

Homework Exercises for Chapter 7
Solutions

EXERCISE 7.1

Not assigned.

EXERCISE 7.2

Integrate the Nu' term by parts:

$$\Pi_{HR} = \int_0^L \left(Nu' - \frac{N^2}{2EA} \right) dx - f_1 u_1 - f_2 u_2 = - \int_0^L \frac{N^2}{2EA} dx - \int_0^L u N' dx + Nu|_0^L - f_1 u_1 - f_2 u_2 \quad (\text{E7.1})$$

The exact solution is constant N , which is assumed in the element. Thus $N' = 0$ and the functional reduces to

$$\Pi_{HR} = - \int_0^L \frac{N^2}{2EA} dx + N(u_2 - u_1) - f_1 u_1 - f_2 u_2. \quad (\text{E7.2})$$

The internal $u(x)$ has disappeared, and the axial displacements only come in through their end values u_1 and u_2 . Therefore, it does not matter what is taken for the displacement field inside the element as long as a constant N is assumed. This is a variational freak since it applies only to that specific example problem.

EXERCISE 7.3 The weak links are (cf. Figure E7.1):

$$e_{ij}^u - e_{ij}^\sigma = 0 \quad \text{in } V, \quad \hat{u}_i - u_i = 0 \quad \text{on } S_u, \quad (\text{E7.3})$$

where $e_{ij}^u = \frac{1}{2}(u_{i,j} + u_{j,i})$ and $e_{ij}^\sigma = C_{ijkl}\sigma_{kl}$. Multiply the residuals (E7.3) by $\delta\sigma_{ij}$ and $\delta t_i = \delta\sigma_{ij}n_j$ and integrate over V and S_u , respectively:

$$\int_V (e_{ij}^u - e_{ij}^\sigma) \delta\sigma_{ij} dV + \int_{S_u} (\hat{u}_i - u_i) \delta\sigma_{ij}n_j dS = 0. \quad (\text{E7.4})$$

Apply the divergence theorem to the first term on the left:

$$\int_V e_{ij}^u \delta\sigma_{ij} dV = \int_V \frac{1}{2}(u_{i,j} + u_{j,i}) \delta\sigma_{ij} dV = - \int_V u_i \delta\sigma_{ij,j} dV + \int_S u_i \delta\sigma_{ij}n_j dS, \quad (\text{E7.5})$$

in which the indicated term vanishes because the BE are strongly satisfied: $\delta(\sigma_{ij,j} + b_i) = \delta\sigma_{ij,j} = 0$ in V . Replacing into (E7.4) gives

$$\delta\Pi_{\text{TCPE}} = \int_V -C_{ijkl}\sigma_{kl} \delta\sigma_{ij} dV + \int_S u_i \delta\sigma_{ij}n_j dS + \int_{S_u} (\hat{u}_i - u_i) \delta\sigma_{ij}n_j dS = 0. \quad (\text{E7.6})$$

Split the S integral over S_t and S_u . Over S_t we can set $\delta\sigma_{ij}n_j = \delta(\sigma_{ij}n_j) = \delta\hat{t}_i = 0$ because the FBCs are strongly satisfied. The integrals of $u_i \delta\sigma_{ij}n_j$ over S_u cancel out and we are left with

$$\delta\Pi_{\text{TCPE}} = - \int_V C_{ijkl}\sigma_{kl} \delta\sigma_{ij} dV + \int_{S_u} \hat{u}_i \delta\sigma_{ij}n_j dS = 0. \quad (\text{E7.7})$$

This is the exact first variation of the complementary energy functional (E7.1).

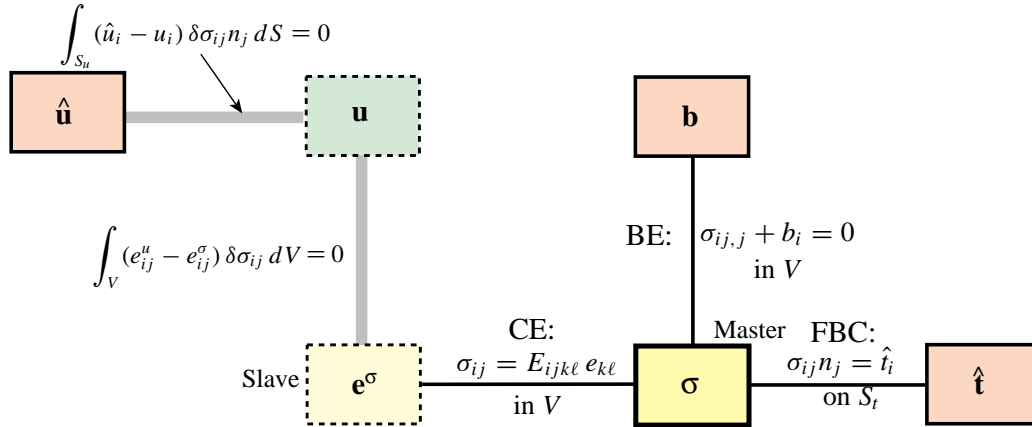


Figure E7.1. Departure Weak Form to derive the TCPE functional.

