6

The HR Variational Principle of Elastostatics

TABLE OF CONTENTS

TABLE OF CONTENTS

§6.1. Introduction

In Chapter 3, a *multifield* variational principle was defined as one that has more than one master field. That is, more than one unknown field is subject to independent variations. The present Chapter begins the study of such functionals within the context of elastostatics. Following a classification of the so-called canonical functionals, the Hellinger-Reissner (HR) mixed functional is derived.

The HR principle is applied to the derivation of a couple of 1D elements in the text, and others are provided in the Exercises.

§6.1.1. Mixed Versus Hybrid

The terminology pertaining to multifield functionals is not uniform across applied mechanics and FEM literature. Sometimes all multifield principles are called mixed; sometimes this term is restricted to specific cases. This book takes a middle ground:

A *Mixed principle* is one where all master fields are internal fields (volume fields in 3D).

A *Hybrid principle* is one where master fields are of different dimensionality. For example one internal volume field and one surface field.

Hybrid principles will be studied in Chapter 8 and 9. They are intrinsically important for FEM discretizations but have only a limited role outside of FEM.

§6.1.2. The Canonical Functionals

If hybrid functionals are excluded, three unknown internal fields of linear elastostatics are candidates for master fields to be varied: displacements u_i, strains e_{ij}, and stresses σ_{ij}. Seven combinations, listed in Table 6.1, may be chosen as masters. These are called the *canonical* functionals of elasticity.

Table 6.1 The Seven Canonical Functionals of Linear Elastostatics

#	Type	Master fields	Name
(I)	Single-field	Displacements	Total Potential Energy (TPE)
(II)	Single-field	Stresses	Total Complementary Potential Energy (TCPE)
(III)	Single-field	Strains	No name
(IV)	Mixed 2 field	Displacements & stresses	Hellinger-Reissner (HR)
(V)	Mixed 2-field	Displacements & strains	No agreed upon name
(VI)	Mixed 2-field	Strains & stresses	No name
(VII)	Mixed 3-field	Displacements, stresses & strains	Veubeke-Hu-Washizu (VHW)

Four of the canonical functionals: (I), (II), (IV) and (VII), have identifiable names. From the standpoint of finite element development those four, plus (V), are most important although they are not equal in importance. By far (I) and (IV) have been the most seminal, distantly followed by (II), (V) and (VII). Functionals (III) and (VI) are mathematical curiosities.

The construction of mixed functionals involves more expertise than single-field ones. And their FEM implementation requires more care and patience.[1]

[1] Strang's famous dictum is "mixed elements lead to mixed results." In other words: more master fields are not necessarily better than one. Some general guides as to when mixed functionals pay off will appear as byproduct of examples.

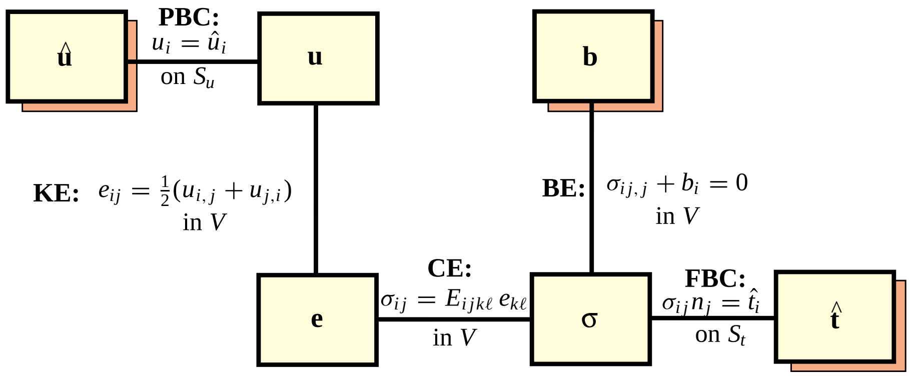

FIGURE 6.1. The Strong Form Tonti diagram for linear elastostatics, reproduced for convenience.

For convenience the Strong Form Tonti diagram of linear elastostatics is shown in Figure 6.1.

§6.2. The Hellinger-Reissner (HR) Principle

§6.2.1. Assumptions

The Hellinger-Reissner (HR) canonical functional of linear elasticity allows displacements and stresses to be varied separately. This establishes the master fields. Two slave strain fields appear, one coming from displacements and one from stresses:

$$e^u_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \qquad e^\sigma_{ij} = C_{ijk\ell}\, \sigma_{k\ell} \tag{6.1}$$

Here $C_{ijk\ell}$ are the entries of the compliance tensor or strain-stress tensor $\underline{\mathbf{C}}$, which is the inverse of $\underline{\mathbf{E}}$. In matrix form this is $\mathbf{e}^\sigma = \mathbf{C}\boldsymbol{\sigma}$, where $\mathbf{C} = \mathbf{E}^{-1}$ is a 6×6 matrix of elastic compliances.

At the exact solution of the elasticity problem, the two strain fields coalesce point by point. But when these fields are obtained by an approximation procedure such as FEM, strains recovered from displacements and strains computed from stresses will not generally agree.

Three weak links appear: BE and FBC (as in the Total Potential Energy principle derived in the previous Chapter), plus the link between the two slave strain fields, which is identified as EE. Figure 6.2 depicts the resulting Weak Form.

Remark 6.1. The weak connection between $\mathbf{e}^u$ and $\mathbf{e}^\sigma$ could have been substituted by a weak connection between $\boldsymbol{\sigma}^u$ and $\boldsymbol{\sigma}$. The results would be the same because the constitutive equation links are strong. The choice of $\mathbf{e}^u$ and $\mathbf{e}^\sigma$ simplifies slightly the derivations below.

