

17

The Quadratic Tetrahedron

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§17.1. Introduction

The technique used in Chapter 15 for the derivation of the linear triangle is now extended to the 10-node tetrahedron, also called the quadratic tetrahedron. This element can have both curved faces and edges. The extension is based on the isoparametric technique and the partial-derivative construction procedure followed the IFEM course for the 6-node triangle.

§17.2. The Quadratic Tetrahedron

The ten node tetrahedron shown in Figure 17.1 is the next complete-polynomial member of the isoparametric tetrahedron family.

The element has four corners with *local* numbers 1 through 4, which must be traversed following the same convention as the four node tetrahedron. It has six side nodes, with local numbers 5 through 10. Nodes 5,6,7 are located on sides 1-2, 2-3 and 3-1, whereas nodes 8,9,10 are located on sides 1-4, 2-4, and 3-4. The side nodes may be located arbitrarily subjected to positive-Jacobian-determinant constraints. Each element face is defined by six nodes. These do not necessarily lie on a plane, but they should not deviate too much from it.

The tetrahedral natural coordinates ζ_1 through ζ_4 are introduced in a fashion similar to that described for the linear tetrahedron in the previous Chapter. The main difference is that $\zeta_i = \text{constant}$ is not necessarily the equation of a plane.

The isoparametric element definition is

$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ x_1 & x_2 & x_3 & x_4 & \dots & x_{10} \\ y_1 & y_2 & y_3 & y_4 & \dots & y_{10} \\ z_1 & z_2 & z_3 & z_4 & \dots & z_{10} \\ u_{x1} & u_{x2} & u_{x3} & u_{x4} & \dots & u_{x10} \\ u_{y1} & u_{y2} & u_{y3} & u_{y4} & \dots & u_{y10} \\ u_{z1} & u_{z2} & u_{z3} & u_{z4} & \dots & u_{z10} \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \\ N_4^e \\ \vdots \\ N_{10}^e \end{bmatrix}. \quad (17.1)$$

The conventional (non-hierarchical) shape functions are given by

$$\begin{aligned} N_1^e &= \zeta_1(2\zeta_1 - 1), & N_2^e &= \zeta_2(2\zeta_2 - 1) \\ N_3^e &= \zeta_3(2\zeta_3 - 1), & N_4^e &= \zeta_4(2\zeta_4 - 1) \\ N_5^e &= 4\zeta_1\zeta_2, & N_6^e &= 4\zeta_2\zeta_3 \\ N_7^e &= 4\zeta_3\zeta_1, & N_8^e &= 4\zeta_1\zeta_4 \\ N_9^e &= 4\zeta_2\zeta_4, & N_{10}^e &= 4\zeta_3\zeta_4 \end{aligned}. \quad (17.2)$$

These shape functions are similar in form to those of the six node quadratic triangle discussed in IFEM. If the element is curved (that is, the six nodes that define each face are not on a plane), the tetrahedron coordinates no longer fall on planes, but form a curvilinear system.

