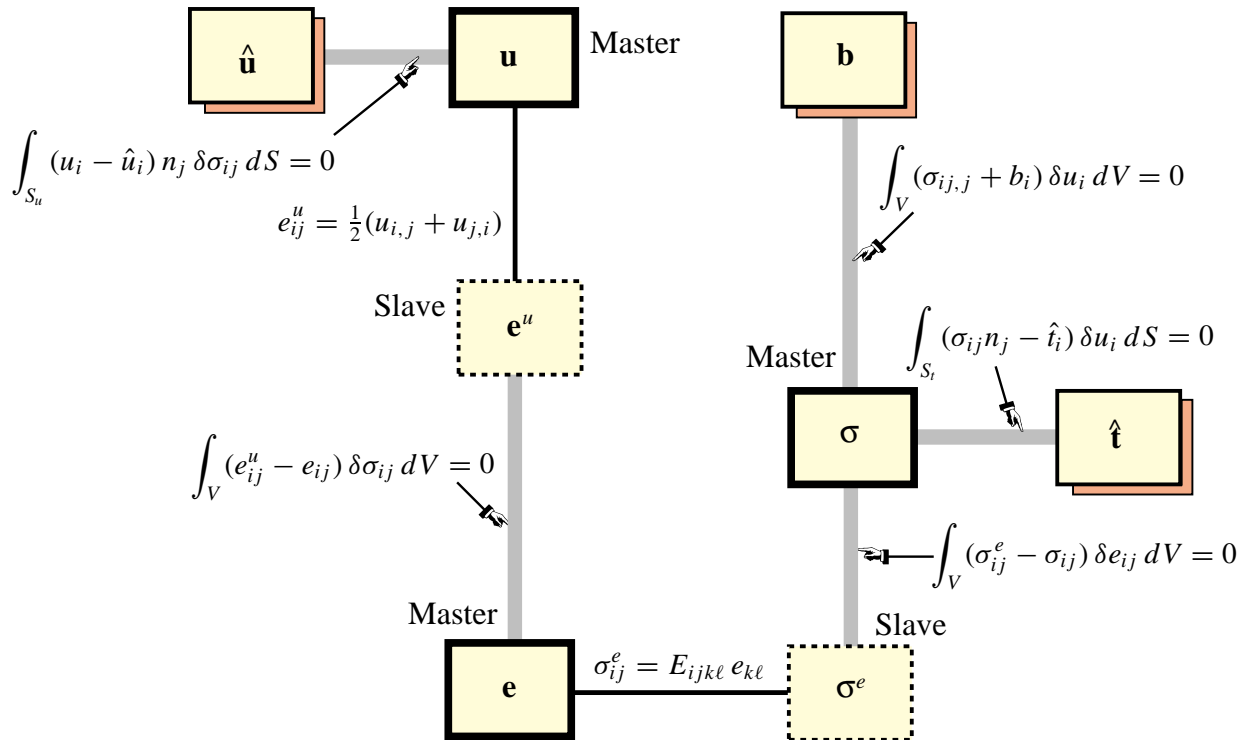


7

The Three-Field Mixed Principle of Elastostatics

Weak Form for Derivation of the VHW Mixed Functional

Advanced FEM



Derivation of the Generalized VHW Mixed Functional

Fast track: integrate residuals, set work pairs, add weak links:

$$\begin{aligned} \delta \Pi_{\text{VHW}}^s = & \int_V (e_{ij}^u - e_{ij}) \delta \sigma_{ij} dV + \int_V (\sigma_{ij}^e - \sigma_{ij}) \delta e_{ij} dV - \int_V (\sigma_{ij,j} + b_i) \delta u_i dV \\ & + \int_{S_t} (\sigma_{ij} n_j - \hat{t}_i) \delta u_i dS - \int_{S_u} (u_i - \hat{u}_i) n_j \delta \sigma_{ij} dS. \end{aligned}$$

Use the Divergence Theorem to get rid of stress derivatives:

$$\begin{aligned} - \int_V \sigma_{ij,j} \delta u_i & \stackrel{\text{DT}}{=} \int_V \sigma_{ij} \delta e_{ij}^u - \int_S \sigma_{ij} n_j \delta u_i dS \\ & = \int_V \sigma_{ij} \delta e_{ij}^u - \int_{S_u} \sigma_{ij} n_j \delta u_i dS - \int_{S_t} \sigma_{ij} n_j \delta u_i dS. \end{aligned}$$

Derivation of the Generalized VHW Mixed Functional (cont'd)