§6.2.2. The Weak Equations

We follow the weigting residual technique used in Chapter 3 for the TPE derivation. Take the residuals of the three weak connections shown in Figure 6.2, multiply them by Lagrange multiplier fields and integrate over the respective domains:

$$\int_V (e^u_{ij} - e^\sigma_{ij})\, w_{ij}\, dV + \int_V (\sigma_{ij,j} + b_i)\, w^*_i\, dV + \int_S (\sigma_{ij} n_j - \hat{t}_i)\, w^{**}_i\, dS = 0 \tag{6.2}$$

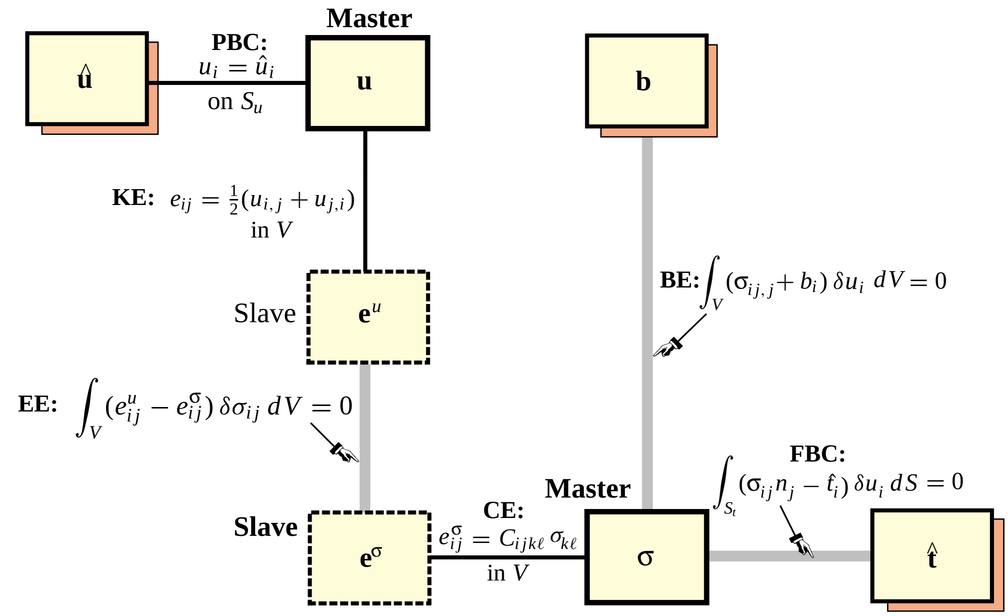

FIGURE 6.2. The starting Weak Form for derivation of the HR principle.

For conformity, w_{ij} must be a second order tensor, whereas w_i^* and w_i^{**} are 3-vectors. These weights must be expressed as variations of either master: either displacements u_i or stresses σ_{ij}, based on work pairing considerations. The residuals of KE are volume forces integrated over V, and those of FBC are surface forces integrated over S. Hence w_i^* and w_i^{**} must be displacement variations to obtain energy density. The residuals of EE are strains integrated over V; consequently w_{ij} must be stress variations. Based on these considerations we set $w_{ij} = \delta\sigma_{ij}$, $w_i^* = -\delta u_i$, $w_i^{**} = \delta u_i$, where the minus sign in the second one is chosen to anticipate eventual cancellation in the surface integrals. Adding the weak link contributions gives

$$\int_V (e^u_{ij} - e^\sigma_{ij})\,\delta\sigma_{ij}\,dV - \int_V (\sigma_{ij,j} + b_i)\,\delta u_i\,dV + \int_S (\sigma_{ij} n_j - \hat{t}_i)\,\delta u_i\,dS = 0. \qquad (6.3)$$

Next, integrate the $\sigma_{ij,j}\,\delta u_i$ term by parts to eliminate the stress derivatives, split the surface integral into $S_u \cup S_t$, and enforce the strong link $u_i = \hat{u}_i$ over S_u:

$$\begin{aligned} -\int_V \sigma_{ij,j}\,\delta u_i\,dV &= \int_V \sigma_{ij}\,\delta e^u_{ij}\,dV - \int_S \sigma_{ij} n_j\,\delta u_i\,dS \\ &= \int_V \sigma_{ij}\,\delta e^u_{ij}\,dV - \int_{S_u} \sigma_{ij} n_j\,\cancelto{0}{\delta u_i}\,dS - \int_{S_t} \sigma_{ij} n_j\,\delta u_i\,dS \\ &= \int_V \sigma_{ij}\,\delta e^u_{ij}\,dV - \int_{S_t} \sigma_{ij} n_j\,\delta u_i\,dS. \end{aligned} \qquad (6.4)$$

in which δe^u_{ij} means the variation of $\delta\frac{1}{2}(u_{i,j} + u_{j,i}) = \frac{1}{2}(\delta u_{i,j} + \delta u_{j,i})$, as in §5.9.2.

Upon simplification of the cancelling terms $\sigma_{ij}\,n_j\,\delta u_i$ on S_t we end up with the following variational statement, written hopefully as the exact variation of a functional $\Pi_{\rm HR}$:

$$\delta\Pi_{\rm HR} = \int_V \left[\, (e^u_{ij} - e^\sigma_{ij})\,\delta\sigma_{ij} + \sigma_{ij}\,\delta e^u_{ij} - b_i\,\delta u_i \,\right] dV - \int_{S_t} \hat{t}_i\,\delta u_i\,dS. \qquad (6.5)$$

§6.2.3. The Variational Form

And indeed (6.5) is the exact variation of

$$\boxed{\Pi_{\rm HR}[u_i,\sigma_{ij}] = \int_V \left(\sigma_{ij} e^u_{ij} - \tfrac{1}{2}\sigma_{ij} C_{ijk\ell}\sigma_{k\ell} - b_i u_i\right) dV - \int_{S_t} \hat{t}_i u_i \, dS.} \tag{6.6}$$

This is called the *Hellinger-Reissner functional*, abbreviated HR.[2] It is often stated in the literature as

$$\Pi_{\rm HR}[u_i,\sigma_{ij}] = \int_V [-\mathcal{U}^*(\sigma_{ij}) + \sigma_{ij}\tfrac{1}{2}(u_{i,j}+u_{j,i}) - b_i u_i]\, dV - \int_{S_t} \hat{t}_i u_i \, dS, \tag{6.7}$$

in which

$$\mathcal{U}^*(\sigma_{ij}) = \tfrac{1}{2}\sigma_{ij} C_{ijk\ell}\sigma_{k\ell} = \tfrac{1}{2}\sigma_{ij} e^\sigma_{ij}, \tag{6.8}$$

is the *complementary energy density* in terms of the master stress field.

In FEM work the functional is usually written in the split form

$$\boxed{\begin{aligned} &\Pi_{\rm HR} = U_{\rm HR} - W_{\rm HR}, \qquad \text{in which} \\ &U_{\rm HR} = \int_V \left(\sigma_{ij} e^u_{ij} - \tfrac{1}{2}\sigma_{ij} C_{ijk\ell}\sigma_{k\ell}\right) dV, \quad W_{\rm HR} = \int_V b_i u_i \, dV + \int_{S_t} \hat{t}_i u_i \, dS. \end{aligned}} \tag{6.9}$$

The HR principle states that stationarity of the total variation

$$\boxed{\delta \Pi_{\rm HR} = 0} \tag{6.10}$$

provides the KE and EE strong links as Euler-Lagrange equations, whereas the FBC strong link appears as a natural boundary condition.