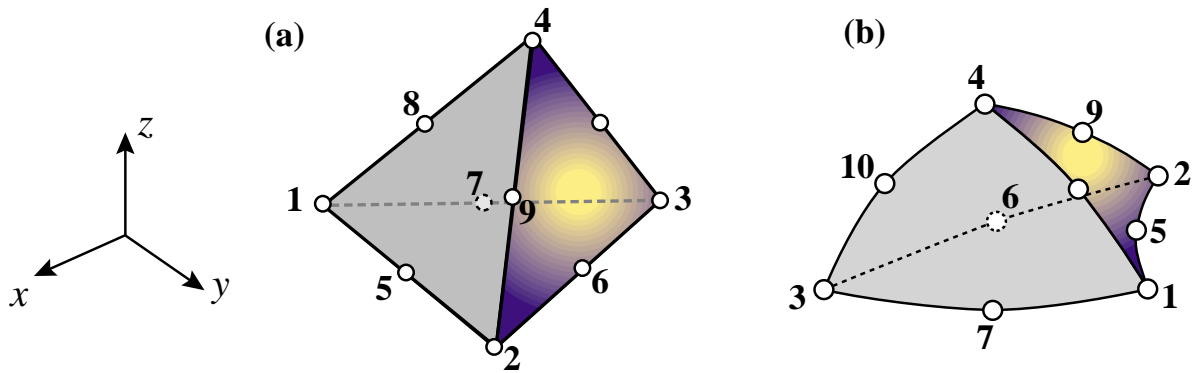


FIGURE 17.1. The ten-node (quadratic) tetrahedron. (a): element with planar faces, (b): element with curved faces.

§17.3. Partial Derivative Calculations

The main task involved in writing the shape function subroutine for the 10-node tetrahedron is the computation of the partial derivatives of the functions in (17.1) with respect to x , y and z at any point in the element. For this purpose consider a generic scalar function, $w(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$, that is quadratically interpolated over the ten-node tetrahedron with the shape functions (17.2):

$$w = [w_1 \quad w_2 \quad w_3 \quad w_4 \quad w_5 \quad w_6 \quad \dots \quad w_{10}] \begin{bmatrix} \zeta_1(2\zeta_1 - 1) \\ \zeta_2(2\zeta_2 - 1) \\ \zeta_3(2\zeta_3 - 1) \\ \zeta_4(2\zeta_4 - 1) \\ 4\zeta_1\zeta_2 \\ 4\zeta_2\zeta_3 \\ \vdots \\ 4\zeta_3\zeta_4 \end{bmatrix}. \quad (17.3)$$

Symbol w may represent x , y , z , u_x , u_y or u_z in the isoparametric representation (7.1), or other element-varying quantities such as body forces or temperatures.

Taking partials with respect to x , y and z , and applying the chain rule twice we get

$$\begin{aligned} \frac{\partial w}{\partial x} &= \sum w_i \frac{\partial N_i}{\partial x} = \sum w_i \left(\frac{\partial N_i}{\partial \zeta_1} \frac{\partial \zeta_1}{\partial x} + \frac{\partial N_i}{\partial \zeta_2} \frac{\partial \zeta_2}{\partial x} + \frac{\partial N_i}{\partial \zeta_3} \frac{\partial \zeta_3}{\partial x} + \frac{\partial N_i}{\partial \zeta_4} \frac{\partial \zeta_4}{\partial x} \right), \\ \frac{\partial w}{\partial y} &= \sum w_i \frac{\partial N_i}{\partial y} = \sum w_i \left(\frac{\partial N_i}{\partial \zeta_1} \frac{\partial \zeta_1}{\partial y} + \frac{\partial N_i}{\partial \zeta_2} \frac{\partial \zeta_2}{\partial y} + \frac{\partial N_i}{\partial \zeta_3} \frac{\partial \zeta_3}{\partial y} + \frac{\partial N_i}{\partial \zeta_4} \frac{\partial \zeta_4}{\partial y} \right), \\ \frac{\partial w}{\partial z} &= \sum w_i \frac{\partial N_i}{\partial z} = \sum w_i \left(\frac{\partial N_i}{\partial \zeta_1} \frac{\partial \zeta_1}{\partial z} + \frac{\partial N_i}{\partial \zeta_2} \frac{\partial \zeta_2}{\partial z} + \frac{\partial N_i}{\partial \zeta_3} \frac{\partial \zeta_3}{\partial z} + \frac{\partial N_i}{\partial \zeta_4} \frac{\partial \zeta_4}{\partial z} \right), \end{aligned} \quad (17.4)$$

where all sums are understood to run from $i = 1$ through 10, and element superscripts on the shape

functions have been suppressed for clarity. In matrix form:

$$\begin{bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial \zeta_1}{\partial x} & \frac{\partial \zeta_2}{\partial x} & \frac{\partial \zeta_3}{\partial x} & \frac{\partial \zeta_4}{\partial x} \\ \frac{\partial \zeta_1}{\partial y} & \frac{\partial \zeta_2}{\partial y} & \frac{\partial \zeta_3}{\partial y} & \frac{\partial \zeta_4}{\partial y} \\ \frac{\partial \zeta_1}{\partial z} & \frac{\partial \zeta_2}{\partial z} & \frac{\partial \zeta_3}{\partial z} & \frac{\partial \zeta_4}{\partial z} \end{bmatrix} \begin{bmatrix} \sum w_i \frac{\partial N_i}{\partial \zeta_1} \\ \sum w_i \frac{\partial N_i}{\partial \zeta_2} \\ \sum w_i \frac{\partial N_i}{\partial \zeta_3} \\ \sum w_i \frac{\partial N_i}{\partial \zeta_4} \end{bmatrix} \quad (17.5)$$

Transposing both sides of (17.5) while switching the left and right hand sides, yields a form exploited below:

$$\begin{bmatrix} \sum w_i \frac{\partial N_i}{\partial \zeta_1} & \sum w_i \frac{\partial N_i}{\partial \zeta_2} & \sum w_i \frac{\partial N_i}{\partial \zeta_3} & \sum w_i \frac{\partial N_i}{\partial \zeta_4} \end{bmatrix} \begin{bmatrix} \frac{\partial \zeta_1}{\partial x} & \frac{\partial \zeta_1}{\partial y} & \frac{\partial \zeta_1}{\partial z} \\ \frac{\partial \zeta_2}{\partial x} & \frac{\partial \zeta_2}{\partial y} & \frac{\partial \zeta_2}{\partial z} \\ \frac{\partial \zeta_3}{\partial x} & \frac{\partial \zeta_3}{\partial y} & \frac{\partial \zeta_3}{\partial z} \\ \frac{\partial \zeta_4}{\partial x} & \frac{\partial \zeta_4}{\partial y} & \frac{\partial \zeta_4}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix}. \quad (17.6)$$

Now make $w \equiv x, y, z$ and stack the results row-wise:

$$\begin{bmatrix} \sum x_i \frac{\partial N_i}{\partial \zeta_1} & \sum x_i \frac{\partial N_i}{\partial \zeta_2} & \sum x_i \frac{\partial N_i}{\partial \zeta_3} & \sum x_i \frac{\partial N_i}{\partial \zeta_4} \\ \sum y_i \frac{\partial N_i}{\partial \zeta_1} & \sum y_i \frac{\partial N_i}{\partial \zeta_2} & \sum y_i \frac{\partial N_i}{\partial \zeta_3} & \sum y_i \frac{\partial N_i}{\partial \zeta_4} \\ \sum z_i \frac{\partial N_i}{\partial \zeta_1} & \sum z_i \frac{\partial N_i}{\partial \zeta_2} & \sum z_i \frac{\partial N_i}{\partial \zeta_3} & \sum z_i \frac{\partial N_i}{\partial \zeta_4} \end{bmatrix} \begin{bmatrix} \frac{\partial \zeta_1}{\partial x} & \frac{\partial \zeta_1}{\partial y} & \frac{\partial \zeta_1}{\partial z} \\ \frac{\partial \zeta_2}{\partial x} & \frac{\partial \zeta_2}{\partial y} & \frac{\partial \zeta_2}{\partial z} \\ \frac{\partial \zeta_3}{\partial x} & \frac{\partial \zeta_3}{\partial y} & \frac{\partial \zeta_3}{\partial z} \\ \frac{\partial \zeta_4}{\partial x} & \frac{\partial \zeta_4}{\partial y} & \frac{\partial \zeta_4}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} & \frac{\partial z}{\partial z} \end{bmatrix}. \quad (17.7)$$

