

11

Axisymmetric Solid Iso-P Elements

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§11.1. Introduction

In this Chapter we consider the finite element discretization of axisymmetric solids that form structures of revolution (SOR). The focus is on isoparametric (iso-P) elements. This focus links up with matter covered in the Introduction to Finite Elements (IFEM) book.

The dimensionality reduction process described in the previous Chapter “folds” the three-dimensional problem into integrals taken over the generating cross section and its boundaries. The finite element discretization can be therefore confined to the $\{r, z\}$ plane. The circumferential (θ) dimension conceptually disappears from the FEM discretization. The resulting finite elements are called *axisymmetric solid elements*, SOR elements, or “ring” elements in the literature. In this and following chapters, the term *ring element* will be often used for brevity.

A ring element is completely defined geometry of its generating cross section in the (r, z) plane, as illustrated in Figure 11.1.

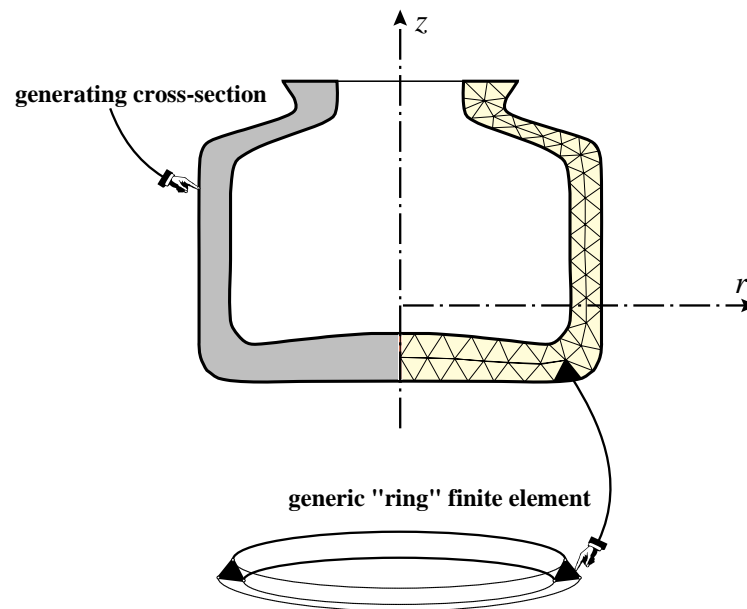


FIGURE 11.1. Axisymmetric solid “ring” element.

Since this cross section is plane, the element geometry definition is two-dimensional. It follows that the two-dimensional geometrical configurations previously studied in IFEM (for example, 3 and 6-node triangles, 4-, 8-, and 9-node quadrilaterals, etc) for the plane stress case can be reused.

The major modeling difference with respect to the plane stress case is the appearance of the circumferential or “hoop” strain and stress, which together contribute a term

$$\frac{1}{2}\sigma_{\theta\theta}\epsilon_{\theta\theta} \quad (11.1)$$

to the strain energy density. This energy term injects an extra row in the strain-displacement matrix **B**. Furthermore the constitutive matrix must be adjusted to reflect the presence of hoop effects as well as absence of the plane stress assumption. Taken together, these changes must be accounted for in the formation of the element stiffness matrix as well as the displacement-to-stress recovery.

§11.2. Isoparametric Definition

Ring finite elements for axisymmetric solids are developed in this and next two Chapters using the displacement-based isoparametric (iso-P) formulation, which was introduced in Chapter 16 of IFEM. The key concept is that both the element cross section geometry and the displacement field are interpolated by the same shape functions. Two degrees of freedom: the radial and axial displacements $\{u_r, u_z\}$ are specified at each node. The iso-P definition of an element with n nodes is

$$\begin{bmatrix} 1 \\ r \\ z \\ u_r \\ u_z \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & \cdots & r_n \\ z_1 & z_2 & \cdots & z_n \\ u_{r1} & u_{r2} & \cdots & u_{rn} \\ u_{z1} & u_{z2} & \cdots & u_{zn} \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ \vdots \\ N_n^e \end{bmatrix}. \quad (11.2)$$

In the left-hand side, $u_r = u_r(r, z)$ and $u_z = u_z(r, z)$ are element displacements in terms of their node values u_{ri} and u_{zi} , respectively and of the shape functions N_i^e collected on the rightmost column vector. The element geometry, represented by the position coordinates r and z in the left hand side, are defined by the same interpolation. The first row of (11.2) expresses that the sum of shape functions $N_1^e + N_2^e + \dots + N_n^e$ is identically one.

The element shape functions N_i are written in terms of the element natural coordinates introduced in IFEM: $\{\zeta_1, \zeta_2, \zeta_3\}$ for triangles and $\{\xi, \eta, \zeta\}$ for quadrilaterals. The same continuity and completeness requirements discussed in that course apply with minor differences (see remarks below).

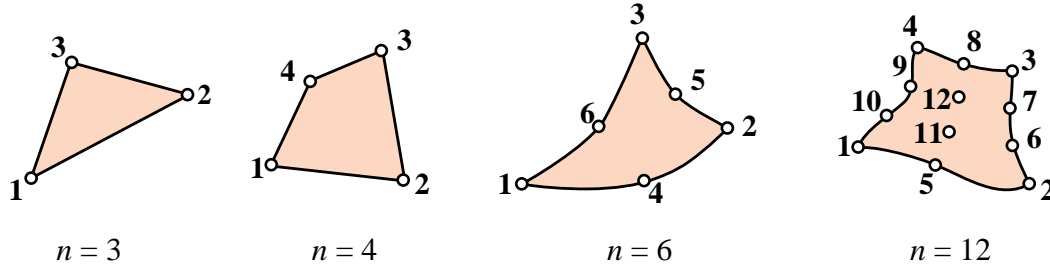
Remark 11.1. Displacement based elements are based on the TPE functional. The variational index for the only master field: displacements, is one. Consequently the interelement continuity required is C^0 . That is, the displacement components u_r and u_z must be continuous between adjacent elements. Element boundaries lying on the axis of revolution are special: at such points the radial displacement u_r must vanish if the structure is continuous there (that is, a tiny hole at $r = 0$ is precluded) although there is no need to make that condition explicit at the element level. That constraint is applied through displacement boundary conditions at the assembly level.