Remark 6.2. To verify the assertion about (6.5) being the first variation of $\Pi_{\rm HR}$, note that

$$\delta(\sigma_{ij} e^u_{ij}) = e^u_{ij}\,\delta\sigma_{ij} + \sigma_{ij}\,\delta e^u_{ij}, \qquad \delta(\tfrac{1}{2}\sigma_{ij} C_{ijk\ell}\sigma_{k\ell}) = C_{ijk\ell}\sigma_{k\ell}\,\delta\sigma_{ij} = e^\sigma_{ij}\,\delta\sigma_{ij}. \tag{6.11}$$

§6.2.4. Variational Indices and FEM Continuity Requirements

For a single-field functional, the *variational index* of its primary variable is the highest derivative m of that field that appears in the variational principle. The connection between variational index and required continuity in FEM shape functions was presented (as recipe) in the introductory FEM course (IFEM). That course considered only the single-field TPE functional, in which the primary variable, and only master, is the displacement field. It was stated that displacement shape functions

[2] The basic idea was contained in the work of Hellinger: E. Hellinger, Die allgemeine Ansätze der Mechanik der Kontinua, Encyklopœdia der Mathematische Wissenschaften, Vol 4^4, ed. by F. Klein and C. Müller, Teubner, Leipzig, 1914. As a proven theorem for the traction specified problem (no PBC) it was first given by Prange: G. Prange, Der Variations- und MinimalPrinzipe der Statik der Baukonstruktionen, Habilitationsschrift, Tech. Univ. Hanover, 1916. As a complete theorem containing both PBC and FBC it was given much later by Reissner: E. Reissner, On a variational theorem in elasticity, *J. Math. Phys.*, **29**, 90–95, 1950.

must be C^{m-1} continuous between elements and C^m inside. For the bar and plane stress problem covered in IFEM, $m = 1$, whereas for the Bernoulli-Euler beam $m = 2$.

In multifield functionals the variational index concept applies to *each varied field*. Thus there are as many variational indices as master fields. In the HR functional (6.9) of 3D elasticity, the variational index m_u of the displacements is 1, because first order derivatives appear in the slave strains e^u_{ij}. The variational index m_σ of the stresses is 0 because no stress derivatives appear. The required continuity of FEM shape functions for displacements and stresses is dictated by these indices. More precisely, if $\Pi_{\rm HR}$ is used as source functional for element derivation:

1. Displacement shape functions must be C^0 (continuous) between elements and C^1 inside (continuous and differentiable).
2. Stress shape functions can be C^{-1} (discontinuous) between elements, and C^0 (continuous) inside.

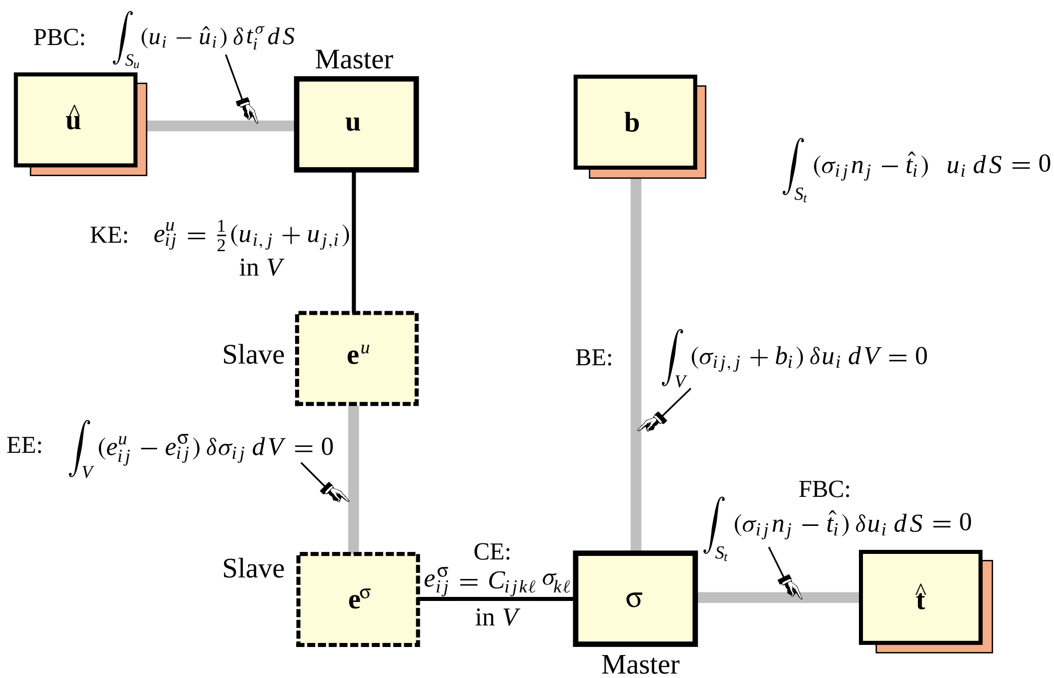

Figure 6.3. WF diagram for displacement-BC-generalized HR, in which PBC is weakened.

§6.2.5. Displacement-BC Generalized HR

If the PBC link (displacement BCs) between u_i and $\hat{u}_i$ is weakened as illustrated in Figure 6.3, the functional $\Pi_{\rm HR}$ generalizes to

$$\boxed{\Pi^g_{\rm HR} = \Pi_{\rm HR} - \int_{S_u} \sigma_{ij} n_i (u_i - \hat{u}_i)\, dS = \Pi_{\rm HR} - \int_{S_u} t^\sigma_i (u_i - \hat{u}_i)\, dS.} \tag{6.12}$$

in which $\sigma_{ij} n_j = t^\sigma_i$ is the surface traction associated with the master stress field.

§6.3. Application Example 1: Tapered Bar Element

In this section the use of the HR functional to construct a very simple finite element is illustrated. Consider a tapered bar made up of isotropic elastic material, as depicted in Figure 6.4(a). The $x_1 \equiv x$ axis is placed along the longitudinal direction. The bar cross section area A varies linearly between the end node areas A_1 and A_2. The element has length L and constant elastic modulus E. Body forces are ignored.

