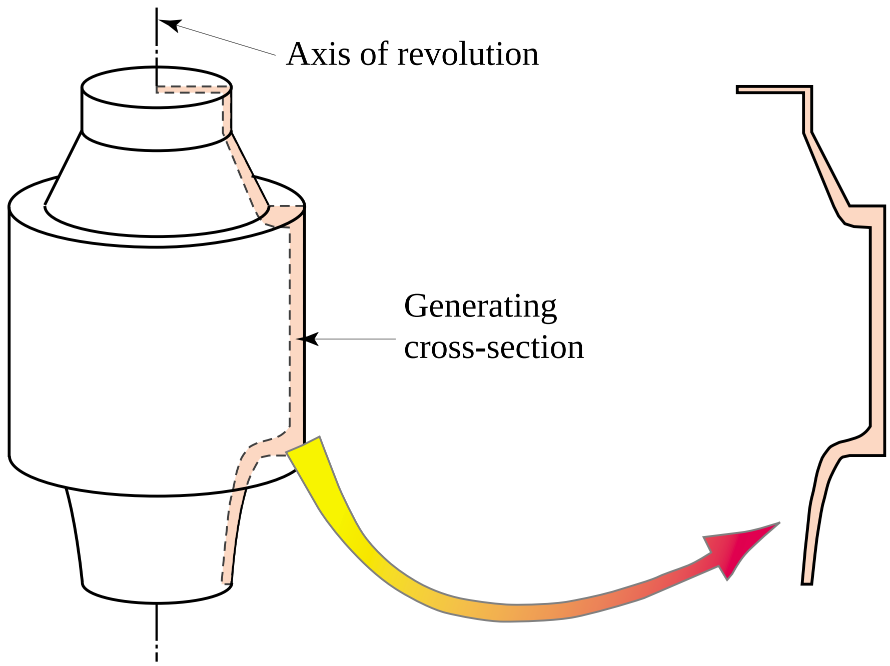
10

Axisymmetric Solids (Structures of Revolution)

Structures of Revolution are Produced by Rotating a Generating Cross Section Through 360 Degrees



For Problem to be Axisymmetric, both Loads and Support BCs must be Rotationally Symmetric

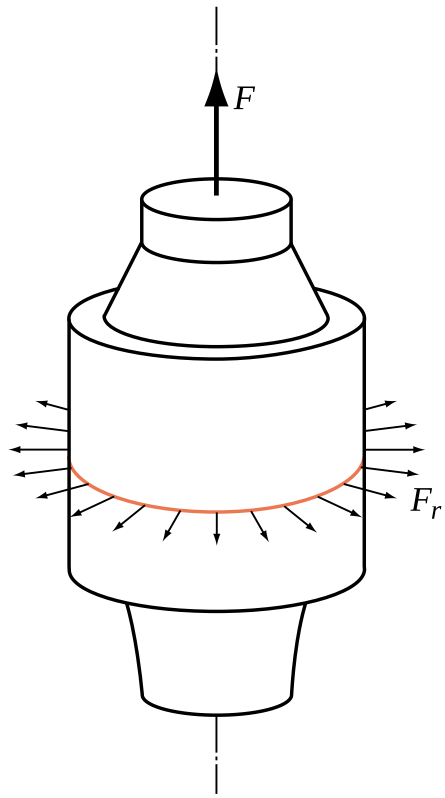
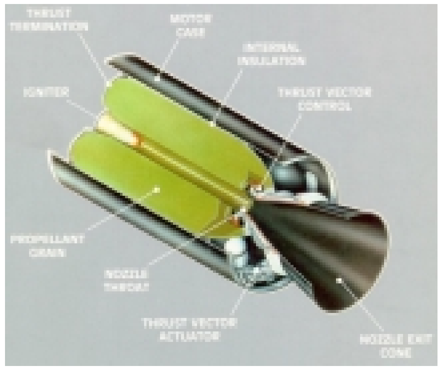
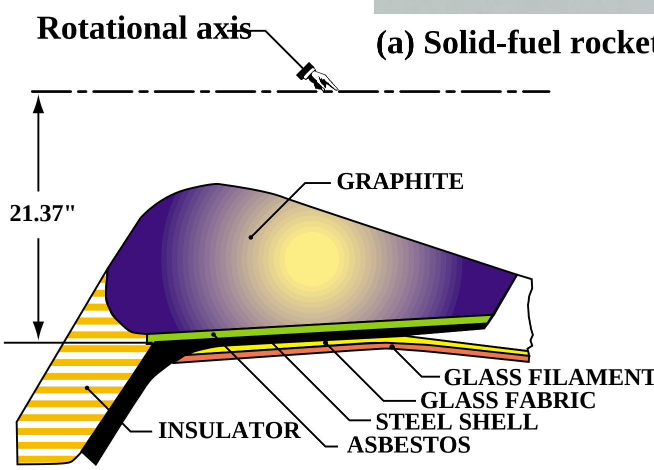