Substitute into above and collect terms

$$\begin{aligned} \delta \Pi_{\text{VHW}}^g = & \int_V \left[(e_{ij}^u - e_{ij}) \delta \sigma_{ij} + (\sigma_{ij}^e - \sigma_{ij}) \delta e_{ij} + \sigma_{ij} \delta e_{ij}^u - b_i \delta u_i \right] dV \\ & - \int_{S_t} \hat{t}_i \delta u_i dS - \int_{S_u} [(u_i - \hat{u}_i) n_j \delta \sigma_{ij} + \sigma_{ij} n_j \delta u_i] dS \end{aligned}$$

This is the exact first variation of

$$\Pi_{\text{VHW}}^g = \int_V [\sigma_{ij}(e_{ij}^u - e_{ij}) + \mathcal{U}(e_{ij}) - b_i u_i] dV - \int_{S_t} \hat{t}_i u_i dS - \int_{S_u} (u_i - \hat{u}_i) \sigma_{ij} n_j dS$$

in which

$$\mathcal{U}(e_{ij}) = \frac{1}{2} e_{ij} E_{ijkl} e_{kl} = \frac{1}{2} \sigma_{ij}^e e_{ij}$$

is the strain energy density in terms of master strains

Derivation of the Generalized VHW Mixed Functional (cont'd)

Variational principle:

$$\delta \Pi_{\text{VHW}}^g = 0$$

This principle yields the weak internal connections as Euler-Lagrange equations, and the weak BCs as natural BCs

Standard Form of VHW Mixed Functional

Standard form of the functional is obtained by enforcing displacement BCs as strong link:

$$\Pi_{\text{VHW}}[u_i, \sigma_{ij}, e_{ij}] = \int_V [\sigma_{ij}(e_{ij}^u - e_{ij}) + \mathcal{U}(e_{ij}) - b_i u_i] dV - \int_{S_t} \hat{t}_i u_i dS.$$

For FEM work it is usually split into internal energy and external work:

$$\Pi_{\text{VHW}} = U_{\text{VHW}} - W_{\text{VHW}}$$

$$U_{\text{VHW}} = \int_V [\sigma_{ij}(e_{ij}^u - e_{ij}) + \mathcal{U}(e_{ij})] dV$$

$$W_{\text{VHW}} = \int_V b_i u_i dV + \int_{S_t} \hat{t}_i u_i dS$$

Variational Indices of Master Fields of VHW Functional

Displacement variational index: 1

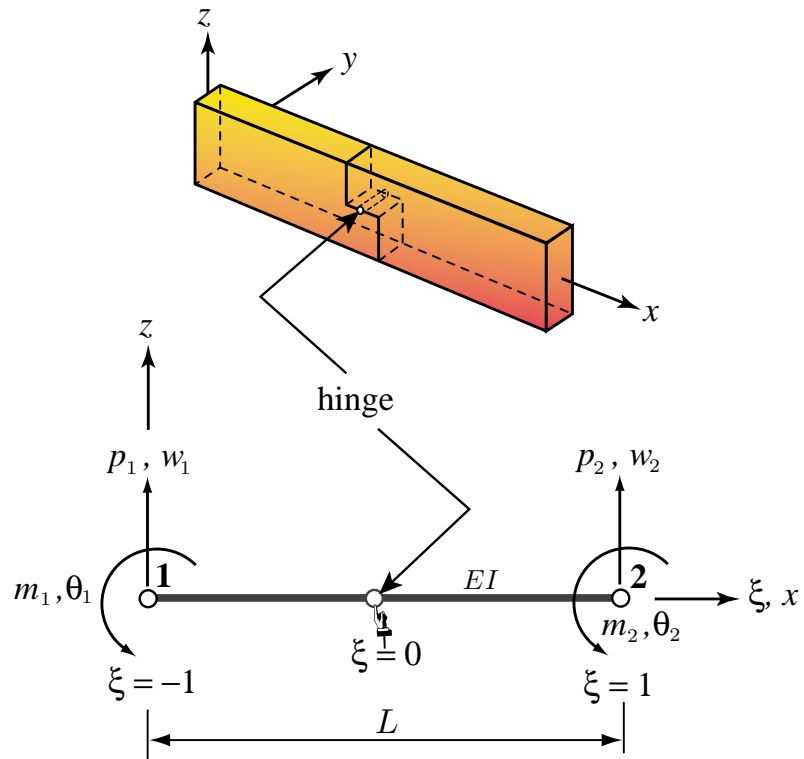
Stress and ***strain*** variational index: 0

Consequence for FEM model development:

Displacements must be interelement continuous, that is, C^0 , but derivatives need not be

Stresses and ***strains*** may be discontinuous between elements, that is, C^{-1}

VHW Application Example: Hinged Plane Beam Element



VHW Formulation of Hinged Plane Beam Element

VHW Functional in terms of displacement, bending moment and curvature masters for Bernoulli-Euler beam (Chapter 4)

$$\Pi_{\text{HW}}[w, M, \kappa] = \int_L \left[M(\kappa^w - \kappa) + \frac{1}{2}EI\kappa^2 \right] dx - f_1 w_1 - f_2 w_2 - m_1 \theta_1 - m_2 \theta_2$$