Remark 11.2. The completeness criterion demands that all rigid body modes and strain states be exactly represented. This is met if the element shape functions can represent any displacement field that is linear in r and z . (In fact, a more detailed analysis shows that this is a slight overkill because the hoop strain is not a displacement differential, but will do for now.)

§11.3. The Element Stiffness Matrix

We continue to assume a generic iso-P ring element with n nodes. Some specific geometric configurations are shown in Figure 11.2. Two freedoms are chosen at each node: the radial and axial displacement components. Thus the element has $2n$ degrees of freedom. As usual, the node displacement vector \mathbf{u}^e is a column $2n$ -vector constructed by arranging those freedoms node by node:

$$\mathbf{u}^e = [u_{r1} \quad u_{z1} \quad u_{r2} \quad u_{z2} \quad \cdots \quad u_{rn} \quad u_{zn}]^T \quad (11.3)$$

FIGURE 11.2. Ring element cross sections characterized by their number of nodes n .

§11.3.1. Displacement Interpolation

As displayed in the iso-P definition (11.2), both displacement components $u_r(r, z)$ and $u_z(r, z)$ are interpolated over the element by the same shape functions. It is convenient to express that interpolation in terms of the node displacement vector (11.3). Extracting the corresponding rows of (11.2) and transposing we obtain

$$\begin{bmatrix} u_r \\ u_z \end{bmatrix} = \begin{bmatrix} N_1^e & 0 & N_2^e & 0 & \dots & N_n^e & 0 \\ 0 & N_1^e & 0 & N_2^e & \dots & 0 & N_n^e \end{bmatrix} \mathbf{u}^e = \mathbf{N} \mathbf{u}^e. \quad (11.4)$$

This \mathbf{N} (with superscript e omitted to reduce clutter) is called the *shape function matrix*. It has dimensions $2 \times 2n$. For example, if the element has 4 nodes (as in the second one from the left in Figure 11.2), \mathbf{N} is 2×8 .

§11.3.2. The Strain-Displacement Matrix

Differentiating the finite element displacement interpolation (11.4) to get the strains $e_{rr} = \partial u_r / \partial r$, $e_{zz} = \partial u_z / \partial z$, $e_{\theta\theta} = u_r / r$ and $2e_{rz} = \partial u_r / \partial z + \partial u_z / \partial r$ yields the strain-displacement relations:

$$\mathbf{e} = \begin{bmatrix} e_{rr} \\ e_{zz} \\ e_{\theta\theta} \\ 2e_{rz} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_1^e}{\partial r} & 0 & \frac{\partial N_2^e}{\partial r} & 0 & \dots & \frac{\partial N_n^e}{\partial r} & 0 \\ 0 & \frac{\partial N_1^e}{\partial z} & 0 & \frac{\partial N_2^e}{\partial z} & \dots & 0 & \frac{\partial N_n^e}{\partial z} \\ \frac{N_1^e}{r} & 0 & \frac{N_2^e}{r} & 0 & \dots & \frac{N_n^e}{r} & 0 \\ \frac{\partial N_1^e}{\partial z} & \frac{\partial N_1^e}{\partial r} & \frac{\partial N_2^e}{\partial z} & \frac{\partial N_2^e}{\partial r} & \dots & \frac{\partial N_n^e}{\partial z} & \frac{\partial N_n^e}{\partial r} \end{bmatrix} \mathbf{u}^e = \mathbf{B} \mathbf{u}^e. \quad (11.5)$$

This $\mathbf{B} = \mathbf{D}\mathbf{N}$ is called the *strain-displacement matrix*. It is dimensioned $3 \times 2n$. For example, if the element has 6 nodes, \mathbf{B} is 3×12 . Note that terms in the third row of \mathbf{B} incur division by zero if $r = 0$, which happens at points on the axis of revolution. This can be the source of numerical difficulties that are discussed later.

The stresses are given in terms of strains and displacements by

$$\boldsymbol{\sigma} = \mathbf{E} \mathbf{e} = \mathbf{E} \mathbf{B} \mathbf{u}^e. \quad (11.6)$$

The expression of the 4×4 elasticity matrix \mathbf{E} is given in the previous Chapter for a general anisotropic material and its isotropic specialization.

Both (11.5) and (11.6) are assumed to hold at all points of the element domain, since they correspond to the strong links in the TPE diagram.

Comparing these expressions with those in Chapter 14 of IFEM, we see that \mathbf{N} is the same but \mathbf{B} has an extra row, which is associated with the hoop strain.

§11.3.3. The Element Stiffness Equations

Inserting the foregoing expressions into the TPE functional and integrating gives the quadratic matrix form

$$\Pi^e = \frac{1}{2} \mathbf{u}^{eT} \mathbf{K}^e \mathbf{u}^e - \mathbf{u}^{eT} \mathbf{f}^e. \quad (11.7)$$

Here the stiffness matrix is

$$\mathbf{K}^e = \int_{\Omega^e} r \mathbf{B}^T \mathbf{E} \mathbf{B} d\Omega, \quad (11.8)$$

while the consistent element nodal force vector combines body force and surface traction effects:

$$\mathbf{f}^e = \mathbf{f}_b^e + \mathbf{f}_t^e = \int_{\Omega^e} r \mathbf{N}^T \mathbf{b} d\Omega + \int_{\Gamma^e} r \mathbf{N}^T \hat{\mathbf{t}} d\Gamma. \quad (11.9)$$

In these expressions Ω^e and Γ^e denote the element domain and boundary, respectively. To justify the names given to \mathbf{K}^e and \mathbf{f}^e , take the variation of (11.7) with respect to the node displacements:

$$\delta \Pi^e = (\delta \mathbf{u}^e)^T [\mathbf{K}^e \mathbf{u}^e - \mathbf{f}^e] = \mathbf{0}. \quad (11.10)$$

Since $\delta \mathbf{u}^e$ is arbitrary, the expression in brackets must vanish, which yields the element stiffness equations

$$\mathbf{K}^e \mathbf{u}^e = \mathbf{f}^e. \quad (11.11)$$

Remark 11.3. Comparing (11.8) and (11.9) with the expressions given for the plane stress problem in Chapter 14 of IFEM, we note that the plate thickness h has been replaced by the radial coordinate r . Unlike h , this coordinate cannot generally be taken out of the integral. Note also that the circumferential factor 2π has been dropped from the integrals that appear (11.8) and (11.9); the main consequence being that axial point along the axis of revolution forces must be divided by 2π as discussed in the previous Chapter.