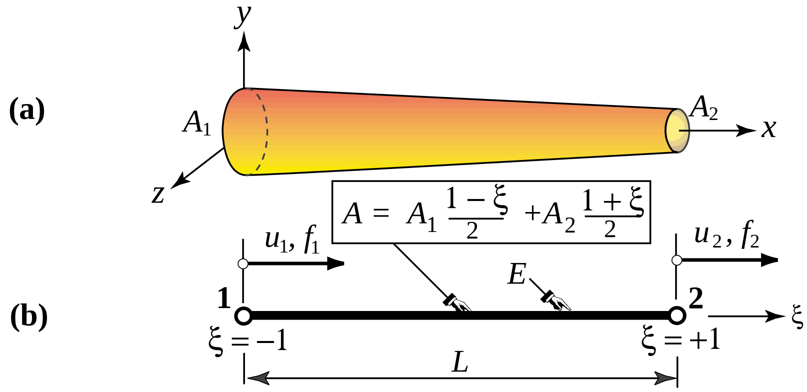

Figure 6.4. Two-node tapered bar element by HR: (a) shows the bar as a 3D object and (b) as a FEM model.

The reduction of the HR functional (6.6) to the bar case furnishes an instructive example of the derivation of a *structural* model based on stress resultants and Mechanics of Materials approximations.

In the theory of bars, the only nonzero stress is $\sigma_{11} \equiv \sigma_{xx}$, which will be denoted by σ for simplicity. The only internal force is the bar axial force $N = A\sigma_{xx}$. The only displacement component that participates in the functional is the axial displacement u_x, which is only a function of x and will be simply denoted by $u(x)$. The value of the axial displacement at end sections 1 and 2 is denoted by u_1 and u_2, respectively. The axial strain is $e_{11} \equiv e_{xx}$, which will be denoted by e. The strong links are $e^u = du(x)/dx = u'$, where primes denote derivatives with respect to x, and $e^\sigma = \sigma/E = N/(EA)$. We call $N^u = EA\,e^u = EA\,u'$, etc.

As for as boundary conditions, for a free (unconnected) element S_t embodies the whole surface of the bar. But according to bar theory the lateral surface is traction free and thus drops off from the surface integral. That leaves the two end sections, at which uniform longitudinal surface tractions $\hat{t}_x$ are prescribed whereas the other component vanishes. On assuming a uniform traction distribution over the end cross sections, we find that the node forces are $f_1 = -t_{x1}A_1$ at section 1 and $f_2 = t_{x2}A_2$ at section 2. (The negative sign in the first one arises because at section 1 the external normal points along $-x$.)

Plugging these relations into the HR functional (6.6) and integrating over the cross section gives

$$\Pi_{\rm HR}[u, N] = \int_L \left(N u' - \frac{N^2}{2EA} \right) dx - f_1 u_1 - f_2 u_2. \tag{6.13}$$

This is an example of a functional written in term of *stress resultants* rather than actual stresses. The theory of beams, plates and shells leads also to this kind of functionals.

§6.3.1. Formulation of the Tapered Bar Element

We now proceed to construct the two-node bar element (e) depicted in Figure 6.3(b), from the functional (6.13). Define ξ is a natural coordinate that varies from $\xi = -1$ at node 1 to $\xi = 1$ at node 2. Assumptions must be made on the variation of displacements and axial forces. Displacements are taken to vary linearly whereas the axial force will be assumed to be constant over the element:

$$u(x) \approx u_1^{(e)} \frac{1-\xi}{2} + u_2^{(e)} \frac{1+\xi}{2}, \qquad N(x) \approx \bar{N}^{(e)} \tag{6.14}$$

These assumptions comply with the C^0 and C^{-1} continuity requirements for displacements and stresses, respectively, stated in §6.2.4. Inserting (6.13) and (6.14) into the functional (6.12) and carrying out the necessary integral over the element length yields[3]

$$\Pi_{\rm HR}^{(e)} = \tfrac{1}{2} \begin{bmatrix} \bar{N}^{(e)} \\ u_1^{(e)} \\ u_2^{(e)} \end{bmatrix}^T \begin{bmatrix} -\dfrac{\gamma L}{E A_m} & -1 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{N}^{(e)} \\ u_1^{(e)} \\ u_2^{(e)} \end{bmatrix} - \begin{bmatrix} 0 \\ f_1^{(e)} \\ f_2^{(e)} \end{bmatrix}^T \begin{bmatrix} \bar{N}^{(e)} \\ u_1^{(e)} \\ u_2^{(e)} \end{bmatrix} \tag{6.15}$$

in which

$$A_m = \tfrac{1}{2}(A_1 + A_2), \qquad \gamma = \frac{A_m}{A_2 - A_1} \log \frac{A_2}{A_1}. \tag{6.16}$$

Note that if the element is prismatic, $A_1 = A_2 = A_m$, and $\gamma = 1$ (take the limit of the Taylor series for γ).

For this discrete form of $\Pi_{\rm HR}^{(e)}$, the Euler-Lagrange equations are simply the stationarity conditions

$$\frac{\partial \Pi_{\rm HR}^{(e)}}{\partial \bar{N}^{(e)}} = \frac{\partial \Pi_{\rm HR}^{(e)}}{\partial u_1^{(e)}} = \frac{\partial \Pi_{\rm HR}^{(e)}}{\partial u_2^{(e)}} = 0, \tag{6.17}$$

which supply the finite element equations

$$\begin{bmatrix} -\dfrac{\gamma L}{E A_m} & -1 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{N}^{(e)} \\ u_1^{(e)} \\ u_2^{(e)} \end{bmatrix} = \begin{bmatrix} 0 \\ f_1^{(e)} \\ f_2^{(e)} \end{bmatrix} \tag{6.18}$$

This is an example of a *mixed finite element*, where the qualifier "mixed" implies that approximations are made in more than one unknown internal quantity; here axial forces and axial displacements.

Because the axial-force degree of freedom $\bar{N}^{(e)}$ is not continuous across elements (recall that C^{-1} continuity for stress variables is allowed), it may be eliminated or "condensed out" at the element level. The static condensation process studied in IFEM yields

$$\frac{E A_m}{\gamma L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1^{(e)} \\ u_2^{(e)} \end{bmatrix} = \begin{bmatrix} f_1^{(e)} \\ f_2^{(e)} \end{bmatrix}, \tag{6.19}$$

or

$$\boxed{\mathbf{K}^{(e)} \mathbf{u}^{(e)} = \mathbf{f}^{(e)}.} \tag{6.20}$$

These are the *element stiffness equations*, obtained here through the HR principle. Had these equations been derived through the TPE principle, one would have obtained a similar expression except that $\gamma = 1$ for any end-area ratio. Thus if the element is prismatic ($A_1 = A_2 = A_m$) the HR and TPE functionals lead to the same element stiffness equations.

