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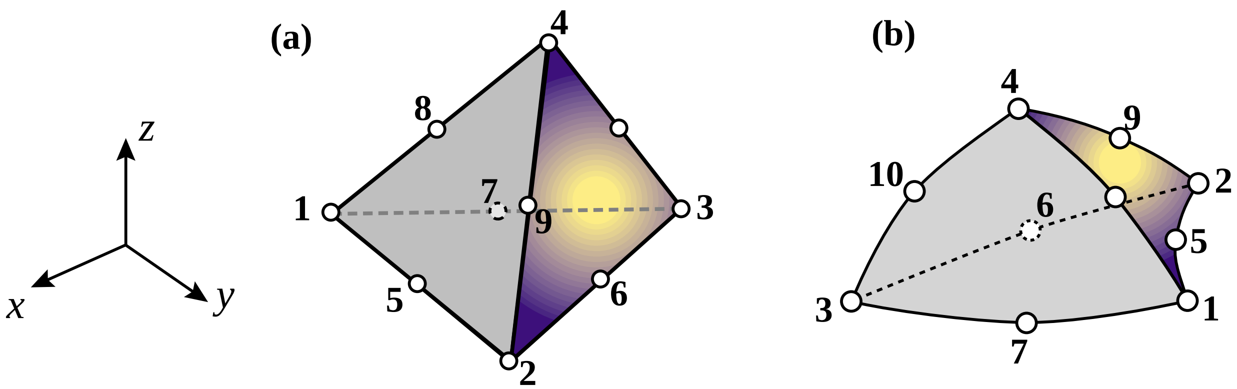
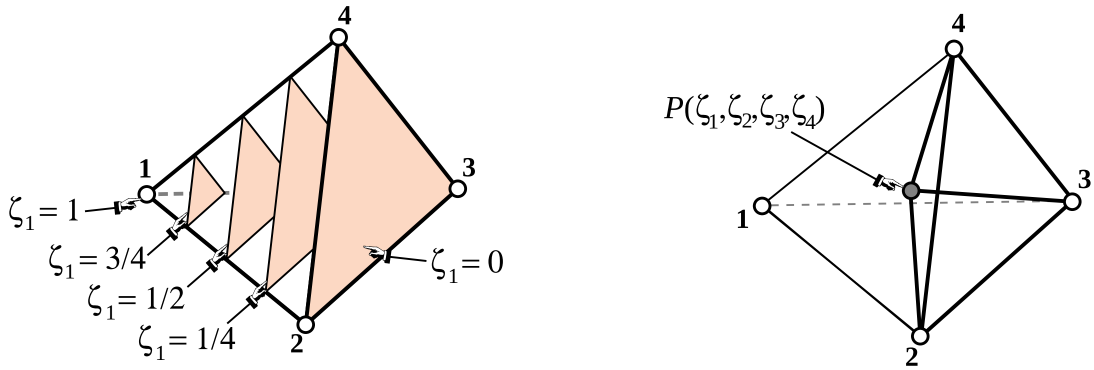
The Quadratic Tetrahedron

The Quadratic (10-Node) Tetrahedron



May have curved edges and faces, as pictured in (b)

Tetrahedron Natural Coordinates same as in 4-node tetrahedron:



interpretation as a moving surface

interpretation as volume ratio

OK, except surface may be curved

No longer valid if element has curved edges and faces

Iso-P Definition of Quadratic Tetrahedron

$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ x_1 & x_2 & x_3 & x_4 & \dots & x_{10} \\ y_1 & y_2 & y_3 & y_4 & \dots & y_{10} \\ z_1 & z_2 & z_3 & z_4 & \dots & z_{10} \\ u_{x1} & u_{x2} & u_{x3} & u_{x4} & \dots & u_{x10} \\ u_{y1} & u_{y2} & u_{y3} & u_{y4} & \dots & u_{y10} \\ u_{z1} & u_{z2} & u_{z3} & u_{z4} & \dots & u_{z10} \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \\ N_4^e \\ \vdots \\ N_{10}^e \end{bmatrix}$$