Illustration for line load F_r and point load F

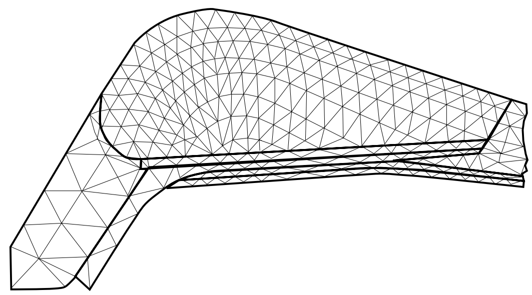
First FEM Analysis of Axisymmetric Solid (Wilson, 1963)



(a) Solid-fuel rocket schematics

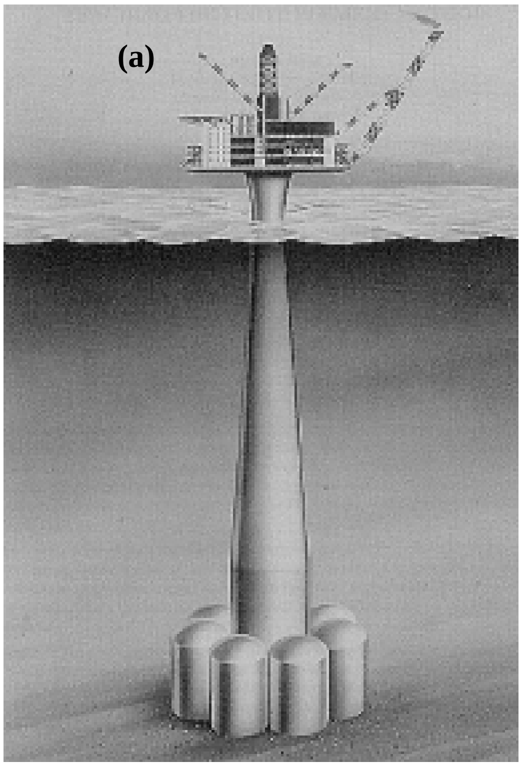


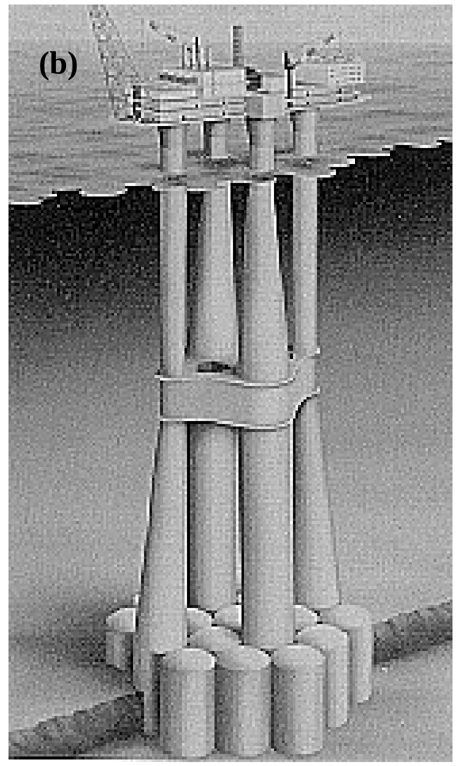
(b) Nozzle exit cone



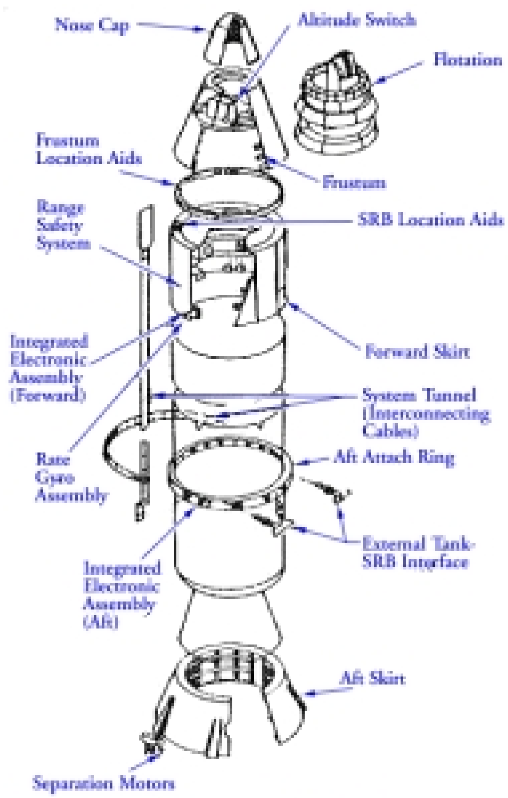
(c) Finite element idealization

"Quasi-Axisymmetric" Marine Structures



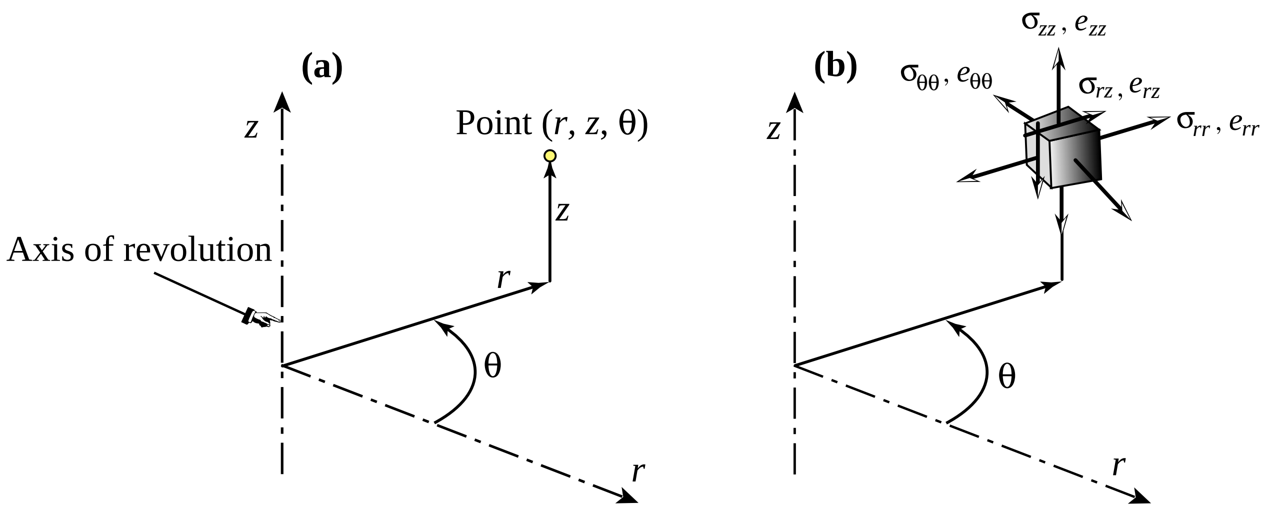


Rockets are Also Quasi-Axisymmetric



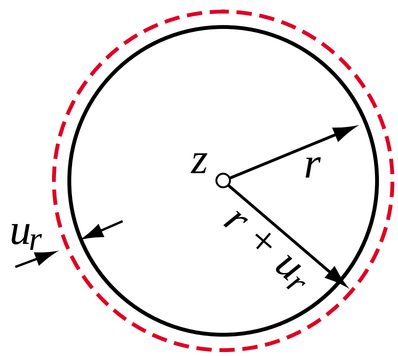
Solid Rocket Booster - Exploded View

Cylindrical Coordinate System (r, z, θ)



Global coordinate system **Nonvanishing strains and stresses**

A Feature Difference from Plane Stress: the Circumferential or "Hoop" Strain and Associated Stress



The length of the original circumference is $2\pi r$, which grows to $2\pi(r+u_r)$, inducing a hoop strain of $2\pi u_r/(2\pi r) = u_r/r$

Kinematic Equations (KE)
Strain-Displacement Relations

$$e_{rr} = \frac{\partial u_r}{\partial r} \qquad e_{zz} = \frac{\partial u_z}{\partial z} \qquad e_{\theta\theta} = \frac{u_r}{r}$$

$$\gamma_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} = e_{rz} + e_{zr} = 2e_{rz}$$

Note that hoop strain is **not** given by partial derivative

In matrix form

$$\mathbf{e} = \begin{bmatrix} e_{rr} \\ e_{zz} \\ e_{\theta\theta} \\ \gamma_{rz} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ \frac{1}{r} & 0 \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \end{bmatrix} \begin{bmatrix} u_r \\ u_z \end{bmatrix} = \mathbf{D}\mathbf{u}$$

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Constitutive (Stress-Strain) Equations

