Advanced FEM

8
Hybrid Variational Principles: Formulation

Hybrid Variational Principles: Where Do They Fit?

Variational Principles
- **Single-field**
- **Multifield**
  - **Mixed: all masters are internal**
  - **Hybrid**
    - **Internally single field**
    - **Internally multifield**

Hybrid principle has multiple masters

One or more masters are interface fields, a.k.a. connector fields or simply connectors

Unlike Traditional Variational Principles, Hybrid Formulations Are Relatively Young

Continuum Mechanics Context

Prager 1967, to treat discontinuities (jumps) at physical interfaces

Finite Element Context

Pian 1964
Pian and Tong 1969
Atluri 1975
Pian and Sumihara 1984

Neither Prager nor Pian were good salesmen,
so idea didnt get much attention until the mid 1980s

Practically Important Because

The best performing finite elements in many commercial codes are hybrid or trans-hybrid

However, theory is not fully developed and documentation is lagging

Motivation for Hybrid Functionals in FEM

Relaxed continuity requirements

Balanced accuracy (displacements and stresses)

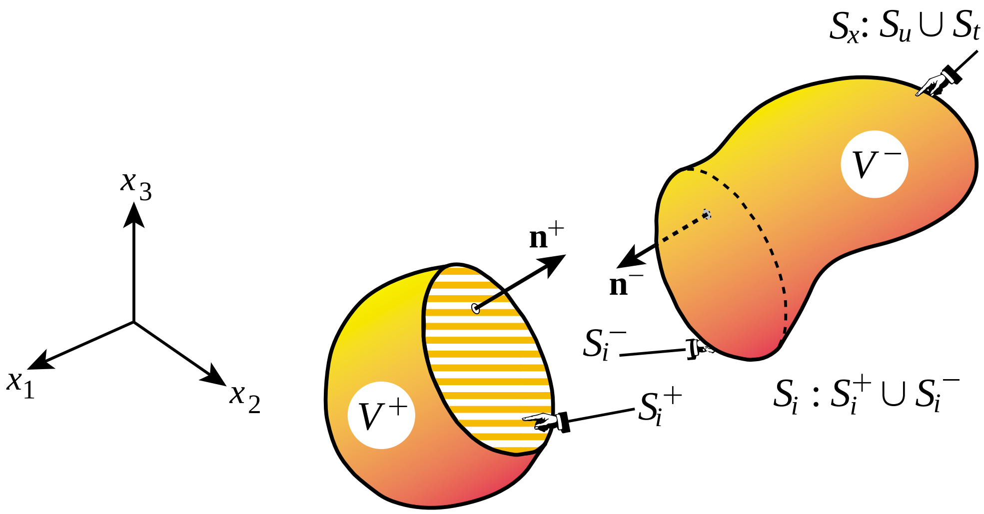
Fit naturally in Direct Stiffness Method framework: from the outside, stress-assumed hybrids look exactly like displacement elements

How about mixed functionals?

In 2D or 3D may require non-condensable DOFs and hence dont fit well into DSM-based commercial codes. Fine for 1D or special purpose programs, for example those treating incompressible media

Combination of mixed+hybrid has not been fully explored

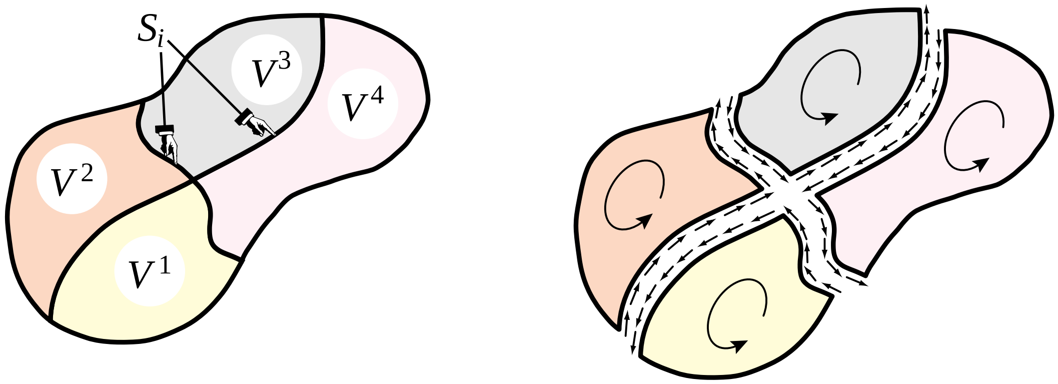
Slicing the Potato through an Internal Boundary