with the shape functions

$$\begin{aligned} N_1^e &= \zeta_1(2\zeta_1 - 1); & N_2^e &= \zeta_2(2\zeta_2 - 1) \\ N_3^e &= \zeta_3(2\zeta_3 - 1); & N_4^e &= \zeta_4(2\zeta_4 - 1) \\ N_5^e &= 4\zeta_1\zeta_2, & N_6^e &= 4\zeta_2\zeta_3 \\ N_7^e &= 4\zeta_3\zeta_1, & N_8^e &= 4\zeta_1\zeta_4 \\ N_9^e &= 4\zeta_2\zeta_4, & N_{10}^e &= 4\zeta_3\zeta_4 \end{aligned}$$

Partial Derivative Computation

Interpolate an arbitrary function *w* in terms of node values:

$$w = [\, w_1 \quad w_2 \quad w_3 \quad w_4 \quad w_5 \quad w_6 \quad \ldots \quad w_{10} \,] \begin{bmatrix} \zeta_1(2\zeta_1 - 1) \\ \zeta_2(2\zeta_2 - 1) \\ \zeta_3(2\zeta_3 - 1) \\ \zeta_4(2\zeta_4 - 1) \\ 4\zeta_1\zeta_2 \\ 4\zeta_2\zeta_3 \\ \vdots \\ 4\zeta_3\zeta_4 \end{bmatrix}$$

Using the chain rule (sum over *i* =1 to 10)

$$\frac{\partial w}{\partial x} = \sum w_i \frac{\partial N_i}{\partial x} = \sum w_i \left(\frac{\partial N_i}{\partial \zeta_1}\frac{\partial \zeta_1}{\partial x} + \frac{\partial N_i}{\partial \zeta_2}\frac{\partial \zeta_2}{\partial x} + \frac{\partial N_i}{\partial \zeta_3}\frac{\partial \zeta_3}{\partial x} + \frac{\partial N_i}{\partial \zeta_4}\frac{\partial \zeta_4}{\partial x} \right)$$

$$\frac{\partial w}{\partial y} = \sum w_i \frac{\partial N_i}{\partial y} = \sum w_i \left(\frac{\partial N_i}{\partial \zeta_1}\frac{\partial \zeta_1}{\partial y} + \frac{\partial N_i}{\partial \zeta_2}\frac{\partial \zeta_2}{\partial y} + \frac{\partial N_i}{\partial \zeta_3}\frac{\partial \zeta_3}{\partial y} + \frac{\partial N_i}{\partial \zeta_4}\frac{\partial \zeta_4}{\partial x} \right)$$

$$\frac{\partial w}{\partial z} = \sum w_i \frac{\partial N_i}{\partial z} = \sum w_i \left(\frac{\partial N_i}{\partial \zeta_1}\frac{\partial \zeta_1}{\partial z} + \frac{\partial N_i}{\partial \zeta_2}\frac{\partial \zeta_2}{\partial z} + \frac{\partial N_i}{\partial \zeta_3}\frac{\partial \zeta_3}{\partial z} + \frac{\partial N_i}{\partial \zeta_4}\frac{\partial \zeta_4}{\partial x} \right)$$

Partial Derivative Computation (cont'd)

Reorganize in matrix form

$$\begin{bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial \zeta_1}{\partial x} & \frac{\partial \zeta_2}{\partial x} & \frac{\partial \zeta_3}{\partial x} & \frac{\partial \zeta_4}{\partial x} \\ \frac{\partial \zeta_1}{\partial y} & \frac{\partial \zeta_2}{\partial y} & \frac{\partial \zeta_3}{\partial y} & \frac{\partial \zeta_4}{\partial y} \\ \frac{\partial \zeta_1}{\partial z} & \frac{\partial \zeta_2}{\partial z} & \frac{\partial \zeta_3}{\partial z} & \frac{\partial \zeta_4}{\partial z} \end{bmatrix} \begin{bmatrix} \sum w_i \frac{\partial N_i}{\partial \zeta_1} \\ \sum w_i \frac{\partial N_i}{\partial \zeta_2} \\ \sum w_i \frac{\partial N_i}{\partial \zeta_3} \\ \sum w_i \frac{\partial N_i}{\partial \zeta_4} \end{bmatrix}$$