This is a linear system with the required unknowns in the second matrix, but its coefficient matrix is not square. To achieve that, differentiate both sides of the identity $\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 = 1$ with respect to x, y and z , and insert as first row:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ \sum x_i \frac{\partial N_i}{\partial \zeta_1} & \sum x_i \frac{\partial N_i}{\partial \zeta_2} & \sum x_i \frac{\partial N_i}{\partial \zeta_3} & \sum x_i \frac{\partial N_i}{\partial \zeta_4} \\ \sum y_i \frac{\partial N_i}{\partial \zeta_1} & \sum y_i \frac{\partial N_i}{\partial \zeta_2} & \sum y_i \frac{\partial N_i}{\partial \zeta_3} & \sum y_i \frac{\partial N_i}{\partial \zeta_4} \\ \sum z_i \frac{\partial N_i}{\partial \zeta_1} & \sum z_i \frac{\partial N_i}{\partial \zeta_2} & \sum z_i \frac{\partial N_i}{\partial \zeta_3} & \sum z_i \frac{\partial N_i}{\partial \zeta_4} \end{bmatrix} \begin{bmatrix} \frac{\partial \zeta_1}{\partial x} & \frac{\partial \zeta_1}{\partial y} & \frac{\partial \zeta_1}{\partial z} \\ \frac{\partial \zeta_2}{\partial x} & \frac{\partial \zeta_2}{\partial y} & \frac{\partial \zeta_2}{\partial z} \\ \frac{\partial \zeta_3}{\partial x} & \frac{\partial \zeta_3}{\partial y} & \frac{\partial \zeta_3}{\partial z} \\ \frac{\partial \zeta_4}{\partial x} & \frac{\partial \zeta_4}{\partial y} & \frac{\partial \zeta_4}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial 1}{\partial x} & \frac{\partial 1}{\partial y} & \frac{\partial 1}{\partial z} \\ \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} & \frac{\partial z}{\partial z} \end{bmatrix}. \quad (17.8)$$

But $\partial x/\partial x = \partial y/\partial y = \partial z/\partial z = 1$ and $\partial 1/\partial x = \partial 1/\partial y = \partial 1/\partial z = \partial x/\partial y = \partial x/\partial z = \partial y/\partial x = \partial y/\partial z = \partial z/\partial x = \partial z/\partial y = 0$ because x, y and z are independent coordinates. Consequently we arrive at a system of linear equations of order 4 with three right-hand sides. Using the summation

convention to get rid of sum symbols the system is

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ x_i \frac{\partial N_i}{\partial \zeta_1} & x_i \frac{\partial N_i}{\partial \zeta_2} & x_i \frac{\partial N_i}{\partial \zeta_3} & x_i \frac{\partial N_i}{\partial \zeta_4} \\ y_i \frac{\partial N_i}{\partial \zeta_1} & y_i \frac{\partial N_i}{\partial \zeta_2} & y_i \frac{\partial N_i}{\partial \zeta_3} & y_i \frac{\partial N_i}{\partial \zeta_4} \\ z_i \frac{\partial N_i}{\partial \zeta_1} & z_i \frac{\partial N_i}{\partial \zeta_2} & z_i \frac{\partial N_i}{\partial \zeta_3} & z_i \frac{\partial N_i}{\partial \zeta_4} \end{bmatrix} \begin{bmatrix} \frac{\partial \zeta_1}{\partial x} & \frac{\partial \zeta_1}{\partial y} & \frac{\partial \zeta_1}{\partial z} \\ \frac{\partial \zeta_2}{\partial x} & \frac{\partial \zeta_2}{\partial y} & \frac{\partial \zeta_2}{\partial z} \\ \frac{\partial \zeta_3}{\partial x} & \frac{\partial \zeta_3}{\partial y} & \frac{\partial \zeta_3}{\partial z} \\ \frac{\partial \zeta_4}{\partial x} & \frac{\partial \zeta_4}{\partial y} & \frac{\partial \zeta_4}{\partial z} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (17.9)$$

