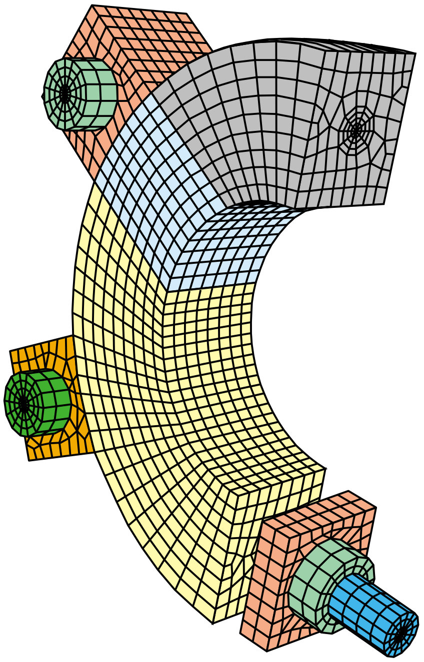
# 15

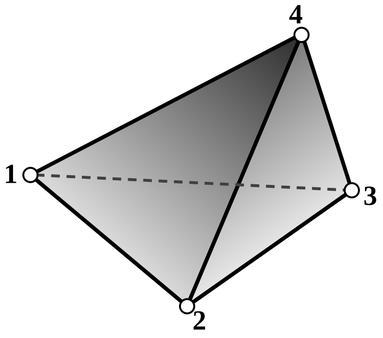
# Solid Elements: Overview

# One advantage of solid elements is that FEM models look like the real thing ...

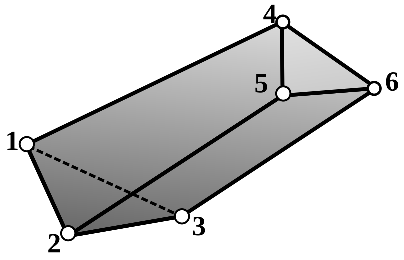


# or like the real McCow

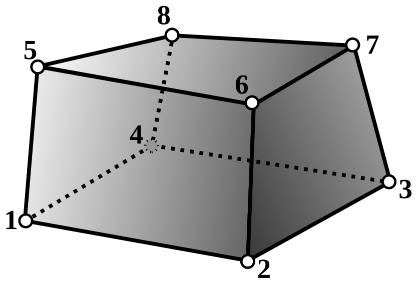
# Standard Solid Elements
# (Corner Nodes Only)



tetrahedron
aka "tet"

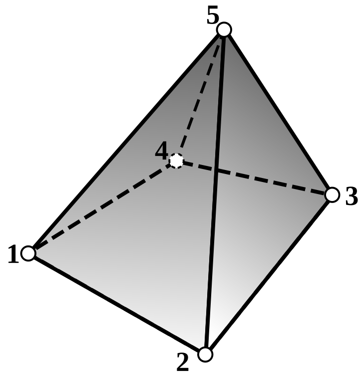


pentahedron
aka "wedge"

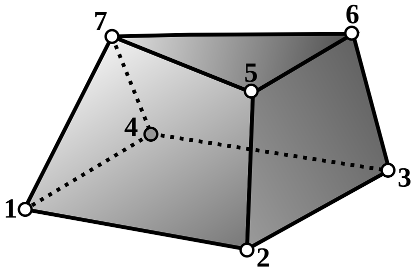


hexahedron
aka "brick"

# Nonstandard Solid Elements (Corner Nodes Only)

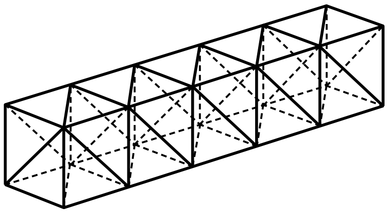


**Pyramid**

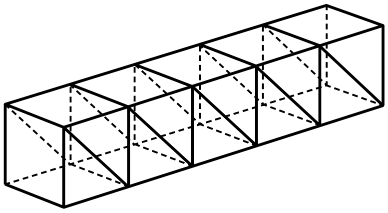


**Wrick**

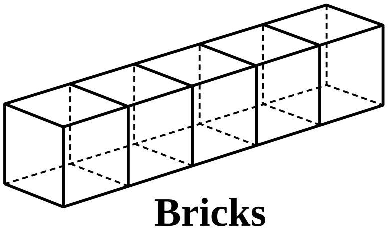
# The Standard Geometries Can Be Used for Repeating Meshes