Bending moment and curvature assumptions (both must vanish at hinge $\xi = 0$)

$$M = \bar{M} \xi \quad \kappa = \bar{\kappa} \xi$$

Transverse displacement assumption

$$w = \frac{1}{4}(1 - \xi)^2(2 + \xi) w_1 + \frac{1}{8}L(1 - \xi)^2(1 + \xi) \theta_1 + \frac{1}{4}(1 + \xi)^2(2 - \xi) w_2 - \frac{1}{8}L(1 + \xi)^2(1 - \xi) \theta_2$$

(element superscripts omitted to reduce clutter)

VHW Formulation of Hinged Beam (cont'd)

Slave curvature derived from transverse displacement

$$\kappa^w = w'' = [(6\xi/L) w_1 + (3\xi - 1)\theta_1 - (6\xi/L) w_2 + (3\xi + 1)\theta_2]/L$$

Remaining calculations are conveniently done through *Mathematica*, using the following script:

```

ClearAll[EI,L,n,m]; n=1; m=1;
kw= 6*\xi/L^2*w1+(3*\xi-1)/L*\theta1+(-6*\xi/L^2*w2)+(3*\xi+1)/L*\theta2;
k= k0*\xi^m; M=M0*\xi^n; W=f1*w1+m1*\theta1+f2*w2+m2*\theta2;
Pi=(L/2)*Integrate[M*(kw-k)+EI*k^2/2,{\xi,-1,1}]-W;
Pi=Simplify[Pi]; Print["VHW functional Pi=", Pi];
r={D[Pi,k0],D[Pi,M0],D[Pi,w1],D[Pi,\theta1],D[Pi,w2],D[Pi,\theta2]};
K={D[r,k0],D[r,M0],D[r,w1],D[r,\theta1],D[r,w2],D[r,\theta2]};
Print["Full Ke=",K//MatrixForm];
K11=Table[K[[i,j]],{i,1,2},{j,1,2}];
K12=Table[K[[i,j]],{i,1,2},{j,3,6}];
K22=Table[K[[i,j]],{i,3,6},{j,3,6}];
Ke = Simplify[K22-Transpose[K12].Inverse[K11].K12];
Print["Condensed Ke=",Ke//MatrixForm];
Print["Eigenvalues of Cond Ke=",Eigenvalues[Ke]];

```

VHW Formulation of Hinged Beam (cont'd)

Using the *Mathematica* script we obtain the complete finite element equations involving six DOFs:

$$\begin{bmatrix}
 \frac{1}{6}EIL & -\frac{1}{6}L & 0 & 0 & 0 & 0 \\
 -\frac{1}{6}L & 0 & \frac{1}{L} & \frac{1}{2} & -\frac{1}{L} & \frac{1}{2} \\
 0 & \frac{1}{L} & 0 & 0 & 0 & 0 \\
 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
 0 & -\frac{1}{L} & 0 & 0 & 0 & 0 \\
 0 & \frac{1}{2} & 0 & 0 & 0 & 0
 \end{bmatrix}
 \begin{bmatrix}
 \bar{\kappa} \\
 \bar{M} \\
 \hline
 w_1 \\
 \theta_1 \\
 w_2 \\
 \theta_2
 \end{bmatrix}
 =
 \begin{bmatrix}
 0 \\
 0 \\
 \hline
 f_1 \\
 m_1 \\
 f_2 \\
 m_2
 \end{bmatrix}$$

VHW Formulation of Hinged Beam (cont'd)

On condensing the moment and curvature DOFs we obtain the stiffness equations in terms of node displacements only:

$$\frac{3EI}{2L^3} \begin{bmatrix} 4 & 2L & -4 & 2L \\ 2L & L^2 & -2L & L^2 \\ -4 & -2L & 4 & -2L \\ 2L & L^2 & -2L & L^2 \end{bmatrix} \begin{bmatrix} w_1^{(e)} \\ \theta_1^{(e)} \\ w_2^{(e)} \\ \theta_2^{(e)} \end{bmatrix} = \begin{bmatrix} f_1^{(e)} \\ m_1^{(e)} \\ f_2^{(e)} \\ m_2^{(e)} \end{bmatrix}$$

or

$$\mathbf{K}^{(e)} \mathbf{u}^{(e)} = \mathbf{f}^{(e)}$$

The condensed 4 x 4 stiffness has rank 1 and rank deficiency 3. Two zero eigenvalues correspond to the standard RBMs of the plane beam. One additional singular mode, corresponding to "hinged motion", appears.