In compact matrix notation

$$\mathbf{JP} = \mathbf{I}_{aug}. \quad (17.10)$$

where

$$\mathbf{J} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ J_{x1} & J_{x2} & J_{x3} & J_{x4} \\ J_{y1} & J_{y2} & J_{y3} & J_{y4} \\ J_{z1} & J_{z2} & J_{z3} & J_{z4} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_i \frac{\partial N_i}{\partial \zeta_1} & x_i \frac{\partial N_i}{\partial \zeta_2} & x_i \frac{\partial N_i}{\partial \zeta_3} & x_i \frac{\partial N_i}{\partial \zeta_4} \\ y_i \frac{\partial N_i}{\partial \zeta_1} & y_i \frac{\partial N_i}{\partial \zeta_2} & y_i \frac{\partial N_i}{\partial \zeta_3} & y_i \frac{\partial N_i}{\partial \zeta_4} \\ z_i \frac{\partial N_i}{\partial \zeta_1} & z_i \frac{\partial N_i}{\partial \zeta_2} & z_i \frac{\partial N_i}{\partial \zeta_3} & z_i \frac{\partial N_i}{\partial \zeta_4} \end{bmatrix}. \quad (17.11)$$

and \mathbf{I}_{aug} is the 3×3 identity matrix augmented with a zero first row. Taking the partials of the shape functions (17.2) with respect to the tetrahedron coordinates and substituting into the above yields

$$\begin{aligned} J_{x1} &= x_1(4\zeta_1 - 1) + 4x_5\zeta_2 + 4x_7\zeta_3 + 4x_8\zeta_4, \\ J_{x2} &= x_2(4\zeta_2 - 1) + 4x_6\zeta_3 + 4x_5\zeta_1 + 4x_9\zeta_4, \\ J_{x3} &= x_3(4\zeta_3 - 1) + 4x_7\zeta_1 + 4x_6\zeta_2 + 4x_{10}\zeta_4, \\ J_{x4} &= x_4(4\zeta_4 - 1) + 4x_8\zeta_1 + 4x_9\zeta_2 + 4x_{10}\zeta_3. \end{aligned} \quad (17.12)$$

For J_{yi} and J_{zi} replace x_i by y_i and z_i , respectively, into the above formulas.

Summarizing, to compute the x - y - z partials for a function w interpolated as per (17.3) the recipe is: form the linear system (17.10) from the geometric data, solve for the 12 tetrahedron coordinates partials, and substitute these into (17.5).

Remark 17.1. By analogy with the isoparametric brick elements, matrix \mathbf{J} of (17.10) may be called a Jacobian matrix. However, the factor J that appears in the element-of-volume transformation

$$dV^e = J d\zeta_1 d\zeta_2 d\zeta_3 d\zeta_4, \quad (17.13)$$

is not $\det \mathbf{J} = |\mathbf{J}|$, but $J = \frac{1}{6}|\mathbf{J}|$. If the element has straight sides with side nodes at the midpoints, J is constant and equal to the volume \mathcal{V} of the tetrahedron, which is given by the usual formula

$$J = \mathcal{V} = \frac{1}{6} \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}, \quad (17.14)$$

as in Chapter 6. In this case J is constant over the element.

§17.3.1. Implementation Considerations

To speed up computations in the shape function subroutine it is important to observe that the Jacobian matrix has the special structure (17.11). We can take advantage of this by subtracting the first column of \mathbf{J} from the last three columns. This manipulation reduces (17.10) to the solution of a 3×3 linear system:

$$\begin{bmatrix} J_{x2} - J_{x1} & J_{x3} - J_{x1} & J_{x4} - J_{x1} \\ J_{y2} - J_{y1} & J_{y3} - J_{y1} & J_{y4} - J_{y1} \\ J_{z2} - J_{z1} & J_{z3} - J_{z1} & J_{z4} - J_{z1} \end{bmatrix} \begin{bmatrix} \frac{\partial(\zeta_2 - \zeta_1)}{\partial x} & \frac{\partial(\zeta_2 - \zeta_1)}{\partial y} & \frac{\partial(\zeta_2 - \zeta_1)}{\partial z} \\ \frac{\partial(\zeta_3 - \zeta_1)}{\partial x} & \frac{\partial(\zeta_3 - \zeta_1)}{\partial y} & \frac{\partial(\zeta_3 - \zeta_1)}{\partial z} \\ \frac{\partial(\zeta_4 - \zeta_1)}{\partial x} & \frac{\partial(\zeta_4 - \zeta_1)}{\partial y} & \frac{\partial(\zeta_4 - \zeta_1)}{\partial z} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (17.15)$$