Ignoring temperature and prestress effects:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{rr} \\ \sigma_{zz} \\ \sigma_{\theta\theta} \\ \sigma_{rz} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} \\ E_{12} & E_{22} & E_{23} & E_{24} \\ E_{13} & E_{23} & E_{33} & 0 \\ E_{14} & E_{24} & 0 & E_{44} \end{bmatrix} \begin{bmatrix} e_{rr} \\ e_{zz} \\ e_{\theta\theta} \\ \gamma_{rz} \end{bmatrix} = \mathbf{E}\,\mathbf{e}$$

If material is isotropic

$$\mathbf{E} = \frac{E}{(1+\nu)\,(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix}$$

Notice that if ν -> 1/2 (incompressible material, such as a solid rocket propellant) the foregoing elasticity matrix "blows up"

Equilibrium (Balance) Equations

General (3D) stress equilibrium equations in cylindrical coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rr}) + \frac{1}{r}\frac{\partial}{\partial \theta}\sigma_{r\theta} + \frac{\partial}{\partial z}\sigma_{rz} - \frac{\sigma_{\theta\theta}}{r} + b_r = 0$$
$$\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{zr}) + \frac{1}{r}\sigma_{z\theta} + \frac{\partial}{\partial z}\sigma_{zz} + b_z = 0$$
$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\sigma_{\theta r}) + \frac{1}{r}\frac{\partial}{\partial \theta}\sigma_{\theta\theta} + \frac{\partial}{\partial z}\sigma_{\theta z} + b \;\; = 0$$

For the axisymmetric problem shear stresses $\sigma_{r\theta}$ and $\sigma_{z\theta}$ as well as the hoop body force b_θ vanish, and $\sigma_{\theta\theta}$ is independent of θ, whence

$$\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rr}) + \frac{\partial}{\partial z}\sigma_{rz} - \frac{\sigma_{\theta\theta}}{r} + b_r = 0$$
$$\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{zr}) + \frac{\partial}{\partial z}\sigma_{zz} + b_z = 0$$

Total Potential Energy (TPE) Functional in Terms of Original Body Volume and Surface

$$\Pi[\mathbf{u}] = U[\mathbf{u}] - W[\mathbf{u}]$$

$$U[\mathbf{u}] = \frac{1}{2}\int_V \boldsymbol{\sigma}^T \mathbf{e}\, dV = \frac{1}{2}\int_V \mathbf{e}^T \mathbf{E}\, \mathbf{e}\, dV$$

$$= \frac{1}{2}\int_V \begin{bmatrix} e_{rr} \\ e_{zz} \\ e_{\theta\theta} \\ 2e_{rz} \end{bmatrix}^T \begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} \\ E_{12} & E_{22} & E_{23} & E_{24} \\ E_{13} & E_{23} & E_{33} & 0 \\ E_{14} & E_{24} & 0 & E_{44} \end{bmatrix} \begin{bmatrix} e_{rr} \\ e_{zz} \\ e_{\theta\theta} \\ 2e_{rz} \end{bmatrix} dV$$

where strains are derived from displacements (master-field superscript u omitted to reduce clutter)

$$W[\mathbf{u}] = W_b[\mathbf{u}] + W_t[\mathbf{u}]$$

$$W_b[\mathbf{u}] = \int_V \mathbf{b}^T \mathbf{u}\, dV = \int_V [\, b_r \quad b_z \,] \begin{bmatrix} u_r \\ u_z \end{bmatrix} dV$$

$$W_t[\mathbf{u}] = \int_{S_t} \hat{\mathbf{t}}^T \mathbf{u}\, dS = \int_{S_t} [\, \hat{t}_r \quad \hat{t}_z \,] \begin{bmatrix} u_r \\ u_z \end{bmatrix} dS$$

Problem Dimensionality Reduction 3D -> 2D

Element of volume:

$dV = 2\pi r\, dA$ reduces

$$U = \tfrac{1}{2} 2\pi \int_A r\, \mathbf{e}^T \mathbf{E}\, \mathbf{e}\, dA$$

$$W_b = 2\pi \int_A r\, \mathbf{b}^T \mathbf{u}\, dA$$

Element of surface:

$dS = 2\pi r\, ds$ reduces

$$W_t = 2\pi \int_{S_t} r\, \mathbf{t}^T \mathbf{u}\, ds$$

Most FEM implementations cancel out the 2π factor

Elimination of 2π Factor Works Fine Except for Point Load

Line load with components

$$W_F = 2\pi r(F_r u_r + F_z u_z)$$

→ no problem

But **watch out** for point load F along z

Correct work term: $W_F = F u_z$

fits 2π-cancellation if F is divided by 2π: $W_F = 2\pi \left(\frac{F}{2\pi}\right) u_z$