**Tetrahedra**

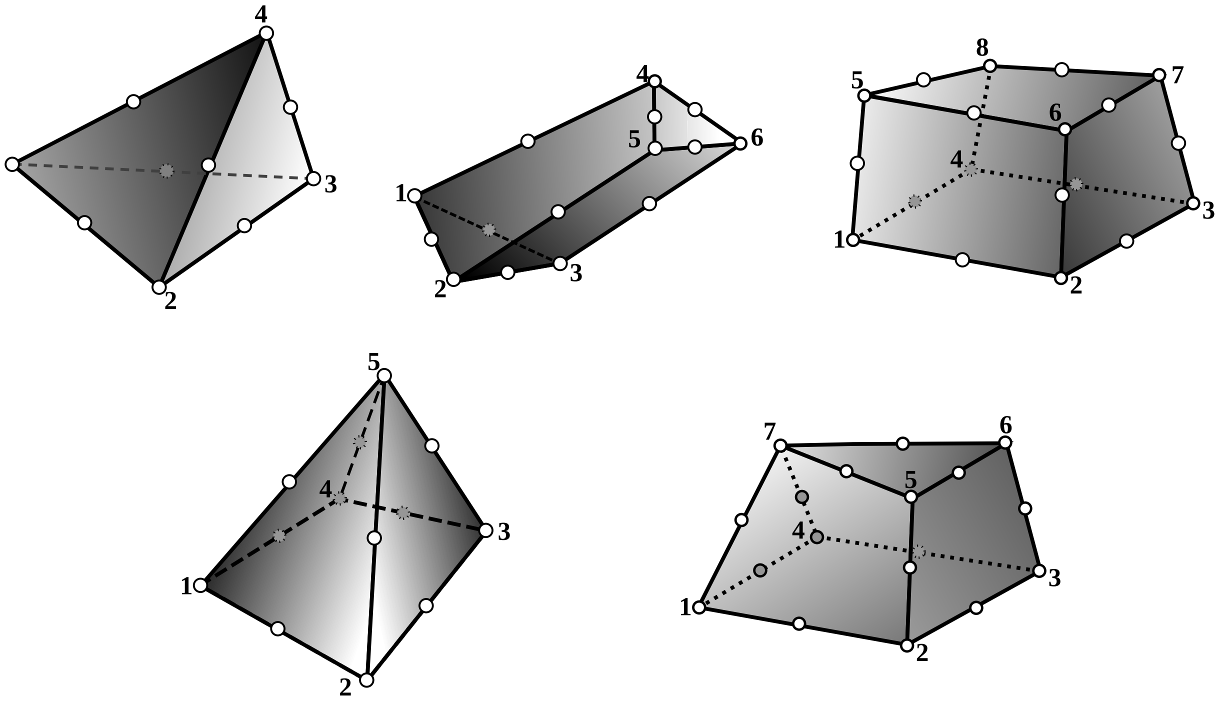


**Wedges**



**Bricks**

# Solid Elements Can Have Midside Nodes



**Midside nodes need not be at the edge midpoints.**
**This facilitates fitting curved-face boundaries**

# IsoParametric Formulation

## For general solid element with $n$ nodes

**Geometry definition**

$$\begin{bmatrix} 1 \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \\ z_1 & z_2 & \dots & z_n \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ \vdots \\ N_n \end{bmatrix}$$

**where $N_i$ are the shape functions (kept generic in this Chapter) The four rows of this matrix relation say that**

$$1 = \sum_{i=1}^{n} N_i \qquad x = \sum_{i=1}^{n} x_i N_i \qquad y = \sum_{i=1}^{n} y_i N_i \qquad z = \sum_{i=1}^{n} z_i N_i$$