or

$$\bar{\mathbf{J}}\bar{\mathbf{P}} = \mathbf{I} \quad (17.16)$$

where \mathbf{I} is the 3×3 identity matrix, and the modified Jacobian matrix is shown on the left hand side. Therefore $\bar{\mathbf{P}}$ is just the inverse of $\bar{\mathbf{J}}$, which can be readily calculated by Cramer's rule. Finally, the first row of \mathbf{P} is recovered from the constraints

$$\frac{\partial\zeta_1}{\partial x} + \frac{\partial\zeta_2}{\partial x} + \frac{\partial\zeta_3}{\partial x} + \frac{\partial\zeta_4}{\partial x} = 0, \quad (17.17)$$

and similarly for the y and z partials. This is done as follows: take the column sum s_x of the computed first column of $\bar{\mathbf{P}}$; then

$$\frac{\partial\zeta_1}{\partial x} = -\frac{1}{4}s_x, \quad (17.18)$$

from which the other x can be recovered. This operation is repeated over the other columns.

To tie up with the notation used in Chapter 6 for the four-node tetrahedron, we may denote the partials as

$$\begin{bmatrix} \frac{\partial\zeta_1}{\partial x} & \frac{\partial\zeta_1}{\partial y} & \frac{\partial\zeta_1}{\partial z} \\ \frac{\partial\zeta_2}{\partial x} & \frac{\partial\zeta_2}{\partial y} & \frac{\partial\zeta_2}{\partial z} \\ \frac{\partial\zeta_3}{\partial x} & \frac{\partial\zeta_3}{\partial y} & \frac{\partial\zeta_3}{\partial z} \\ \frac{\partial\zeta_4}{\partial x} & \frac{\partial\zeta_4}{\partial y} & \frac{\partial\zeta_4}{\partial z} \end{bmatrix} = \frac{1}{J} \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{bmatrix} \quad (17.19)$$

For the 4-node tetrahedron, $J = \frac{1}{6}\mathcal{V}$ was constant over the element, but now it generally will vary with position unless the tetrahedron has planar faces.

§17.4. Gauss Rules over Tetrahedra

We mention the first two numerical integration rules, which find applications in the evaluation of element stiffness matrix and consistent force vector.

One point rule (exact for constant and linear polynomials over plane-face tetrahedra):

$$\frac{1}{\mathcal{V}} \int_{V^e} F(\zeta_1, \zeta_2, \zeta_3, \zeta_4) dV^e \approx F\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right). \quad (17.20)$$

Four-point rule (exact for constant through quadratic polynomials over plane-face tetrahedra):

$$\frac{1}{\mathcal{V}} \int_{V^e} F(\zeta_1, \zeta_2, \zeta_3, \zeta_4) dV^e \approx \frac{1}{4}F(\alpha, \beta, \beta, \beta) + \frac{1}{4}F(\beta, \alpha, \beta, \beta) + \frac{1}{4}F(\beta, \beta, \alpha, \beta) + \frac{1}{4}F(\beta, \beta, \beta, \alpha), \quad (17.21)$$

in which $\alpha = (5 + 3\sqrt{5})/20 = 0.58541020$, $\beta = (5 - \sqrt{5})/20 = 0.13819660$. More details on this and other integration rules are posted in Chapter 17: A Compendium of Gauss Integration Rules for FEM.