Transpose both sides

$$\begin{bmatrix} \sum w_i \frac{\partial N_i}{\partial \zeta_1} & \sum w_i \frac{\partial N_i}{\partial \zeta_2} & \sum w_i \frac{\partial N_i}{\partial \zeta_3} & \sum \frac{\partial N_i}{\partial \zeta_4} \end{bmatrix} \begin{bmatrix} \frac{\partial \zeta_1}{\partial x} & \frac{\partial \zeta_1}{\partial y} & \frac{\partial \zeta_1}{\partial z} \\ \frac{\partial \zeta_2}{\partial x} & \frac{\partial \zeta_2}{\partial y} & \frac{\partial \zeta_2}{\partial z} \\ \frac{\partial \zeta_3}{\partial x} & \frac{\partial \zeta_3}{\partial y} & \frac{\partial \zeta_3}{\partial z} \\ \frac{\partial \zeta_4}{\partial x} & \frac{\partial \zeta_4}{\partial y} & \frac{\partial \zeta_4}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix}$$

Partial Derivative Computation (cont'd)

Make $w = x,y,z$ and stack row-wise

$$\begin{bmatrix} \sum x_i \frac{\partial N_i}{\partial \zeta_1} & \sum x_i \frac{\partial N_i}{\partial \zeta_2} & \sum x_i \frac{\partial N_i}{\partial \zeta_3} & \sum x_i \frac{\partial N_i}{\partial \zeta_4} \\ \sum y_i \frac{\partial N_i}{\partial \zeta_1} & \sum y_i \frac{\partial N_i}{\partial \zeta_2} & \sum y_i \frac{\partial N_i}{\partial \zeta_3} & \sum y_i \frac{\partial N_i}{\partial \zeta_4} \\ \sum z_i \frac{\partial N_i}{\partial \zeta_1} & \sum z_i \frac{\partial N_i}{\partial \zeta_2} & \sum z_i \frac{\partial N_i}{\partial \zeta_3} & \sum z_i \frac{\partial N_i}{\partial \zeta_4} \end{bmatrix} \begin{bmatrix} \frac{\partial \zeta_1}{\partial x} & \frac{\partial \zeta_1}{\partial y} & \frac{\partial \zeta_1}{\partial z} \\ \frac{\partial \zeta_2}{\partial x} & \frac{\partial \zeta_2}{\partial y} & \frac{\partial \zeta_2}{\partial z} \\ \frac{\partial \zeta_3}{\partial x} & \frac{\partial \zeta_3}{\partial y} & \frac{\partial \zeta_3}{\partial z} \\ \frac{\partial \zeta_4}{\partial x} & \frac{\partial \zeta_4}{\partial y} & \frac{\partial \zeta_4}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} & \frac{\partial z}{\partial z} \end{bmatrix}$$