(This was Prager's concept; S_i was a surface of discontinuity)

Traversing an Internal Boundary

(pictured in 2D for convenient visualization)



There are **two sides** to an internal boundary $\dot{S}$, and **two exterior normals** at each point of S_i , which go in opposite directions

Breaking up Integrals Over the Sliced Potato

$$\int_V f\, dV = \sum_{m=1}^{M} \int_{V^m} f\, dV$$

$$\int_S g\, dS = \int_{S_u} g\, dS + \int_{S_t} g\, dS + \int_{S_i} g\, dS$$

If g is the projection over normal: g = **f.n**, and **f** is continuous over S_i, the interface integral vanishes

Hybridization

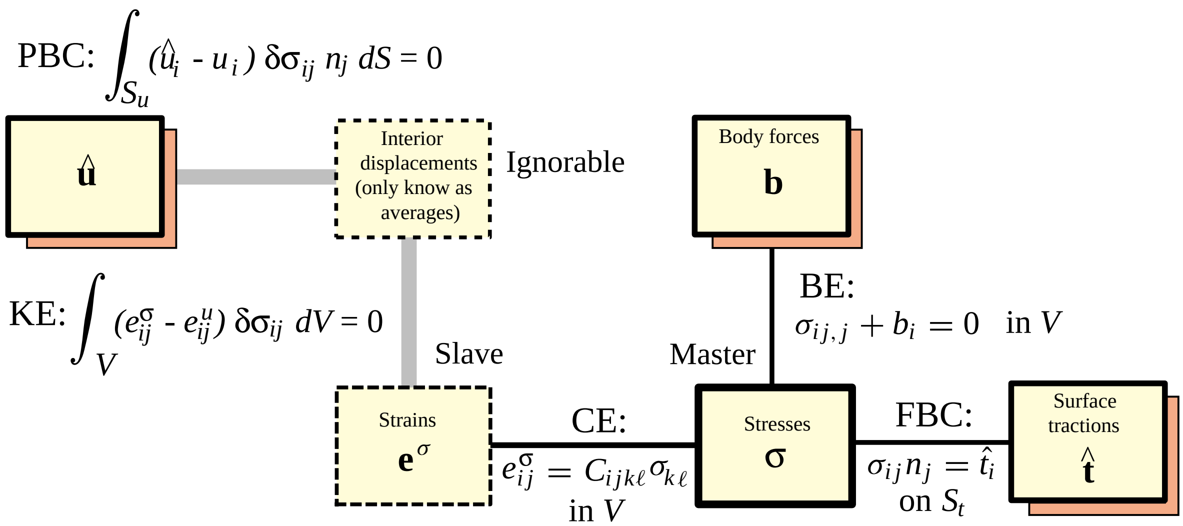
General rule:

Hybrid Principle = Interior Functional + Interface Potential

For the ***Equilibrium Stress Hybrid Functional***
(one of the simplest) the **interior functional** is the
Total Complementary Potential Energy:

$$\Pi_C[\sigma_{ij}] = -\tfrac{1}{2}\int_V \sigma_{ij} C_{ijk\ell}\, \sigma_{k\ell}\, dV + \int_{S_u} \hat{u}_i \sigma_{ij} n_j\, dS = -U_C + W_C$$

Recall the Weak Form Diagram of TCPE for Linear Elastostatics



The Equilibrium Stress Hybrid Functional

Add an **interface potential** term to TPCE. This term is an integral over the interior boundary:

Interface Potential

$$\Pi_C^d[\sigma_{ij}, d_i] = \Pi_C[\sigma_{ij}] + \pi_d[\sigma_{ij}, d_i] = \Pi_C[\sigma_{ij}] + \int_{S_i} d_i \, \sigma_{ij} n_j \, dS$$

Here d_i , the **second master**, is the **interface displacement field.** Rewrite above as

$$\Pi_C^d = -U_C + W_d$$

where W_d is the **work potential:**

$$W_d = \int_{S_u} \hat{u}_i \, \sigma_{ij} \, n_j \, dS + \int_{S_i} d_i \, \sigma_{ij} \, n_j \, dS$$

The Equilibrium Stress Hybrid Functional (cont'd)

(no textbook has this derivation right)