§11.3.4. *Thermal Effects

Suppose the temperature of the structure changes axisymmetrically by $\Delta T(r, z)$ from a reference temperature. The constitutive equations become

$$\boldsymbol{\sigma} = \mathbf{E}(\mathbf{e} - \boldsymbol{\alpha} \Delta T) \quad (11.12)$$

where $\boldsymbol{\alpha}$ is an array of dilatation coefficients that provide thermal strains in a mechanically unconstrained body. (A linear relation between strains and temperature changes is assumed.) For isotropic materials

$$\boldsymbol{\alpha}^T = [\alpha \quad \alpha \quad \alpha \quad 0] \quad (11.13)$$

where α is the usual coefficient of dilatation of the material. On inserting (11.13) into the potential energy formulation for a generic element yields element stiffness equations that account for thermal effects:

$$\mathbf{K}^e \mathbf{u}^e = \mathbf{f}^e = \mathbf{f}_M^e + \mathbf{f}_T^e \quad (11.14)$$

where \mathbf{f}_M are the mechanical forces considered so far, and \mathbf{f}_T , called the *thermal forces* or *equivalent thermal loads*, account for contribution from the temperature changes:

$$\mathbf{f}_T^e = \int_A \mathbf{B}^T \alpha \Delta T r dA \quad (11.15)$$

This term can be evaluated by numerical integration.

The sum of mechanical and thermal node forces is sometimes called the *effective node force vector* or simply *effective forces* in the FEM literature.

§11.4. Element Computations

The entries of the element stiffness matrix \mathbf{K}^e and consistent node force vector \mathbf{f}^e are usually computed by numerical integration techniques that rely on the Gauss quadrature formulas covered in Chapters 17, 23 and 24 of IFEM. (Some of that material is reproduced in §11.5 for convenience.)

§11.4.1. Stiffness Matrix Integration

For specificity suppose that the stiffness matrix of a *quadrilateral* isoparametric element is to be numerically integrated by a p -point Gauss quadrature rule along each isoparametric coordinate. Denote by ξ_k and η_ℓ the Gauss points abscissae whereas w_k and w_ℓ denote the corresponding integration weights, with indices k and ℓ running from 1 through p . Then

$$\mathbf{K}^e = \sum_{k=1}^p \sum_{\ell=1}^p w_k w_\ell \mathbf{B}^T(\xi_k, \eta_\ell) \mathbf{E} \mathbf{B}(\xi_k, \eta_\ell) r(\xi_k, \eta_\ell) J_\Omega(\xi_k, \eta_\ell). \quad (11.16)$$

Here $\mathbf{B}(\xi_k, \eta_\ell)$ means $\mathbf{B} = \mathbf{B}(\xi, \eta)$ evaluated at the Gauss point; likewise for $r(\xi_k, \eta_\ell)$ and $J_\Omega(\xi_k, \eta_\ell)$. The latter is the Jacobian determinant that maps the $\{r, z\}$ area element to that in the natural coordinates $\{\xi, \eta\}$ as discussed in Chapter 17 of IFEM. Usually the elasticity matrix \mathbf{E} is assumed constant over the element, which explains its lack of arguments in (11.16).

§11.4.2. Consistent Body Force Vector Integration

Body forces (also called volume forces) arise frequently in analysis of SOR. The most important loads of this type are

1. Gravity (own weight). This effect is important in massive SOR, as encountered in civil, geophysical and nuclear applications.
2. Centrifugal forces in rotating structures. These are important in aerospace and mechanical structures (for example, high-speed rotating machinery such as turbines).
3. Thermal, shrinkage and prestress effects. These may be important depending on fabrication techniques, material, and the environment to which the structure will be exposed.

The consistent force vector for body forces may be also evaluated by numerical Gauss quadrature resulting in an expression similar to (11.16):

$$\mathbf{f}_b^e = \sum_{k=1}^p \sum_{\ell=1}^p w_k w_\ell \mathbf{N}^T(\xi_k, \eta_\ell) \mathbf{b}(\xi_k, \eta_\ell) r(\xi_k, \eta_\ell) J_\Omega(\xi_k, \eta_\ell). \quad (11.17)$$

Note that here we evaluate the shape function matrix \mathbf{N} and the body force vector \mathbf{b} at the Gauss point. While the former is supplied as part of the iso-P definition (11.9), the question arises on how to interpolate \mathbf{b} to get $\mathbf{b}(\xi_k, \eta_\ell)$. Two possibilities:

Element Level Specification. \mathbf{b} is defined element by element; for example constant b_r and b_z that may jump from element to element.

Node Level Specification. \mathbf{b} is defined by nodal values, and interpolated by the same shape functions as displacements.

Both choices are common in practice, but the first is more flexible in accomodating interelement force discontinuities. For example the body force due to gravity or centrifugal effects jumps between materials of different density. The second choice is appropriate for a continuous body force field, and especially when \mathbf{b} can be defined as a function of the position coordinates.

§11.4.3. Consistent Surface Traction Integration

The consistent force vector associated with surface tractions is often simple enough that is can be precomputed by hand using NbN or EbE lumping, as explained in Chapter 7 of IFEM, and fed to the FEM program as node forces.

For more difficult cases involving for example space-varying pressures or oblique surfaces, unidimensional numerical integration may also be applied:

$$\mathbf{f}_t^e = \sum_{k=1}^p w_k \mathbf{N}^T(\xi_k) \hat{\mathbf{t}}(\xi_k) r(\xi_k) J_\Gamma(\xi_k). \quad (11.18)$$

Here ξ is an isoparametric arlength coordinate, customized to traverse element sides on which a nonzero traction vector $\hat{\mathbf{t}}$ is given, and J_Γ the associated arlength Jacobian. If this procedure is implemented, $\hat{\mathbf{t}}$ is generally defined at designated elements traversed side by side.

§11.4.4. Preventing Division by Zero

One glance at the strain displacement matrix \mathbf{B} in (11.5) shows that the third row terms involve a division by zero if $r = 0$, that is, at points on the axis of revolution. It follows that one should avoid evaluating that matrix there, as that could abort the calculations. This may happen if one or more elements extends to the axis, as is the case in the lower portion of the structure shown in Figure 11.1. Fortunately it is not difficult to avoid trouble by sticking to two common sense rules:

1. When numerically integrating over element areas, as in the computation of \mathbf{K}^e in (11.16), use only quadrature rules with *internal* sample points. For the quadrilateral geometry, all Gauss-product rules comply with this restriction.
2. In the postprocessing stage, avoid evaluating \mathbf{B} directly at element nodes for strain recovery from displacements. Instead, evaluate at interior points (usually Gauss points) and extrapolate to nodes as discussed in Chapter 30 of IFEM. The stress recovery routines described in the next two Chapters make use of this avoidance technique.