EXERCISE 7.4 The only weak connection is $\sigma_{ij} = E_{ijkl} e_{kl}$. We begin as above, trying

$$\delta \Pi_S = \int_V (\sigma_{ij} - E_{ijkl} e_{kl}) \delta e_{ij} dV, \tag{E7.8}$$

where σ_{ij} must be viewed as a data field.⁶ This is the exact variation of

$$\Pi_S(e_{ij}) = \int_V (\sigma_{ij} - \frac{1}{2} E_{ijkl} e_{kl}) e_{ij} dV. \tag{E7.9}$$

And this is the end. No further progress can be made. This is the only canonical functional of elasticity that contains no boundary integrals.

In the Exercise statement it was noted that functional Π_S has limited practical value. The reason is that all of the difficult field equations are taken as strong. The stress field must satisfy both equilibrium equations and stress BC point by point *a priori*, while the strain field must be compatible with a displacement field that satisfies the displacement BC. The only relaxation of the governing equations pertains to the constitutive equations.

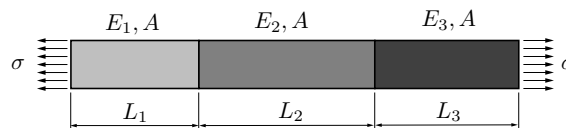


Figure E7.2. Application of the strain-only canonical functional to material homogenization.

Nevertheless the principle may be occasionally useful in material homogenization, as the following simple example illustrates. Consider a bar of uniform cross section A and total length $L = L_1 + L_2 + L_3$ made up of three materials with elastic moduli E_1, E_2 and E_3 , respectively. Using the functional Π_S , find an average modulus E .

⁶ Why? Because the stresses must satisfy the volume equilibrium equations and the surface traction BC *a priori*. Thus the stress field must be known at every point in V .

To carry out the homogenization process, assume that the bar is under a constant axial stress field $\sigma = P/A$ (see Figure E7.2), which obviously satisfies all stress equilibrium equations and surface traction boundary conditions. The average strain $e = E^{-1}\sigma$ is taken as the only unknown to be varied in Π_S :

$$\Pi_S(e) = AL\sigma e - \frac{1}{2}A(E_1L_1 + E_2L_2 + E_3L_3)e^2. \quad (\text{E7.10})$$

The condition $\delta\Pi_S = 0$ gives $\partial\Pi_S/\partial e = 0$, from which

$$e = \frac{L}{E_1L_1 + E_2L_2 + E_3L_3}\sigma, \quad E = \frac{\sigma}{e} = \frac{E_1L_1 + E_2L_2 + E_3L_3}{L}. \quad (\text{E7.11})$$

This technique essentially amounts to equating the strain energies absorbed by the actual and homogenized (fictitious) bars. Note that the displacement field does not appear in this statically determinate problem.

EXERCISE 7.5

Yes, it does if the assumed moment \bar{M} and curvature $\bar{\kappa}$ can be strongly linked by the constitutive equation: $\bar{M} = EI\bar{\kappa}$, point by point. This is the case in the example because both are taken as multiple of ξ . Making κ a slave of M by setting $\kappa = \kappa^M = M/EI$ in VHW collapses the functional to HR.

EXERCISE 7.6

Setting $m=n=0$ the script yields the complete stiffness of the shearless beam

$$\begin{bmatrix} EIL & -L & 0 & 0 & 0 & 0 \\ -L & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (\text{E7.12})$$

Upon condensing the curvature and moment DOFs:

$$\frac{EI}{L} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \quad (\text{E7.13})$$

EXERCISE 7.7 Fitting moment and curvature variations to pass through two noncoincident zero points (the hinges) produces $N = 0$ and $\kappa = 0$ over the element. The full stiffness is the null matrix, which cannot be statically condensed.

EXERCISE 7.8 The functional collapses to the TPE.