Eliminate interior integral (Interface Potential) from the identity

$$\int_{S_i} d_i \, \sigma_{ij} n_j \, dS = \int_S d_i \sigma_{ij} \, n_j \, dS - \int_{S_u} d_i \sigma_{ij} n_j \, dS - \int_{S_t} d_i \sigma_{ij} \, n_j \, dS$$

Interface Potential

Replace in the work potential, and simplify (see explanation in Notes) to get

$$W_d = \int_S d_i \, \sigma_{ij} n_j \, dS \; - \int_{S_t} d_i \, \hat{t}_i \, dS$$

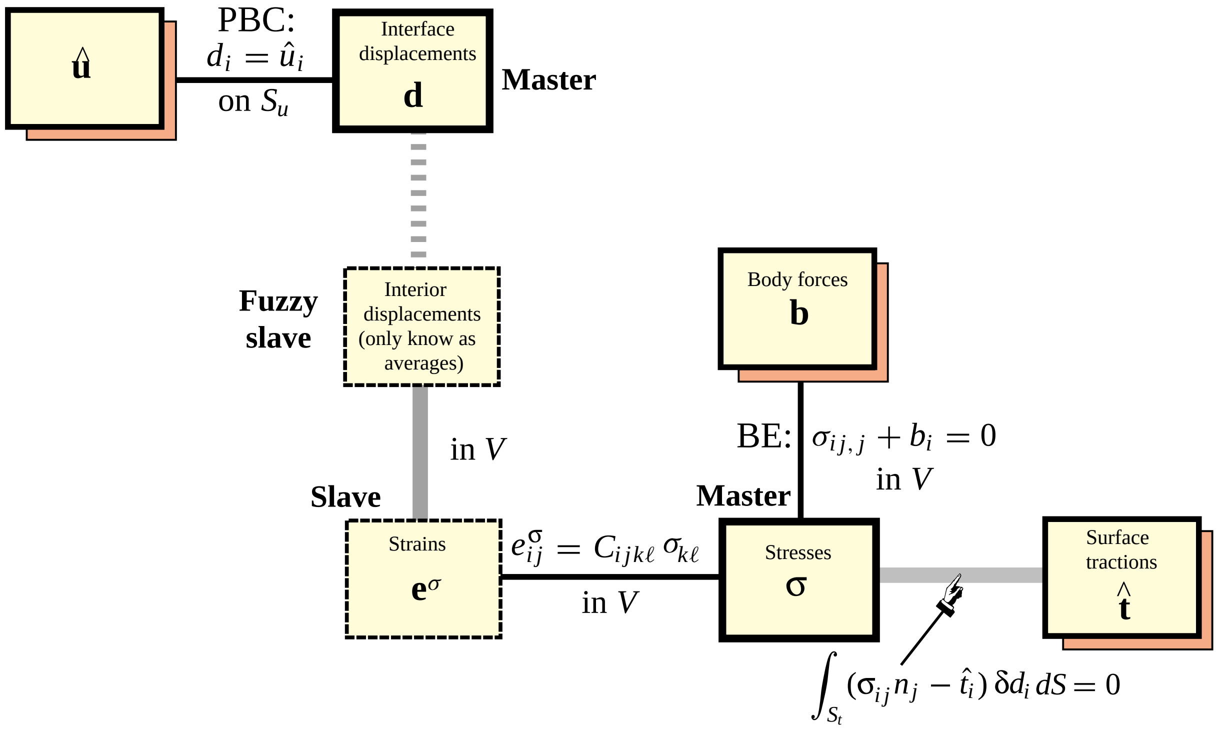
over entire boundary!