§11.5. Numerical Integration by Gauss Rules

The following material on numerical integration is transcribed here from Chapter 17 of the IFEM Notes for convenience. The Gauss quadrature information modules listed below are used in the element implementations covered in the next two Chapters.

The use of numerical integration is essential for practical evaluation of integrals over isoparametric element domains. The standard practice has been to use *Gauss integration* because such rules use a *minimal number of sample points to achieve a desired level of accuracy*. This economy is important for efficient element calculations, since a *matrix product* is evaluated at each sample point. The fact that the location of the sample points in Gauss rules is usually given by non-rational numbers is of no concern in digital computation.

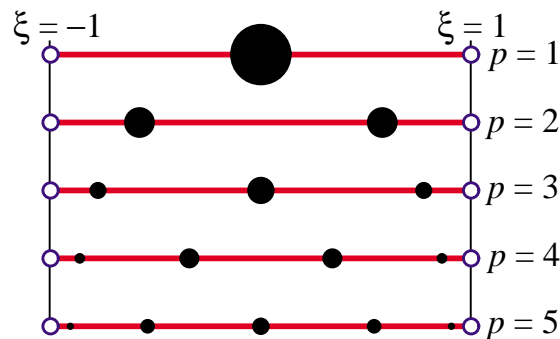


FIGURE 11.3. The first five unidimensional Gauss rules $p = 1, 2, 3, 4, 5$ depicted over the line segment $\xi \in [-1, +1]$. Sample point locations are marked with black circles. The radii of those circles are proportional to the integration weights.

§11.5.1. One Dimensional Rules

The classical Gauss integration rules are defined by

$$\int_{-1}^1 F(\xi) d\xi \approx \sum_{i=1}^p w_i F(\xi_i). \quad (11.19)$$

Here $p \geq 1$ is the number of Gauss integration points (also known as sample points), w_i are the integration weights, and ξ_i are sample-point abscissae in the interval $[-1, 1]$. The use of the canonical interval $[-1, 1]$ is no restriction, because an integral over another range, say from a to b , can be transformed to $[-1, +1]$ via a simple linear transformation of the independent variable, as shown in the Remark below.

The first five unidimensional Gauss rules, illustrated in Figure 11.3, are listed in Table 11.1. These integrate exactly polynomials in ξ of orders up to 1, 3, 5, 7 and 9, respectively. In general a unidimensional Gauss rule with p points integrates exactly polynomials of order up to $2p - 1$. This is called the *degree* of the formula.

Table 11.1 - One Dimensional Gauss Rules with 1 through 5 Sample Points

| Points | Rule |
|--------|---|
| 1 | $\int_{-1}^1 F(\xi) d\xi \approx 2F(0)$ |
| 2 | $\int_{-1}^1 F(\xi) d\xi \approx F(-1/\sqrt{3}) + F(1/\sqrt{3})$ |
| 3 | $\int_{-1}^1 F(\xi) d\xi \approx \frac{5}{9}F(-\sqrt{3/5}) + \frac{8}{9}F(0) + \frac{5}{9}F(\sqrt{3/5})$ |
| 4 | $\int_{-1}^1 F(\xi) d\xi \approx w_{14}F(\xi_{14}) + w_{24}F(\xi_{24}) + w_{34}F(\xi_{34}) + w_{44}F(\xi_{44})$ |
| 5 | $\int_{-1}^1 F(\xi) d\xi \approx w_{15}F(\xi_{15}) + w_{25}F(\xi_{25}) + w_{35}F(\xi_{35}) + w_{45}F(\xi_{45}) + w_{55}F(\xi_{55})$ |

For the 4-point rule, $\xi_{34} = -\xi_{24} = \sqrt{(3 - 2\sqrt{6/5})/7}$, $\xi_{44} = -\xi_{14} = \sqrt{(3 + 2\sqrt{6/5})/7}$, $w_{14} = w_{44} = \frac{1}{2} - \frac{1}{6}\sqrt{5/6}$, and $w_{24} = w_{34} = \frac{1}{2} + \frac{1}{6}\sqrt{5/6}$.

For the 5-point rule, $\xi_{55} = -\xi_{15} = \frac{1}{3}\sqrt{5 + 2\sqrt{10/7}}$, $\xi_{45} = -\xi_{35} = \frac{1}{3}\sqrt{5 - 2\sqrt{10/7}}$, $\xi_{35} = 0$, $w_{15} = w_{55} = (322 - 13\sqrt{70})/900$, $w_{25} = w_{45} = (322 + 13\sqrt{70})/900$ and $w_{35} = 512/900$.

```

LineGaussRuleInfo[{rule_, numer_}, point_] := Module[
  {g2={-1,1}/Sqrt[3], w3={5/9, 8/9, 5/9},
   g3={-Sqrt[3/5], 0, Sqrt[3/5]},
   w4={(1/2)-Sqrt[5/6]/6, (1/2)+Sqrt[5/6]/6,
        (1/2)+Sqrt[5/6]/6, (1/2)-Sqrt[5/6]/6},
   g4={-Sqrt[(3+2*Sqrt[6/5])/7], -Sqrt[(3-2*Sqrt[6/5])/7],
        Sqrt[(3-2*Sqrt[6/5])/7], Sqrt[(3+2*Sqrt[6/5])/7]},
   g5={-Sqrt[5+2*Sqrt[10/7]], -Sqrt[5-2*Sqrt[10/7]], 0,
        Sqrt[5-2*Sqrt[10/7]], Sqrt[5+2*Sqrt[10/7]}/3,
   w5={322-13*Sqrt[70], 322+13*Sqrt[70], 512,
        322+13*Sqrt[70], 322-13*Sqrt[70]}/900,
   i=point, p=rule, info={{Null, Null}, 0}},
  If [p==1, info={0, 2}];
  If [p==2, info={g2[[i]], 1}];
  If [p==3, info={g3[[i]], w3[[i]]}];
  If [p==4, info={g4[[i]], w4[[i]]}];
  If [p==5, info={g5[[i]], w5[[i]]}];
  If [numer, Return[N[info]], Return[Simplify[info]]];
];

```

FIGURE 11.4. A *Mathematica* module that returns the first five unidimensional Gauss rules.