**The first equation says that shape functions must identically add up to unity, which is a completeness requirement**

# IsoParametric Formulation (cont'd)

**Displacement interpolation uses the same shape functions as geometry definition**

$$\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} u_{x1} & u_{x2} & \dots & u_{xn} \\ u_{y1} & u_{y2} & \dots & u_{yn} \\ u_{z1} & u_{z2} & \dots & u_{zn} \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ \vdots \\ N_n \end{bmatrix}$$

**The three rows of this matrix relation say that**

$$u_x = \sum_{i=1}^{n} u_{xi} N_i \qquad u_y = \sum_{i=1}^{n} u_{yi} N_i \qquad u_z = \sum_{i=1}^{n} u_{zi} N_i$$

# IsoParametric Formulation (cont'd)

**The nodal DOF of element are arranged as 3$n$-vector**

$$\mathbf{u}^T = [\, u_{x1} \quad u_{y1} \quad u_{z1} \quad u_{x2} \quad \ldots \quad u_{xn} \quad u_{yn} \quad u_{zn} \,]$$

**whereas strains and stresses are arranged as 6-vectors**

$$\mathbf{e} = \begin{bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ 2e_{xy} \\ 2e_{yz} \\ 2e_{zx} \end{bmatrix} \qquad \boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{bmatrix}$$

# IsoParametric Formulation (cont'd)

**Strain-displacement equations for infinitesimal strains:**

$$\mathbf{e}=\begin{bmatrix}\frac{\partial u_x}{\partial x}\\ \frac{\partial u_y}{\partial y}\\ \frac{\partial u_z}{\partial z}\\ \frac{\partial u_y}{\partial x}+\frac{\partial u_x}{\partial y}\\ \vdots\end{bmatrix}=\begin{bmatrix}N_{x1}&0&0&\dots&N_{xn}&0&0\\ 0&N_{y1}&0&\dots&0&N_{yn}&0\\ 0&0&N_{z1}&\dots&0&0&N_{zn}\\ N_{y1}&N_{x1}&0&\dots&N_{yn}&N_{xn}&0\\ 0&N_{z1}&N_{y1}&\dots&0&N_{yn}&N_{xn}\\ N_{z1}&0&N_{x1}&\dots&N_{zn}&0&N_{xn}\end{bmatrix}\begin{bmatrix}u_{x1}\\ u_{y1}\\ u_{z1}\\ \vdots\\ u_{xn}\\ u_{yn}\\ u_{zn}\end{bmatrix}=\mathbf{Bu}$$

**Constitutive equations for linear elastostatics (no initial stresses):**

$$\boldsymbol{\sigma}=\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{xy}\\ \sigma_{yz}\\ \sigma_{zx}\end{bmatrix}=\begin{bmatrix}E_{11}&E_{12}&E_{13}&E_{14}&E_{15}&E_{16}\\ &E_{22}&E_{23}&E_{24}&E_{25}&E_{26}\\ &&E_{33}&E_{34}&E_{35}&E_{36}\\ &&&E_{44}&E_{45}&E_{46}\\ &&&&E_{55}&E_{56}\\ symm&&&&&E_{66}\end{bmatrix}\begin{bmatrix}e_{xx}\\ e_{yy}\\ e_{zz}\\ 2e_{xy}\\ 2e_{yz}\\ 2e_{zx}\end{bmatrix}=\mathbf{Ee}$$

# IsoParametric Formulation (cont'd)

**The element stiffness matrix is given by**

$$\boxed{\mathbf{K}^e = \int_{V^e} \mathbf{B}^T \mathbf{E} \mathbf{B} \, dV}$$

**which is numerically integrated by a $n_G$-point Gauss rule:**

$$\boxed{\mathbf{K}^e = \sum_{k=1}^{n_G} w_k \, J_k \mathbf{B}_k^T \mathbf{E} \mathbf{B}_k}$$

