**Homework Exercises for Chapter 6**

**Solutions**

### EXERCISE 6.1

Not assigned.

### EXERCISE 6.2

Integrate the $Nu'$ term by parts:

$$\Pi_{HR} = \int_0^L \left( Nu' - \frac{N^2}{2EA} \right) dx - f_1 u_1 - f_2 u_2 = -\int_0^L \frac{N^2}{2EA}\, dx - \int_0^L u N'\, dx + Nu|_0^L - f_1 u_1 - f_2 u_2 \quad \text{(E6.1)}$$

The exact solution is constant $N$, which is assumed in the element. Thus $N' = 0$ and the functional reduces to

$$\Pi_{HR} = -\int_0^L \frac{N^2}{2EA}\, dx + N(u_2 - u_1) - f_1 u_1 - f_2 u_2. \quad \text{(E6.2)}$$

The internal $u(x)$ has disappeared, and the axial displacements only come in through their end values $u_1$ and $u_2$. Therefore, it does not matter what is taken for the displacement field inside the element as long as a constant $N$ is assumed. This is a variational freak since it applies only to that specific example problem.

### EXERCISE 6.3

Not assigned.

### EXERCISE 6.4

The weak links are (cf. Figure E6.1):

$$e^u_{ij} - e^\sigma_{ij} = 0 \quad \text{in } V, \qquad \hat{u}_i - u_i = 0 \quad \text{on } S_u, \quad \text{(E6.3)}$$

where $e^u_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ and $e^\sigma_{ij} = C_{ijk\ell}\sigma_{k\ell}$. Multiply the residuals (E6.3) by $\delta\sigma_{ij}$ and $\delta t_i = \delta\sigma_{ij} n_j$ and integrate over $V$ and $S_u$, respectively:

$$\int_V (e^u_{ij} - e^\sigma_{ij})\, \delta\sigma_{ij}\, dV + \int_{S_u} (\hat{u}_i - u_i)\, \delta\sigma_{ij} n_j\, dS = 0. \quad \text{(E6.4)}$$

Apply the divergence theorem to the first term on the left:

$$\int_V e^u_{ij}\, \delta\sigma_{ij}\, dV = \int_V \tfrac{1}{2}(u_{i,j} + u_{j,i})\, \delta\sigma_{ij}\, dV = -\int_V u_i\, \overbrace{\delta\sigma_{ij,j}}^{0}\, dV + \int_S u_i\, \delta\sigma_{ij} n_j\, dS, \quad \text{(E6.5)}$$

in which the indicated term vanishes because the BE are strongly satisfied: $\delta(\sigma_{ij,j} + b_i) = \delta\sigma_{ij,j} = 0$ in $V$. Replacing into (E6.4) gives

$$\delta\Pi_{\text{TCPE}} = \int_V -C_{ijk\ell}\, \sigma_{k\ell}\, \delta\sigma_{ij}\, dV + \int_S u_i\, \delta\sigma_{ij} n_j\, dS + \int_{S_u} (\hat{u}_i - u_i)\, \delta\sigma_{ij} n_j\, dS = 0. \quad \text{(E6.6)}$$

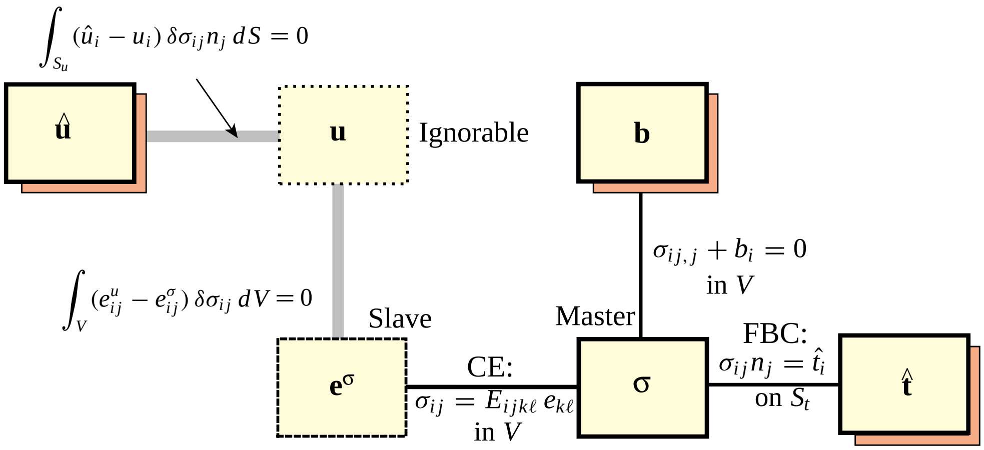


Figure E6.1. Departure Weak Form to derive the TCPE functional.