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[3] Derivation details are worked out in an Exercise.

Table 6.2 Results for one-element analysis of fixed-free tapered bar

Area ratio	u_2 from HR	u_2 from TPE	Exact u_2
$A_1/A_2 = 1$	$PL/(EA_m)$	$PL/(EA_m)$	$PL/(EA_m)$
$A_1/A_2 = 2$	$1.0397PL/(EA_m)$	$PL/(EA_m)$	$1.0397PL/(EA_m)$
$A_1/A_2 = 5$	$1.2071PL/(EA_m)$	$PL/(EA_m)$	$1.2071PL/(EA_m)$

§6.3.2. Numerical Example

To give a simple numerical example, suppose that the bar of Figure 6.2 is fixed at end 1 whereas end 2 is under a given axial force P. Results for sample end area ratios are given in Table 6.2. It can be seen that the HR formulation yields the exact displacement solution for all area ratios.

Also note that the discrepancy of the one-element TPE solution from the exact one grows as the area ratio deviates from one. The TPE elements underestimate the actual deflections, and are therefore on the stiff side. To improve the TPE results we need to divide the bar into more elements.

§6.3.3. The Bar Flexibility

From (6.19) we immediately obtain

$$u_2 - u_1 = \frac{\gamma L}{EA_m}(f_2 - f_1) = F(f_2 - f_1) \tag{6.21}$$

This called a *flexibility equation*. The number $F = \gamma L/(EA_m)$ is the flexibility coefficient or *influence coefficient*. For more complicated elements we would obtain a *flexibility matrix*. Relations such as (6.21) were commonly worked out in older books in matrix structural analysis. The reason is that flexibility equations are closely connected to classical static experiments in which a force is applied, and a displacement or elongation measured.

§6.4. Application Example 2: A Curved Cable Element

§6.4.1. Connector Elements

The HR functional is useful for deriving a class of elements known as *connector elements*.[4] The concept is illustrated in Figure 6.5(a). The *connector nodes* are those through which the element links to other elements through the node displacements. These displacements are the *connector degrees of freedom*, or simple the *connectors*. The box models the intrinsic response of the element; if it is best described in terms of response to forces or stresses, as depicted in Figures 6.5(b,c), it is called a *flexibility box* or F-box.

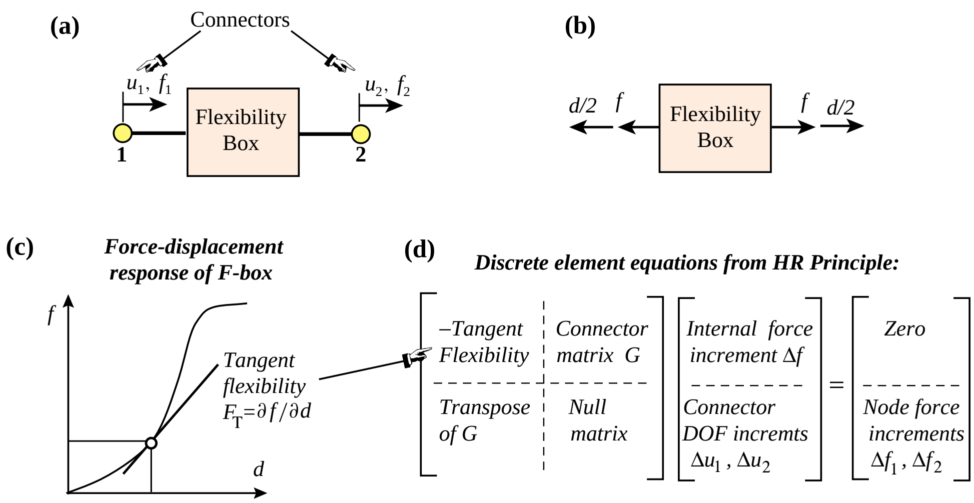

Figure 6.5. A connector element (sketch) developed with the help of the HR principle.

In many applications the box response is nonlinear. Examples are elements modelling contact, friction and joints. If this is the only place where nonlinear behavior occur, the flexilibity element acts as a device to isolate local nonlinearities. This is an effective way to reuse linear FEM programs.

Consider for simplicity a one-dimensional, 2 node flexibility element as the one sketched in Figure 6.5. The connector nodes are 1 and 2. The connector DOF are the axial displacements u_1 and u_2. The relative displacement is $\Delta = u_2 - u_1$. The kernel behavior is described by the response to an axial force f, as pictured in Figure 6.5(c):

$$d = F(f). \tag{6.22}$$

The tangent flexibility is

$$F_T = \frac{\partial d}{\partial f} = \frac{\partial F(f)}{\partial f}. \tag{6.23}$$

Application of the HR principle leads to the tangent equation

$$\begin{bmatrix} -F_T & -1 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta f \\ \Delta u_1 \\ \Delta u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \Delta f_1 \\ \Delta f_2 \end{bmatrix} \tag{6.24}$$

[4] Hybrid elements, covered in Sections 8ff, are also useful in this regard. Often the two approaches lead to identical results.

where Δ denote increments.[5] Condensation of Δf as internal freedom gives the stiffness matrix

$$\begin{bmatrix} K_T & -K_T \\ -K_T & K_T \end{bmatrix} \begin{bmatrix} \Delta u_1 \\ \Delta u_2 \end{bmatrix} = \begin{bmatrix} \Delta f_1 \\ \Delta f_2 \end{bmatrix} \tag{6.25}$$

This result could also been obtained directly from physics, or from the displacement formulation. However, the HR approach remains unchanged when passing to 2 and 3 dimensions.