# IsoParametric Formulation (cont'd)

**To speed up element formation introduce**

$$\mathbf{B}_i = \begin{bmatrix} N_{xi} & 0 & 0 \\ 0 & N_{yi} & 0 \\ 0 & 0 & N_{zi} \\ N_{yi} & N_{xi} & 0 \\ 0 & N_{zi} & N_{yi} \\ N_{zi} & 0 & N_{xi} \end{bmatrix} \qquad \mathbf{B}_j = \begin{bmatrix} N_{xj} & 0 & 0 \\ 0 & N_{yj} & 0 \\ 0 & 0 & N_{zj} \\ N_{yj} & N_{xj} & 0 \\ 0 & N_{zj} & N_{yi} \\ N_{zj} & 0 & N_{xj} \end{bmatrix} \qquad i,\, j = 1, \ldots n \ .$$

**and**

$$\mathbf{C} = \begin{bmatrix} B_{xi}E_{11} + B_{yi}E_{14} + B_{zi}E_{16} & B_{yi}E_{12} + B_{xi}E_{14} + B_{zi}E_{15} & B_{zi}E_{13} + B_{yi}E_{15} + B_{xi}E_{16} \\ B_{xi}E_{12} + B_{yi}E_{24} + B_{zi}E_{26} & B_{yi}E_{22} + B_{xi}E_{24} + B_{zi}E_{25} & B_{zi}E_{23} + B_{yi}E_{25} + B_{xi}E_{26} \\ B_{xi}E_{13} + B_{yi}E_{34} + B_{zi}E_{36} & B_{yi}E_{23} + B_{xi}E_{34} + B_{zi}E_{35} & B_{zi}E_{33} + B_{yi}E_{35} + B_{xi}E_{36} \\ B_{xi}E_{14} + B_{yi}E_{44} + B_{zi}E_{46} & B_{yi}E_{24} + B_{xi}E_{44} + B_{zi}E_{45} & B_{zi}E_{34} + B_{yi}E_{45} + B_{xi}E_{46} \\ B_{xi}E_{15} + B_{yi}E_{45} + B_{zi}E_{56} & B_{yi}E_{25} + B_{xi}E_{45} + B_{zi}E_{55} & B_{zi}E_{35} + B_{yi}E_{55} + B_{xi}E_{56} \\ B_{xi}E_{16} + B_{yi}E_{46} + B_{zi}E_{66} & B_{yi}E_{26} + B_{xi}E_{46} + B_{zi}E_{56} & B_{zi}E_{36} + B_{yi}E_{56} + B_{xi}E_{66} \end{bmatrix}$$

**Then the 3 x 3 {*i*,*j*} stiffness matrix block is given by**

$$\mathbf{Q}_{ij} = \begin{bmatrix} B_{xj}C_{11} + B_{yj}C_{41} + B_{zj}C_{61} & B_{xj}C_{12} + B_{yj}C_{42} + B_{zj}C_{62} & B_{xj}C_{13} + B_{yj}C_{43} + B_{zj}C_{63} \\ B_{yj}C_{21} + B_{xi}C_{41} + B_{zj}C_{51} & B_{yj}C_{22} + B_{xi}C_{42} + B_{zj}C_{52} & B_{yj}C_{23} + B_{xi}C_{43} + B_{zj}C_{53} \\ B_{zj}C_{31} + B_{yj}C_{51} + B_{xj}C_{61} & B_{zj}C_{32} + B_{yj}C_{52} + B_{xj}C_{62} & B_{zj}C_{33} + B_{yj}C_{53} + B_{xj}C_{63} \end{bmatrix}$$

# IsoParametric Formulation (cont'd)

**The element consistent mass matrix is given by**

$$\mathbf{M}_C^e = \int_{V^e} \rho \mathbf{N}\mathbf{N}^T \; dV$$

**in which N denotes the shape function matrix**

$$\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} N_1 & 0 & 0 & \ldots & N_n & 0 & 0 \\ 0 & N_1 & 0 & \ldots & 0 & N_n & 0 \\ 0 & 0 & N_1 & \ldots & 0 & 0 & N_n \end{bmatrix} \mathbf{u} = \mathbf{N}\mathbf{u}$$

**The mass matrix is also usually evaluated by Gauss numerical integration**