Split the $S$ integral over $S_t$ and $S_u$. Over $S_t$ we can set $\delta\sigma_{ij} n_j = \delta(\sigma_{ij} n_j) = \delta\hat{t}_i = 0$ because the FBCs are strongly satisfied. The integrals of $u_i\, \delta\sigma_{ij}\, n_j$ over $S_u$ cancel out and we are left with

$$\delta\Pi_{\rm TCPE} = -\int_V C_{ijk\ell}\sigma_{k\ell}\, \delta\sigma_{ij}\, dV + \int_{S_u} \hat{u}_i\, \delta\sigma_{ij} n_j\, dS = 0. \tag{E6.7}$$

This is the exact first variation of the complementary energy functional (E6.1).

### EXERCISE 6.5

The only weak connection is $\sigma_{ij} = E_{ijk\ell} e_{k\ell}$. We begin as above, trying

$$\delta\Pi_S = \int_V (\sigma_{ij} - E_{ijk\ell} e_{k\ell})\, \delta e_{ij}\, dV, \tag{E6.8}$$

where $\sigma_{ij}$ must be viewed as a data field.[10] This is the exact variation of

$$\Pi_S(e_{ij}) = \int_V (\sigma_{ij} - \tfrac{1}{2} E_{ijk\ell} e_{k\ell})\, e_{ij}\, dV. \tag{E6.9}$$

And this is the end. No further progress can be made. This is the only canonical functional of elasticity that contains no boundary integrals.

In the Exercise statement it was noted that functional $\Pi_S$ has limited practical value. The reason is that all of the difficult field equations are taken as strong. The stress field must satisfy both equilibrium equations and stress BC point by point *a priori*, while the strain field must be compatible with a displacement field that satisfies the displacement BC. The only relaxation of the governing equations pertains to the constitutive equations.

Nevertheless the principle may be occasionally useful in material homogenization, as the following simple example illustrates. Consider a bar of uniform cross section $A$ and total length $L = L_1 + L_2 + L_3$ made up

[10] Why? Because the stresses must satisfy the volume equilibrium equations and the surface traction BC *a priori*. Thus the stress field must be known at every point in $V$.

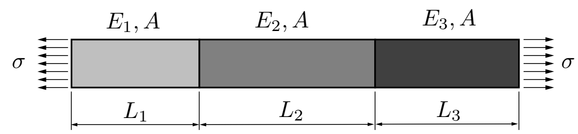


Figure E6.2. Application of the strain-only canonical functional to material homogenization.

of three materials with elastic moduli $E_1$, $E_2$ and $E_3$, respectively. Using the functional $\Pi_S$, find an average modulus $E$.

To carry out the homogenization process, assume that the bar is under a constant axial stress field $\sigma = P/A$ (see Figure E6.2), which obviously satisfies all stress equilibrium equations and surface traction boundary conditions. The average strain $e = E^{-1}\sigma$ is taken as the only unknown to be varied in $\Pi_S$:

$$\Pi_S(e) = AL\sigma e - \tfrac{1}{2}A(E_1L_1 + E_2L_2 + E_3L_3)e^2. \tag{E6.10}$$

The condition $\delta\Pi_S = 0$ gives $\partial\Pi_S/\partial e = 0$, from which

$$e = \frac{L}{E_1L_1 + E_2L_2 + E_3L_3}\sigma, \qquad E = \frac{\sigma}{e} = \frac{E_1L_1 + E_2L_2 + E_3L_3}{L}. \tag{E6.11}$$

This technique essentially amounts to equating the strain energies absorbed by the actual and homogenized (fictitious) bars. Note that the displacement field does not appear in this statically determinate problem.