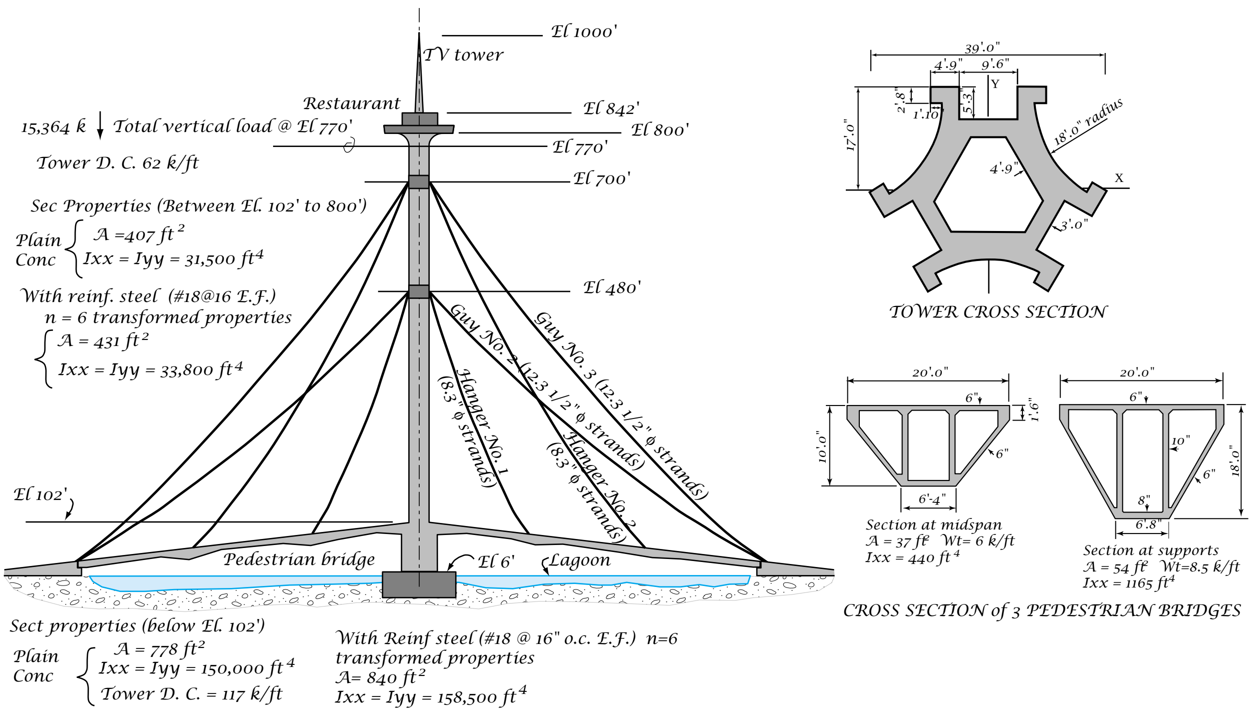

Figure 6.6. 1000-ft guyed tower studied in 1967 for the South Florida coast.

§6.4.2. A Curved Cable Element

As an application consider the development of a curved cable element used to model the guy and hanger members of the tower structure shown in Figure 6.6(a).[6]

Figure 6.7(a) shows a two-dimensional FEM model, with 62 nodes and 3 freedoms per node.[7] To cut down the number of elements along the cable members, a curved cable element, pictured in

[5] The first entry of the right hand side has been set to zero for simplicity. Generally it is not.

[6] A 1000-ft guyed tower proposed for the South Florida coast by a group of rich Cuban expatriates and dubbed the "Tower of Freedom" as it was supposed to serve as a guide beacon for boats escaping Cuba with refugees. The preliminary design of Figure 6.6 was made by a well known structural engineering company and dated June 1967. Ray W. Clough and Joseph Penzien were consultants for the verification against hurricane winds. Analyzed using an ad-hoc 2D FEM code by Mike Shears and the writer, who was then a post-doc at UC Berkeley, July–September 1967. The project was canceled as too costly and plans for a 3D cable analysis code shelved.

[7] The structure has 120° circular symmetry. Reduced to one plane of symmetry (plane of the paper) by appropriate projections of the right-side (windward) portion.

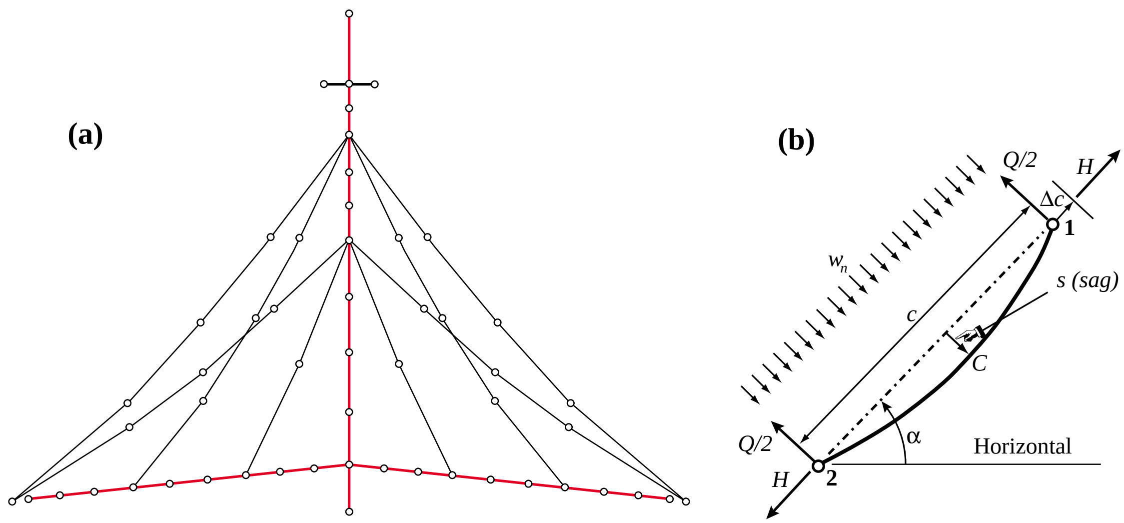

Figure 6.7. (a): 2D FEM model of guyed tower of Figure 6.6 for vibration and dynamic analysis under hurricane wind loads (1967); (b): curved cable element developed to model the guy and hanger cables with few elements along the length.

Figure 6.7(b) was constructed. The method that follows illustrates the application of the flexibility approach to connector elements.

The element has two nodes, 1 and 2. The distance 1–2 is the chord distance c. The actual length of the strained cable element is L, so $L \geq c$. The force H along the chord is called the *thrust*. H and the chord change Δc play the role of f and d, respectively, in the flexibility response sketched in Figure 6.5. The cable is subjected to a uniform transverse load w_n specified per unit of chord length. (The load is usually a combination of self-weight and wind.) The elastic rigidity of the cable is EA_0, where E is the apparent elastic modulus (which depends on the fabrication of the cable) and A_0 the original structural area.

The following simplifying assumptions are made at the element level:

1. The sag is small compared to chord length: $s < c/10$, which characterizes a taut element.[8]
2. The load w_n is uniform. As a consequence, the transverse reaction loads at nodes are $Q/2$, with $Q = w_n c$. See Figure chapdot7(b).
3. $Q = w_n c$ is fixed even if c changes. This is exact for self weight, and approximately verified for wind loads.
4. The effect of tangential loads (along the chord) on the element deformation is neglected.
5. Hooke's law applies in the form $L - L_0 = H/(EA_0)$, where L_0 is the unstrained length of the element.