Remark 11.4. A more general integral, such as $F(x)$ over $[a, b]$ in which $\ell = b - a > 0$, is transformed to the canonical interval $[-1, 1]$ through the mapping $x = \frac{1}{2}a(1 - \xi) + \frac{1}{2}b(1 + \xi) = \frac{1}{2}(a + b) + \frac{1}{2}\ell\xi$, or $\xi = (2/\ell)(x - \frac{1}{2}(a + b))$. The Jacobian of this mapping is $J = dx/d\xi = 1/2\ell$. Thus

$$\int_a^b F(x) dx = \int_{-1}^1 F(\xi) J d\xi = \int_{-1}^1 F(\xi) \frac{1}{2}\ell d\xi. \quad (11.20)$$

Remark 11.5. Higher order Gauss rules are tabulated in standard manuals for numerical computation. For example, the widely used Handbook of Mathematical Functions [1] lists (in Table 25.4) rules with up to 96 points. For $p > 6$ the abscissas and weights of sample points are not expressible as rational numbers or radicals, and can only be given as floating-point numbers.

§11.5.2. Implementation of 1D Rules

The *Mathematica* module shown in Figure 11.4 returns either exact or floating-point information for the first five unidimensional Gauss rules. To get information for the i^{th} point of the p^{th} rule, in which $1 \leq i \leq p$ and $p = 1, 2, 3, 4, 5$, call the module as

$$\{\text{xii}, \text{wi}\} = \text{LineGaussRuleInfo}[\{\text{p}, \text{numer}\}, \text{i}] \quad (11.21)$$

Logical flag *numer* is *True* to get numerical (floating-point) information, or *False* to get exact information. The module returns the sample point abscissa ξ_i in *xii* and the weight w_i in *wi*. If *p* is not in the implemented range 1 through 5, the module returns $\{\text{Null}, 0\}$.

Example 11.1. $\{\text{xi}, \text{w}\} = \text{LineGaussRuleInfo}[\{3, \text{False}\}, 2]$ returns $\text{xi}=0$ and $\text{w}=8/9$, whereas $\{\text{xi}, \text{w}\} = \text{LineGaussRuleInfo}[\{3, \text{True}\}, 2]$ returns (to 16 places) $\text{xi}=0.$ and $\text{w}=0.888888888888889$.

§11.5.3. Two Dimensional Rules

The simplest two-dimensional Gauss rules are called *product rules*. They are obtained by applying the unidimensional rules to each independent variable in turn. To apply these rules we must first reduce the integrand to the canonical form:

$$\int_{-1}^1 \int_{-1}^1 F(\xi, \eta) d\xi d\eta = \int_{-1}^1 d\eta \int_{-1}^1 F(\xi, \eta) d\xi. \quad (11.22)$$

Once this is done we can process numerically each integral in turn:

$$\int_{-1}^1 \int_{-1}^1 F(\xi, \eta) d\xi d\eta = \int_{-1}^1 d\eta \int_{-1}^1 F(\xi, \eta) d\xi \approx \sum_{i=1}^{p_1} \sum_{j=1}^{p_2} w_i w_j F(\xi_i, \eta_j). \quad (11.23)$$

where p_1 and p_2 are the number of Gauss points in the ξ and η directions, respectively. Usually the same number $p = p_1 = p_2$ is chosen if the shape functions are taken to be the same in the ξ and η directions. This is in fact the case for all quadrilateral elements presented here. The first four two-dimensional Gauss product rules with $p = p_1 = p_2$ are illustrated in Figure 11.5.

§11.5.4. Implementation of 2D Gauss Rules

The *Mathematica* module listed in Figure 11.6 implements two-dimensional product Gauss rules having 1 through 5 points in each direction. The number of points in each direction may be the same or different. If the rule has the same number of points p in both directions the module is called in either of two ways:

$$\begin{aligned} \{\{\text{xii}, \text{etaj}\}, \text{wij}\} &= \text{QuadGaussRuleInfo}[\{\text{p}, \text{numer}\}, \{\text{i}, \text{j}\}] \\ \{\{\text{xii}, \text{etaj}\}, \text{wij}\} &= \text{QuadGaussRuleInfo}[\{\text{p}, \text{numer}\}, \text{k}] \end{aligned} \quad (11.24)$$

The first form is used to get information for point $\{i, j\}$ of the $p \times p$ rule, in which $1 \leq i \leq p$ and $1 \leq j \leq p$. The second form specifies that point by a “visiting counter” k that runs from 1 through p^2 ; if so $\{i, j\}$ are internally extracted¹ as $\text{j} = \text{Floor}[(\text{k}-1)/\text{p}] + 1$; $\text{i} = \text{k} - \text{p} * (\text{j} - 1)$.

¹ Indices i and j are denoted by *i1* and *i2*, respectively, inside the module.

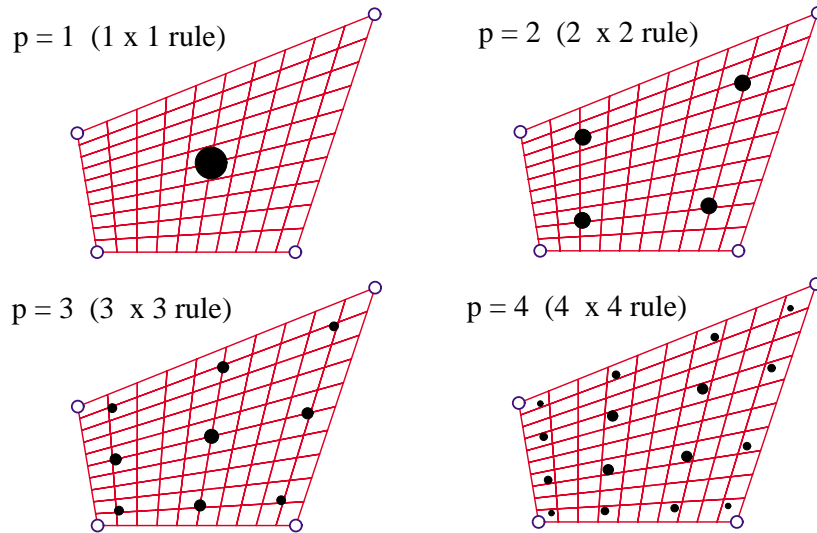


FIGURE 11.5. The first four two-dimensional Gauss product rules $p = 1, 2, 3, 4$ depicted over a straight-sided quadrilateral region. Sample points are marked with black circles. The areas of these circles are proportional to the integration weights.