§17.5. The Element Stiffness Matrix

In three dimensional elasticity this element has $10 \times 3 = 30$ degrees of freedom. For the following derivations we assume that they are arranged as

$$\mathbf{u}^e = [u_{x1} \ u_{y1} \ u_{z1} \ u_{x2} \ u_{y2} \ u_{z2} \ \dots \ u_{x10} \ u_{y10} \ u_{z10}]^T. \quad (17.22)$$

The 6×30 strain-displacement matrix for the 10-node tetrahedron is

$$\mathbf{B} = \begin{bmatrix} q_{x1} & q_{x2} & \dots & q_{x1} & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & q_{y1} & q_{y2} & \dots & q_{y10} & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & q_{z1} & q_{z2} & \dots & q_{z10} \\ q_{y1} & q_{y2} & \dots & q_{y10} & q_{x1} & q_{x2} & \dots & q_{x10} & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & q_{z1} & q_{z2} & \dots & q_{z10} & q_{y1} & q_{y2} & \dots & q_{y10} \\ q_{x1} & q_{x2} & \dots & q_{x10} & 0 & 0 & \dots & 0 & q_{z1} & q_{z2} & \dots & q_{z10} \end{bmatrix}. \quad (17.23)$$

where (summation convention implied on $j = 1, 2, 3, 4$):

$$\begin{aligned} q_{xi} &= \frac{\partial N_i}{\partial \zeta_j} \frac{\partial \zeta_j}{\partial x} = J^{-1} \frac{\partial N_i}{\partial \zeta_j} a_j, \\ q_{yi} &= \frac{\partial N_i}{\partial \zeta_j} \frac{\partial \zeta_j}{\partial y} = J^{-1} \frac{\partial N_i}{\partial \zeta_j} b_j, \\ q_{zi} &= \frac{\partial N_i}{\partial \zeta_j} \frac{\partial \zeta_j}{\partial z} = J^{-1} \frac{\partial N_i}{\partial \zeta_j} c_j, \end{aligned} \quad (17.24)$$

The stiffness matrix \mathbf{K}^e is evaluated by numerical integration

$$\mathbf{K}^e = \sum_{k=1}^p w_k \mathbf{B}^T(\zeta_{ik}) \mathbf{E} \mathbf{B}(\zeta_{ik}) J(\zeta_{ik}) \quad (17.25)$$

where p is the number of Gauss points, (ζ_{ik}) denotes the coordinate quartet $(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ at the k^{th} integration point, and w_k are the corresponding weights. The stress-strain matrix \mathbf{E} is the same as in Chapter 6. The four-point rule integrates this element with the correct rank.

§17.6. The Consistent Node Force Vector

Consider a body force field over the element defined by its components

$$\mathbf{b} = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}. \quad (17.26)$$

The consistent node force vector is given by

$$\mathbf{f}^e = \int_{V^e} \mathbf{N}^T \mathbf{b} dV^e, \quad (17.27)$$

where \mathbf{N} is the 3×30 matrix of shape functions that relates element displacements to node displacements:

$$\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \mathbf{N}\mathbf{u}. \quad (17.28)$$

For this element and the node displacement ordering (17.22)

$$\mathbf{N} = \begin{bmatrix} N_1 & \dots & N_{10} & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & N_1 & \dots & N_{10} & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & N_1 & \dots & N_{10} \end{bmatrix}, \quad (17.29)$$

where the shape functions are given by (17.2).

Homework Exercises for Chapter 17**The Quadratic Tetrahedron**

EXERCISE 17.1 [A:15] The 10-node tetrahedron element is converted into an 11-point tetrahedron by adding node point 11 located at the centroid $\zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = 1/4$. What is the shape function N_{11}^e ? (You do not need to write the full element definition).

EXERCISE 17.2 [A:15] The next full-polynomial, isoparametric member of the tetrahedron family is the cubic tetrahedron, which has 21 node points. Where do you think the nodes are located?

EXERCISE 17.3 [A/C:25] Derive the shape functions for the 21-node tetrahedron.

EXERCISE 17.4 [A:15] Justify the rule (17.18).

EXERCISE 17.5 [A/C:20] Compute \mathbf{f}^e for a straight-face 10-node tetrahedron if the body forces $b_x = b_y = 0$ and b_z is constant, using the 4-point rule (17.21) to evaluate the integral in (17.27). (You need to give only the z force components).