Under the foregoing assumptions, the cable deflection profile is parabolic, and we get

$$s = \frac{Qc}{8H}, \qquad L = L_0\left(1 + \frac{H}{EA_0}\right) = c + \frac{8}{3}\frac{s^2}{c^2}, \qquad \frac{c}{L_0} = \frac{1 + \dfrac{H}{EA_0}}{1 + \dfrac{Q^2}{24H^2}}. \tag{6.26}$$

[8] If this property is not realized, the cable member should be divided into more elements. Dividing one element into two cuts c and s approximately by 2 and 4, respectively, so s/c is roughly halved.

The first equation comes from moment equilibrium at the sagged element midpoint C, the second from the shallow parabola-arclength formula, and the third one from eliminating the sag s between the first two. Differentiation gives the tangent flexibility

$$F_T = \frac{\partial c}{\partial H} = \frac{L_0}{EA_0} \frac{1}{1 + \dfrac{Q^2}{24H^2}} + \frac{\dfrac{Q^2 L_0}{12H^3}\left(1 + \dfrac{H}{EA_0}\right)}{\left(1 + \dfrac{Q^2}{24H^2}\right)^2} \tag{6.27}$$

For most structural cables, $H << EA_0$ and $(Q/H)^2 << 1$. Accordingly the above formula simplifies to

$$F_T = \frac{L_0}{EA_0} + \frac{Q^2 L_0}{12H^3}, \tag{6.28}$$

which was used in the 1967 dynamic analysis at Berkeley. If $Q \to 0$ or $H \to \infty$, (6.28) reduces to the flexibility $L_0/(EA_0)$ of a linear bar element, as can be expected. Replacing into (6.24) and condensing out ΔH gives the tangent local stiffness matrix of the cable element as (6.25), where u_1 and u_2 are axial displacements at nodes 1 and 2 along the chord, and $K_T = 1/F_T$. This matrix relation can be transformed to the global coordinate system in the usual manner.

Homework Exercises for Chapter 6

The HR Variational Principle of Elastostatics

EXERCISE 6.1 [A:10] Derive the Euler-Lagrange equations and natural BCs of $\Pi_{\rm HR}$ given in (6.-3).

EXERCISE 6.2 [A:20] As shown in Table 6.1, the tapered bar HR model derived in §6.3 gives the exact end displacement with one element. But the assumed linear-displacement variation does not agree with the displacement of the exact solution, which is nonlinear in x if $A_1 \neq A_2$. Explain this contradiction. *Hint*: a variational freak; integrate the Nu' term in (6.13) by parts and use the fact that the exact solution has constant N.

EXERCISE 6.3 [A:25] Construct the HR functional for the cable element treated in §6.4.2. Hint: construct the complementary energy function of the F-box.

EXERCISE 6.4 [A:15] Show that if the master σ_{ij} is replaced by the slave $\sigma^u_{ij} = E_{ijk\ell} e^u_{ij}$ with $e^u_{ij} = (u_{i,j} + u_{j,i})/2$, the HR functional reduces to the TPE functional.

EXERCISE 6.5 [A:25] Derive the Total Complementary Potential Energy (TCPE) functional of linear elastostatics

$$\Pi_{\rm TCPE}[\sigma_{ij}] = -\tfrac{1}{2} \int_V \sigma_{ij} C_{ijk\ell} \sigma_{k\ell}\, dV + \int_{S_u} \sigma_{ij} n_j\, \hat{u}_i\, dS, \tag{E6.1}$$

by choosing stresses as the only primary field. Choose BE, FBC, and the right-to-left constitutive equations [the strain-stress relations $e^\sigma_{ij} = C_{ijk\ell}\sigma_{k\ell}$] as strong connections whereas KE and PBC are weak connections. Notice that the displacement field u_i does not appear in this functional; only the prescribed displacements $\hat{u}_i$.

EXERCISE 6.6 [A:30] Derive the anonymous functional $\Pi_S[e_{ij}]$ based on strains as only primary field. Take all connections as strong except the constitutive equations. Note: this functional seems to appear in only one book,[9] and therein only as a curiosity. It is weird looking because of its abnormal simplicity:

$$\Pi_{\rm S}[e_{ij}] = \int_V (\sigma_{ij} - \tfrac{1}{2} E_{ijk\ell} e_{k\ell})\, e_{ij}\, dV. \tag{E6.2}$$

That's right, no boundary terms, and σ_{ij} is here a *data* field!

[9] J. T. Oden and J. N. Reddy, *Variational Methods in Theoretical Mechanics*, Springer-Verlag, Berlin (1982).

Homework Exercises for Chapter 6

Solutions

EXERCISE 6.1

Not assigned.

EXERCISE 6.2

Integrate the Nu' term by parts:

$$\Pi_{HR} = \int_0^L \left(Nu' - \frac{N^2}{2EA} \right) dx - f_1 u_1 - f_2 u_2 = -\int_0^L \frac{N^2}{2EA}\, dx - \int_0^L u N'\, dx + Nu|_0^L - f_1 u_1 - f_2 u_2 \quad \text{(E6.3)}$$

The exact solution is constant N, which is assumed in the element. Thus $N' = 0$ and the functional reduces to

$$\Pi_{HR} = -\int_0^L \frac{N^2}{2EA}\, dx + N(u_2 - u_1) - f_1 u_1 - f_2 u_2. \quad \text{(E6.4)}$$

The internal $u(x)$ has disappeared, and the axial displacements only come in through their end values u_1 and u_2. Therefore, it does not matter what is taken for the displacement field inside the element as long as a constant N is assumed. This is a variational freak since it applies only to that specific example problem.

EXERCISE 6.3

Not assigned.