If the integration rule has p_1 points in the ξ direction and p_2 points in the η direction, the module may be called also in two ways:

$$\begin{aligned} \{ \{ x_{ii}, \eta_{jj} \}, w_{ij} \} &= \text{QuadGaussRuleInfo}[\{ \{ p_1, p_2 \}, \text{numer} \}, \{ i, j \}] \\ \{ \{ x_{ii}, \eta_{jj} \}, w_{ij} \} &= \text{QuadGaussRuleInfo}[\{ \{ p_1, p_2 \}, \text{numer} \}, k] \end{aligned} \quad (11.25)$$

The meaning of the second argument is as follows. In the first form i runs from 1 to p_1 and j from 1 to p_2 . In the second form k runs from 1 to $p_1 p_2$; if so i and j are extracted by $j = \text{Floor}[(k-1)/p_1] + 1$; $i = k - p_1 * (j-1)$. In all four forms, logical flag *numer* is set to True if numerical information is desired and to False if exact information is desired.

The module returns ξ_i and η_j in x_{ii} and η_{jj} , respectively, and the weight product $w_i w_j$ in w_{ij} . This code is used in the Exercises at the end of the chapter. If the inputs are not in range, the module returns $\{ \{ \text{Null}, \text{Null} \}, 0 \}$.

```

QuadGaussRuleInfo[ {rule_, numer_}, point_ ] := Module[
  {ξ, η, p1, p2, i, j, w1, w2, m, info = { {Null, Null}, 0 }},
  If [Length[rule] == 2, {p1, p2} = rule, p1 = p2 = rule];
  If [p1 < 0, Return[QuadNonProductGaussRuleInfo[
    { -p1, numer }, point ]]];
  If [Length[point] == 2, {i, j} = point, m = point;
    j = Floor[(m-1)/p1] + 1; i = m - p1 * (j-1) ];
  {ξ, w1} = LineGaussRuleInfo[ {p1, numer}, i];
  {η, w2} = LineGaussRuleInfo[ {p2, numer}, j];
  info = { {ξ, η}, w1 * w2 };
  If [numer, Return[N[info]], Return[Simplify[info]]];
];

```

FIGURE 11.6. A *Mathematica* module that returns two-dimensional product Gauss rules.

Example 11.2. `{{ xi, eta }, w}=QuadGaussRuleInfo[{3, False}, {2, 3}]` returns `xi=0, eta=Sqrt[3/5]` and `w=40/81`.

Example 11.3. `{{ xi, eta }, w}=QuadGaussRuleInfo[{3, True}, {2, 3}]` returns (to 16-place precision) `xi=0., eta=0.7745966692414834` and `w=0.49382716049382713`.

§11.6. Rank Sufficiency: Linkup with Numerical Quadrature

The following material on numerical integration is transcribed here from Chapter 19 of the IFEM Notes for convenience.

One desirable attribute of the element stiffness matrix is: \mathbf{K}^e should not possess any zero-energy kinematic mode other than rigid body modes.²

This can be mathematically expressed as follows. Let n_F be the number of element degrees of freedom, and n_R be the number of independent rigid body modes. Let r denote the rank of \mathbf{K}^e . The element is called *rank sufficient* if $r = n_F - n_R$ and *rank deficient* if $r < n_F - n_R$. In the latter case, the *rank deficiency* is defined by

$$d = (n_F - n_R) - r \quad (11.26)$$

If an isoparametric element is numerically integrated, let n_G be the number of Gauss points, while n_E denotes the order of the stress-strain matrix \mathbf{E} . Two additional assumptions are made:

- (i) The element shape functions satisfy completeness in the sense that the rigid body modes are exactly captured by them.
- (ii) Matrix \mathbf{E} is of full rank.

Then each Gauss point adds up to n_E to the rank of \mathbf{K}^e , up to a maximum of $n_F - n_R$.³ Assuming that each points does add up exactly n_E , the rank of \mathbf{K}^e will be

$$r = \min(n_F - n_R, n_E n_G) \quad (11.27)$$

To attain rank sufficiency, $n_E n_G$ must equal or exceed $n_F - n_R$:

$$n_E n_G \geq n_F - n_R \quad (11.28)$$

from which the appropriate Gauss integration rule can be selected.

In the axisymmetric solid problem, $n_E = 4$ because \mathbf{E} is a 4×4 matrix of elastic moduli. Also $n_R = 1$. Consequently $r = \min(n_F - 1, 4n_G)$ and $4n_G \geq n_F - 1$.

² In an introductory FEM course, “should” is strengthened to “must”. In an advanced course one can find exceptions to the rule. For example, the reduced-integration Quad8 has a rank deficient stiffness but the element is nonetheless useful.

³ Note the proviso “adds up to.” There are cases when the addition comes up short of n_E . For a somewhat trivial example, think of using the same rule twice: the rank is not affected although the number of Gauss points is doubled. The reduced-integration Quad8 is a nontrivial case in point.

Example 11.4. Consider a 6-node quadratic triangle ring element. Then $n_F = 2 \times 6 = 12$. To attain the proper rank of $12 - n_R = 12 - 1 = 11$, $n_G \geq 3$. A 3-point Gauss rule, such as the midpoint rule defined in Chapter 24 of the IFEM Notes makes the element rank sufficient.

Example 11.5. Consider an 9-node biquadratic quadrilateral ring element. Then $n_F = 2 \times 9 = 18$. To attain the proper rank of $18 - n_R = 18 - 1 = 17$, $n_G \geq 5$. The 2×2 product Gauss rule is insufficient because $n_G = 4$. Hence a 3×3 rule, which yields $n_G = 9$, is required to attain rank sufficiency.

Homework Exercises for Chapter 11
Axisymmetric Solid Iso-P Elements

The following material is largely covered in Chapter 24 of the IFEM Notes, posted at <http://www.colorado.edu/engineering/CAS/courses.d/IFEM.d/IFEM.Ch24.d> but is recapitulated here for completeness.

