and 9. (This kind of element is historically important as a pit stop on the way the “degenerated solid” thick-shell elements developed in the late 1960s.)

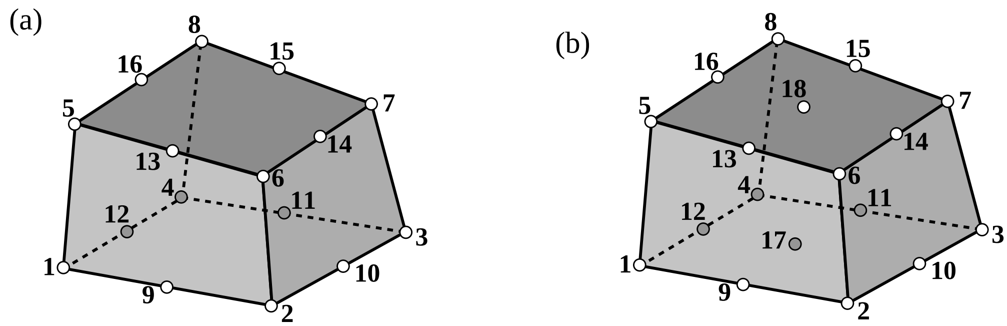

Figure E18.1. (a): 16-node hexahedron for Exercise 18.1;
(b): 18-node hexahedron for Exercise 18.2.

EXERCISE 18.2 [A:20] Find the shape functions associated with the 18-node hexahedron depicted in Figure E18.1(b) for node points 1, 9 and 17. (This node configuration is used for some solid-shell elements discussed in the last part of the course.)

EXERCISE 18.3 [A:15] Which minimum integration rules of Gauss-product type gives a rank sufficient stiffness matrix for (a) the 20-node hexahedron, (b) the 27-node hexahedron, (c) the the 16-node hexahedron of Exercise 18.1 and (d) the 18-node hexahedron of Exercise 18.2. For the last two, would a formula containing less Gauss sample points in the μ direction (for example: $3 \times 3 \times 1$, work, at least on paper?