Coefficient matrix is not square. To make it so differentiate both sides of the identity $\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 = 1$ with respect to *x*, *y* and *z* and insert as first row:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ \sum x_i \frac{\partial N_i}{\partial \zeta_1} & \sum x_i \frac{\partial N_i}{\partial \zeta_2} & \sum x_i \frac{\partial N_i}{\partial \zeta_3} & \sum x_i \frac{\partial N_i}{\partial \zeta_4} \\ \sum y_i \frac{\partial N_i}{\partial \zeta_1} & \sum y_i \frac{\partial N_i}{\partial \zeta_2} & \sum y_i \frac{\partial N_i}{\partial \zeta_3} & \sum y_i \frac{\partial N_i}{\partial \zeta_4} \\ \sum z_i \frac{\partial N_i}{\partial \zeta_1} & \sum z_i \frac{\partial N_i}{\partial \zeta_2} & \sum z_i \frac{\partial N_i}{\partial \zeta_3} & \sum z_i \frac{\partial N_i}{\partial \zeta_4} \end{bmatrix} \begin{bmatrix} \frac{\partial \zeta_1}{\partial x} & \frac{\partial \zeta_1}{\partial y} & \frac{\partial \zeta_1}{\partial z} \\ \frac{\partial \zeta_2}{\partial x} & \frac{\partial \zeta_2}{\partial y} & \frac{\partial \zeta_2}{\partial z} \\ \frac{\partial \zeta_3}{\partial x} & \frac{\partial \zeta_3}{\partial y} & \frac{\partial \zeta_3}{\partial z} \\ \frac{\partial \zeta_4}{\partial x} & \frac{\partial \zeta_4}{\partial y} & \frac{\partial \zeta_4}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial 1}{\partial x} & \frac{\partial 1}{\partial y} & \frac{\partial 1}{\partial z} \\ \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} & \frac{\partial z}{\partial z} \end{bmatrix}$$

Partial Derivative Computation (cont'd)

Since x,y,z are independent variables,

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ x_i \frac{\partial N_i}{\partial \zeta_1} & x_i \frac{\partial N_i}{\partial \zeta_2} & x_i \frac{\partial N_i}{\partial \zeta_3} & x_i \frac{\partial N_i}{\partial \zeta_4} \\ y_i \frac{\partial N_i}{\partial \zeta_1} & y_i \frac{\partial N_i}{\partial \zeta_2} & y_i \frac{\partial N_i}{\partial \zeta_3} & y_i \frac{\partial N_i}{\partial \zeta_4} \\ z_i \frac{\partial N_i}{\partial \zeta_1} & z_i \frac{\partial N_i}{\partial \zeta_2} & z_i \frac{\partial N_i}{\partial \zeta_3} & z_i \frac{\partial N_i}{\partial \zeta_4} \end{bmatrix} \begin{bmatrix} \frac{\partial \zeta_1}{\partial x} & \frac{\partial \zeta_1}{\partial y} & \frac{\partial \zeta_1}{\partial z} \\ \frac{\partial \zeta_2}{\partial x} & \frac{\partial \zeta_2}{\partial y} & \frac{\partial \zeta_2}{\partial z} \\ \frac{\partial \zeta_3}{\partial x} & \frac{\partial \zeta_3}{\partial y} & \frac{\partial \zeta_3}{\partial z} \\ \frac{\partial \zeta_4}{\partial x} & \frac{\partial \zeta_4}{\partial y} & \frac{\partial \zeta_4}{\partial z} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In compact matrix form

$$\mathbf{J}\,\mathbf{P} = \mathbf{I}_{aug}$$

in which

$$\mathbf{J} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ J_{x1} & J_{x2} & J_{x3} & J_{x4} \\ J_{y1} & J_{y2} & J_{y3} & J_{y4} \\ J_{z1} & J_{z2} & J_{z3} & J_{z4} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_i \frac{\partial N_i}{\partial \zeta_1} & x_i \frac{\partial N_i}{\partial \zeta_2} & x_i \frac{\partial N_i}{\partial \zeta_3} & x_i \frac{\partial N_i}{\partial \zeta_4} \\ y_i \frac{\partial N_i}{\partial \zeta_1} & y_i \frac{\partial N_i}{\partial \zeta_2} & y_i \frac{\partial N_i}{\partial \zeta_3} & y_i \frac{\partial N_i}{\partial \zeta_4} \\ z_i \frac{\partial N_i}{\partial \zeta_1} & z_i \frac{\partial N_i}{\partial \zeta_2} & z_i \frac{\partial N_i}{\partial \zeta_3} & z_i \frac{\partial N_i}{\partial \zeta_4} \end{bmatrix}$$

Partial Derivative Computation (cont'd)