The four simplest Gauss integration rules over a triangle of area A use 1, 3, 3 and 7 points. These rules are tabulated below. In the following expressions, $F(\zeta_1, \zeta_2, \zeta_3)$ denotes the function to be integrated over the triangle, expressed in terms of the triangular coordinates ζ_1, ζ_2 and ζ_3 , while A is the area of the element.

One point rule (exact for constant and linear polynomials over straight sided triangles):

$$\frac{1}{A} \int_{A^e} F(\zeta_1, \zeta_2, \zeta_3) dA^e \approx F\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right). \quad (\text{E11.1})$$

Three-point rules (exact for constant through quadratic polynomials over straight sided triangles):

$$\frac{1}{A} \int_{A^e} F(\zeta_1, \zeta_2, \zeta_3) dA^e \approx \frac{1}{3}F\left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right) + \frac{1}{3}F\left(\frac{1}{6}, \frac{2}{3}, \frac{1}{6}\right) + \frac{1}{3}F\left(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}\right). \quad (\text{E11.2})$$

$$\frac{1}{A} \int_{A^e} F(\zeta_1, \zeta_2, \zeta_3) dA^e \approx \frac{1}{3}F\left(\frac{1}{2}, \frac{1}{2}, 0\right) + \frac{1}{3}F\left(0, \frac{1}{2}, \frac{1}{2}\right) + \frac{1}{3}F\left(\frac{1}{2}, 0, \frac{1}{2}\right). \quad (\text{E11.3})$$

The latter is also called the 3-midpoint rule for obvious reasons.

Seven point rule (exact for constant through cubic polynomials over straight sided triangles):

$$\begin{aligned} \frac{1}{A} \int_{A^e} F(\zeta_1, \zeta_2, \zeta_3) dA^e \approx & w_0 F\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) + w_1 [F(\alpha_1, \beta_1, \beta_1) + F(\beta_1, \alpha_1, \beta_1) + F(\beta_1, \beta_1, \alpha_1)] \\ & + w_2 [F(\alpha_2, \beta_2, \beta_2) + F(\beta_2, \alpha_2, \beta_2) + F(\beta_2, \beta_2, \alpha_2)], \end{aligned} \quad (\text{E11.4})$$

where $w_0 = 9/40 = 0.225$, $w_1 = (155 - \sqrt{15})/1200 = 0.1259391805$, $w_2 = (155 + \sqrt{15})/1200 = 0.1323941528$, $\alpha_1 = (9 + 2\sqrt{15})/21 = 0.7974269853$, $\beta_1 = (6 - \sqrt{15})/21 = 0.1012865073$, $\alpha_2 = (9 - 2\sqrt{15})/21 = 0.0597158718$, $\beta_2 = (6 + \sqrt{15})/21 = 0.4701420641$.

A *Mathematica* module that implement these rules is `TrigGaussRuleInfo.nb` posted in the index of this Chapter. The use of this module is explained in Chapter 24 of the IFEM Notes.

EXERCISE 11.1 [A/C:15] Using the minimum quadrature rule necessary for exactness, verify the following polynomial integrals over straight-sided triangles, where indices i, j, k run over 1,2,3, and r is interpolated as $r = r_1\zeta_1 + r_2\zeta_2 + r_3\zeta_3$.

$$\int_{A^e} \zeta_i dA^e = \frac{1}{3}A, \quad (\text{E11.5})$$

$$\int_{A^e} \zeta_i \zeta_j dA^e = \frac{1}{12}A(1 + \delta_{ij}), \quad (\text{E11.6})$$

$$\int_{A^e} \zeta_i \zeta_j \zeta_k dA^e = \frac{1}{60}\gamma_{ijk}A, \quad (\text{E11.7})$$

$$\int_{A^e} r dA^e = \frac{1}{3}A(r_1 + r_2 + r_3) = Ar_0, \quad (\text{E11.8})$$

$$\int_{A^e} \zeta_i r dA^e = \frac{1}{12} A(r_1 + r_2 + r_3 + r_i), \quad (\text{E11.9})$$

$$\int_{A^e} r^2 dA^e = \frac{1}{12} A[(r_1 + r_2 + r_3)^2 + r_1^2 + r_2^2 + r_3^2], \quad (\text{E11.10})$$

$$\int_{A^e} \zeta_i r^2 dA^e = \frac{1}{60} \gamma_{ijk} A r_j r_k. \quad (\text{E11.11})$$

In the above, $\delta_{ij} = 1$ if $i = j$ else 0 (the Kronecker delta); $\gamma_{ijk} = 6$ if $i = j = k$, $\gamma_{ijk} = 1$ if $i \neq j \neq k$, else 2. Note: you can (and should) make use of previous results for expediency; to prove for example (E8.8) substitute (E8.5) as appropriate.

EXERCISE 11.2 [A/C:15] Apply the triangle integration formulas to the non-polynomial integral

$$\int_{A^e} \frac{dA^e}{r} \quad (\text{E11.12})$$

for the two cases

- (a) $r_1 = 0, r_2 = r_3 = a$, and compare to the exact integral $2A/a$.
 (b) $r_1 = 1, r_2 = 2, r_3 = 3$ and compare to the exact integral $\log(27/16) A/a = 0.523238A/a$. (The 7 point rule should be accurate to 4 digits).

EXERCISE 11.3 [A/C:15] Repeat (a) of the previous exercise for the integrals

$$\int_{A^e} \frac{\zeta_1}{r} dA^e, \quad \int_{A^e} \frac{\zeta_2}{r} dA^e, \quad \int_{A^e} \frac{\zeta_3}{r} dA^e. \quad (\text{E11.13})$$

The exact integrals are A/a for the first one, and $\frac{1}{2}A/a$ for the other two. Why are all numerical quadrature formula exact for the latter case?

EXERCISE 11.4 [A/C:20] The isoparametric definition of the 3-node linear SOR triangle is

$$\begin{bmatrix} 1 \\ r \\ z \\ u_r \\ u_z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ z_1 & z_2 & z_3 \\ u_{r1} & u_{r2} & u_{r3} \\ u_{z1} & u_{z2} & u_{z3} \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} \quad (\text{E11.14})$$

Using results stated in Exercise 11.1 as appropriate, compute the entries

$$K_{r1r1}, \quad K_{r1z1}, \quad K_{z1z1} \quad (\text{E11.15})$$

of the element stiffness matrix \mathbf{K}^e for the geometry defined by

$$r_1 = 0, \quad r_2 = r_3 = a, \quad z_1 = z_2 = 0, \quad z_3 = b. \quad (\text{E11.16})$$

The material is isotropic with $\nu = 0$ for which the stress-strain matrix is

$$\mathbf{E} = E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}, \quad (\text{E11.17})$$

Note: For integrals that contain $1/r$, use one of the 3-point rules. Partial answer (if midpoint rule used) $K_{r1r1} = \frac{1}{2}Eb$.