EXERCISE 6.4 The weak links are (cf. Figure E6.1):

$$e^u_{ij} - e^\sigma_{ij} = 0 \quad \text{in } V, \qquad \hat{u}_i - u_i = 0 \quad \text{on } S_u, \quad \text{(E6.5)}$$

where $e^u_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ and $e^\sigma_{ij} = C_{ijk\ell}\sigma_{k\ell}$. Multiply the residuals (E6.5) by $\delta\sigma_{ij}$ and $\delta t_i = \delta\sigma_{ij} n_j$ and integrate over V and S_u, respectively:

$$\int_V (e^u_{ij} - e^\sigma_{ij})\, \delta\sigma_{ij}\, dV + \int_{S_u} (\hat{u}_i - u_i)\, \delta\sigma_{ij} n_j\, dS = 0. \quad \text{(E6.6)}$$

Apply the divergence theorem to the first term on the left:

$$\int_V e^u_{ij}\, \delta\sigma_{ij}\, dV = \int_V \tfrac{1}{2}(u_{i,j} + u_{j,i})\, \delta\sigma_{ij}\, dV = -\int_V u_i\, \overset{0}{\cancel{\delta\sigma_{ij,j}}}\, dV + \int_S u_i\, \delta\sigma_{ij} n_j\, dS, \quad \text{(E6.7)}$$

in which the indicated term vanishes because the BE are strongly satisfied: $\delta(\sigma_{ij,j} + b_i) = \delta\sigma_{ij,j} = 0$ in V. Replacing into (E6.6) gives

$$\delta\Pi_{\text{TCPE}} = \int_V -C_{ijk\ell}\, \sigma_{k\ell}\, \delta\sigma_{ij}\, dV + \int_S u_i\, \delta\sigma_{ij} n_j\, dS + \int_{S_u} (\hat{u}_i - u_i)\, \delta\sigma_{ij} n_j\, dS = 0. \quad \text{(E6.8)}$$

Split the S integral over S_t and S_u. Over S_t we can set $\delta\sigma_{ij} n_j = \delta(\sigma_{ij} n_j) = \delta\hat{t}_i = 0$ because the FBCs are strongly satisfied. The integrals of $u_i\, \delta\sigma_{ij}\, n_j$ over S_u cancel out and we are left with

$$\delta\Pi_{\text{TCPE}} = -\int_V C_{ijk\ell}\sigma_{k\ell}\, \delta\sigma_{ij}\, dV + \int_{S_u} \hat{u}_i\, \delta\sigma_{ij} n_j\, dS = 0. \quad \text{(E6.9)}$$

This is the exact first variation of the complementary energy functional (E6.3).

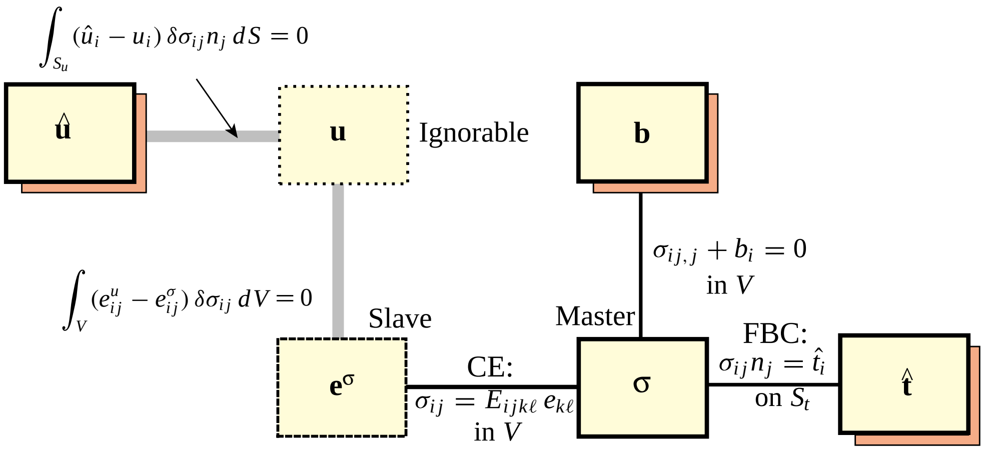

Figure E6.1. Departure Weak Form to derive the TCPE functional.

EXERCISE 6.5 The only weak connection is $\sigma_{ij} = E_{ijk\ell} e_{k\ell}$. We begin as above, trying

$$\delta\Pi_S = \int_V (\sigma_{ij} - E_{ijk\ell} e_{k\ell})\, \delta e_{ij}\, dV, \tag{E6.10}$$

where σ_{ij} must be viewed as a data field.[10] This is the exact variation of

$$\Pi_S(e_{ij}) = \int_V (\sigma_{ij} - \tfrac{1}{2} E_{ijk\ell} e_{k\ell})\, e_{ij}\, dV. \tag{E6.11}$$

And this is the end. No further progress can be made. This is the only canonical functional of elasticity that contains no boundary integrals.

In the Exercise statement it was noted that functional Π_S has limited practical value. The reason is that all of the difficult field equations are taken as strong. The stress field must satisfy both equilibrium equations and stress BC point by point *a priori*, while the strain field must be compatible with a displacement field that satisfies the displacement BC. The only relaxation of the governing equations pertains to the constitutive equations.

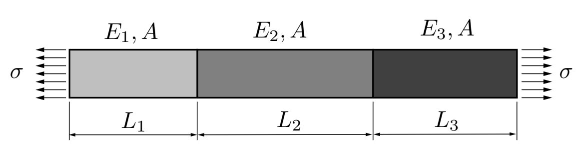

Figure E6.2. Application of the strain-only canonical functional to material homogenization.

Nevertheless the principle may be occasionally useful in material homogenization, as the following simple example illustrates. Consider a bar of uniform cross section A and total length $L = L_1 + L_2 + L_3$ made up of three materials with elastic moduli E_1, E_2 and E_3, respectively. Using the functional Π_S, find an average modulus E.

[10] Why? Because the stresses must satisfy the volume equilibrium equations and the surface traction BC *a priori*. Thus the stress field must be known at every point in V.

To carry out the homogenization process, assume that the bar is under a constant axial stress field $\sigma = P/A$ (see Figure E6.2), which obviously satisfies all stress equilibrium equations and surface traction boundary conditions. The average strain $e = E^{-1}\sigma$ is taken as the only unknown to be varied in Π_S:

$$\Pi_S(e) = AL\sigma e - \tfrac{1}{2}A(E_1L_1 + E_2L_2 + E_3L_3)e^2. \tag{E6.12}$$

The condition $\delta\Pi_S = 0$ gives $\partial\Pi_S/\partial e = 0$, from which

$$e = \frac{L}{E_1L_1 + E_2L_2 + E_3L_3}\sigma, \qquad E = \frac{\sigma}{e} = \frac{E_1L_1 + E_2L_2 + E_3L_3}{L}. \tag{E6.13}$$

This technique essentially amounts to equating the strain energies absorbed by the actual and homogenized (fictitious) bars. Note that the displacement field does not appear in this statically determinate problem.