Inserting the shape functions one computes

$$J_{x1} = x_1(4\zeta_1 - 1) + 4x_5\zeta_2 + 4x_7\zeta_3 + 4x_8\zeta_4$$
$$J_{x2} = x_2(4\zeta_2 - 1) + 4x_6\zeta_3 + 4x_5\zeta_1 + 4x_9\zeta_4$$
$$J_{x3} = x_3(4\zeta_3 - 1) + 4x_7\zeta_1 + 4x_6\zeta_2 + 4x_{10}\zeta_4$$
$$J_{x4} = x_4(4\zeta_4 - 1) + 4x_8\zeta_1 + 4x_9\zeta_2 + 4x_{10}\zeta_3$$

and likewise for *y* and *z*. By analogy with isoP brick elements the matrix J above may be called a Jacobian matrix. The factor J that appears in the element of volume transformation

$$dV^e = J\, d\zeta_1\, d\zeta_2\, d\zeta_3\, d\zeta_4$$

is not det **J** but (det **J**)/6. If the element has straight edges and with side nodes at the midpoints *J* is constant and equal to the tetrahedron volume

$$J = \mathcal{V} = \frac{1}{6}\begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}$$

Partial Derivative Computation (cont'd)

Implementation considerations: subtract off first column from other 3 to get a 3 x 3 system

$$\begin{bmatrix} J_{x2}-J_{x1} & J_{x3}-J_{x1} & J_{x4}-J_{x1} \\ J_{y2}-J_{y1} & J_{y3}-J_{y1} & J_{y4}-J_{y1} \\ J_{z2}-J_{z1} & J_{z3}-J_{z1} & J_{z4}-J_{z1} \end{bmatrix} \begin{bmatrix} \frac{\partial(\zeta_2-\zeta_1)}{\partial x} & \frac{\partial(\zeta_2-\zeta_1)}{\partial y} & \frac{\partial(\zeta_2-\zeta_1)}{\partial z} \\ \frac{\partial(\zeta_3-\zeta_1)}{\partial x} & \frac{\partial(\zeta_3-\zeta_1)}{\partial y} & \frac{\partial(\zeta_3-\zeta_1)}{\partial z} \\ \frac{\partial(\zeta_4-\zeta_1)}{\partial x} & \frac{\partial(\zeta_4-\zeta_1)}{\partial y} & \frac{\partial(\zeta_4-\zeta_1)}{\partial z} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or

$$\bar{\mathbf{J}}\,\bar{\mathbf{P}} = \mathbf{I}$$

Inversion of $\bar{\mathbf{J}}$ gives the partial derivatives sought.

Numerical Integration Over Tetrahedra

One point rule (degree 1)

$$\frac{1}{\mathcal{V}}\int_{V^e} F(\zeta_1,\zeta_2,\zeta_3,\zeta_4)\,dV \approx F(\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{4})$$

Four point rule (degree 2)

$$\frac{1}{\mathcal{V}}\int_{V^e} F(\zeta_1,\zeta_2,\zeta_3,\zeta_4)\,dV \approx \tfrac{1}{4}F(\alpha,\beta,\beta,\beta)+\tfrac{1}{4}F(\beta,\alpha,\beta,\beta) +\tfrac{1}{4}F(\beta,\beta,\alpha,\beta)+\tfrac{1}{4}F(\beta,\beta,\beta,\alpha)$$

where $\alpha = 0.58541020$, $\beta = 0.13819660$

More details on integration rules in Chapter 17, which gives a complete compendium for FEM work.

Element Stiffness Matrix

Evaluate by numerical quadrature

$$\mathbf{K}^e = \sum_{k=1}^{p} w_k \, \mathbf{B}^T \, \mathbf{E} \, \mathbf{B} \, J$$

in which $\mathbf{B}$ and J are evaluated at the k-th integration point

Element Consistent Node Force Vector

For a body force **b** such as gravity:

$$\mathbf{b} = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} \qquad \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \mathbf{N}\mathbf{u}$$

$$\mathbf{f}^e = \int_{V^e} \mathbf{N}^T \mathbf{b} \, dV$$

which may also be treated by numerical quadrature