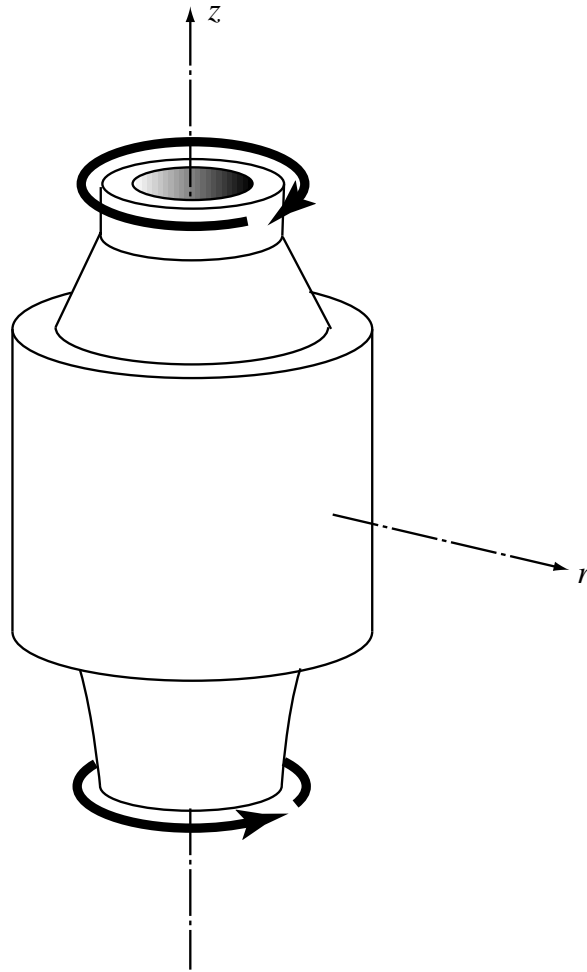


Figure E11.1. Variable-section circular shaft conveying torque: the subject of Exercise 11.11.

EXERCISE 11.5 [A/C:20] The triangle of Exercise 11.4 is subjected to the radial-centrifugal body force field

$$b_r = \rho\omega^2 r, \quad b_z = 0 \quad (\text{E11.18})$$

where ρ and ω are constant (ρ is the mass density while ω is the angular velocity.)

Compute the consistent node force vector \mathbf{f}^e . In doing so interpolate the radial component b_r linearly over the element:

$$b_r(\zeta_1, \zeta_2, \zeta_3) = b_{r1}\zeta_1 + b_{r2}\zeta_2 + b_{r3}\zeta_3, \quad (\text{E11.19})$$

where b_{ri} is $\rho\omega^2 r$ evaluated at corner i . Partial answers: $f_{r1} = \omega a^3/10$, $f_{z1} = 0$.

EXERCISE 11.6 [A/C:20] Repeat the previous Exercise for the own-weight body force field

$$b_r = 0, \quad b_z = -\rho g \quad (\text{E11.20})$$

where ρ and g are constant. Partial answer: $f_{z2} = -ga/4$.

EXERCISE 11.7 [A/C:20] For the triangle geometry of the preceding 3 exercises, find the consistent thermal forces pertaining to a uniform temperature increase ΔT , assuming an isotropic material with zero Poisson's ratio.

EXERCISE 11.8 [A/C:15] Show that $u_z = c$ (c is a constant) is the only possible rigid body mode of a SOR element. (Hint: consider the presence of the circumferential strain). Hence deduce that the correct rank of the stiffness matrix of a SOR element with n nodes and 2 DOFs per node is $2n - 1$.

EXERCISE 11.9 [A/C:15] Find the displacement fields that separately generate the following constant strain states:

$$e_{zz} = c_{zz}, \quad \text{others zero} \quad (\text{E11.21})$$

$$\gamma_{rz} = c_{rz}, \quad \text{others zero} \quad (\text{E11.22})$$

where c_{zz} and c_{rz} are constants.

EXERCISE 11.10 [A/C:15] Show that there are no displacement fields that separately generate the following constant strain states:

$$e_{rr} = c_{rr}, \quad \text{others zero} \quad (\text{E11.23})$$

$$e_{\theta\theta} = c_{\theta\theta}, \quad \text{others zero} \quad (\text{E11.24})$$

where c_{rr} and $c_{\theta\theta}$ are constants. Hint: integrate the appropriate strain-displacement relations.

EXERCISE 11.11 [A/C:25] The SOR sketched in Figure E11.1 (a circular shaft with varying cross section) is subjected to torsional loading as indicated. According to Saint-Venant's torsion theory, the displacement components for this case are entirely circumferential, that is, $u_r = u_z = 0$ and $u_\theta = u_\theta(r, z)$. The torsional shear strains

$$\gamma_{r\theta} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}, \quad \gamma_{z\theta} = \frac{\partial u_\theta}{\partial z}, \quad (\text{E11.25})$$

are nonzero and functions of r, z only; all other strains ($e_{rr}, e_{zz}, e_{\theta\theta}$ and γ_{rz}) vanish. Assuming the shaft is fabricated with an isotropic material, the only nonzero stress components are the shear stresses

$$\sigma_{r\theta} = G \gamma_{r\theta}, \quad \sigma_{z\theta} = G \gamma_{z\theta}, \quad (\text{E11.26})$$

where $G = \frac{1}{2}E/(1 + \nu)$ is the shear modulus.

- Explain why this problem can be discretized by two-dimensional "ring" finite elements by laying out a mesh over the (r, z) plane ($r \geq 0$), although the element type is different from that considered previously in this Chapter. How many degrees of freedoms would these "torqued ring" elements have per node?
- If an isoparametric formulation is used for the torqued-ring elements of (a), the element stiffness matrix, on suppressing the 2π factor, is given by the usual expression

$$\mathbf{K}^e = \int_{A^e} \mathbf{B}^T \mathbf{E} \mathbf{B} r \, dA^e. \quad (\text{E11.27})$$

But how would \mathbf{B} and \mathbf{E} look like?