We thus arrive at the final form of the Equilibrium Stress Hybrid Functional

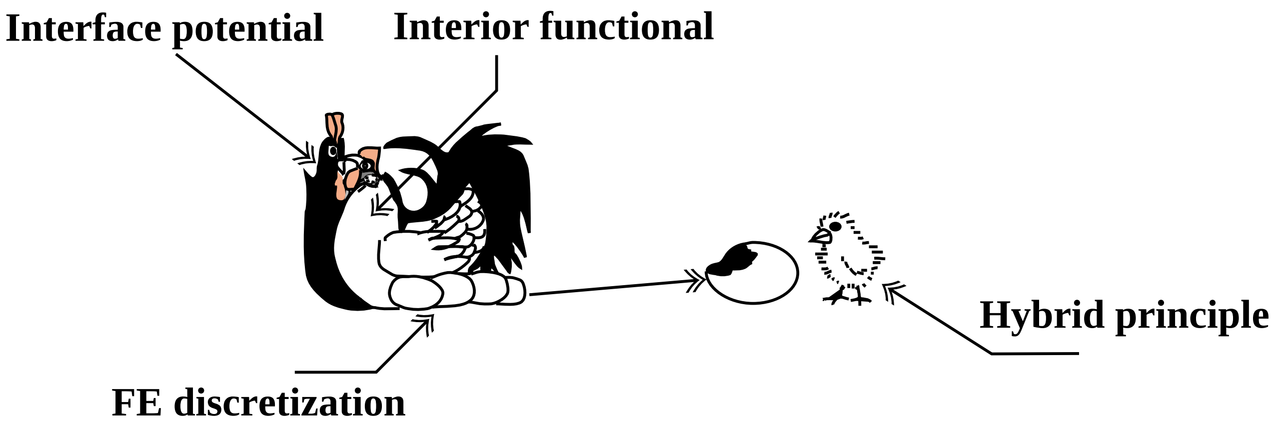
$$\Pi_C^d[\sigma_{ij}, d_i] = -\tfrac{1}{2} \int_V \sigma_{ij} C_{ijk\ell} \, \sigma_{k\ell} \, dV + \int_S d_i \, \sigma_{ij} n_j \, dS - \int_{S_t} d_i \hat{t}_i \, dS$$

Weak Form of Stress Hybrid Functional

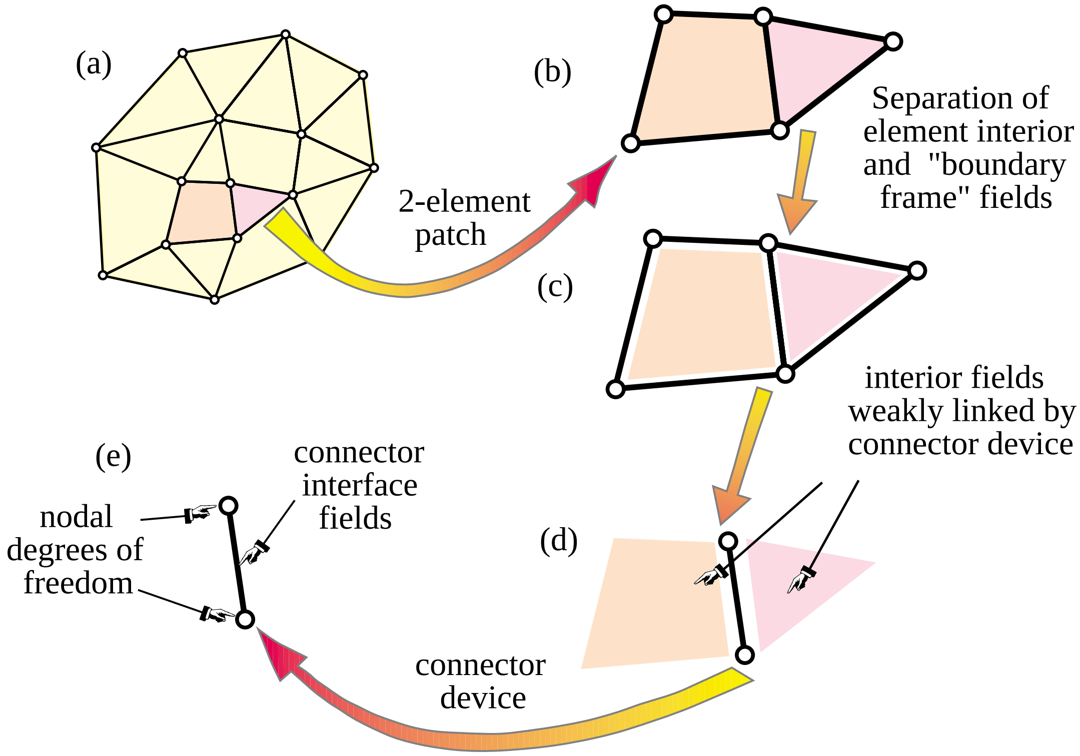
(Note: diagram needs improvement; suggestions welcome)



The Chicken and Egg Story, Revisited



Schematics of FEM Derivation from a Hybrid Variational Principle



The 4-Node Plane Stress Quadrilateral Derived as an Equilibrium Stress Hybrid

(next lecture)