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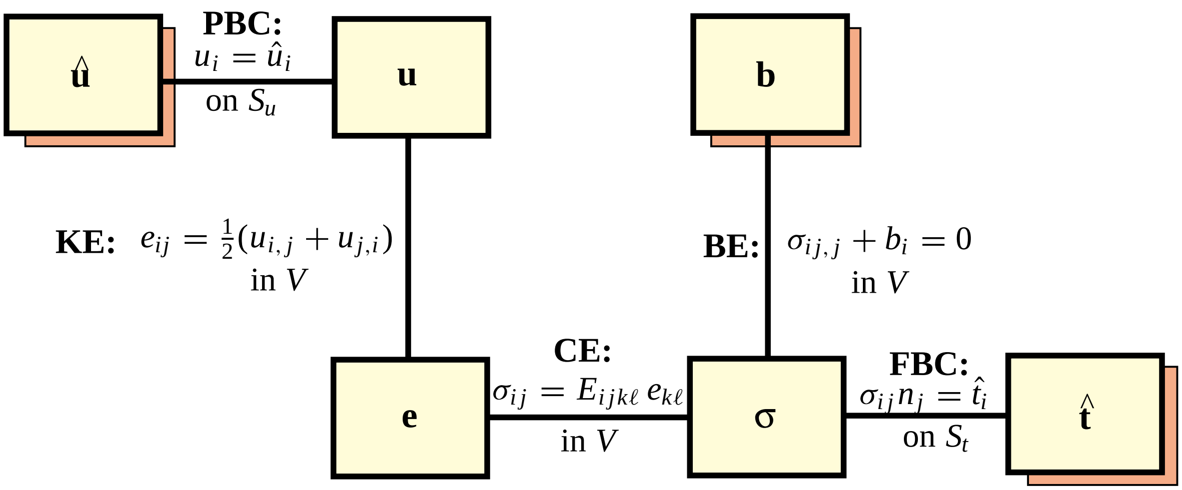
The HR Variational Principle of Elastostatics

The Seven Canonical Functionals of Linear Elastostatics

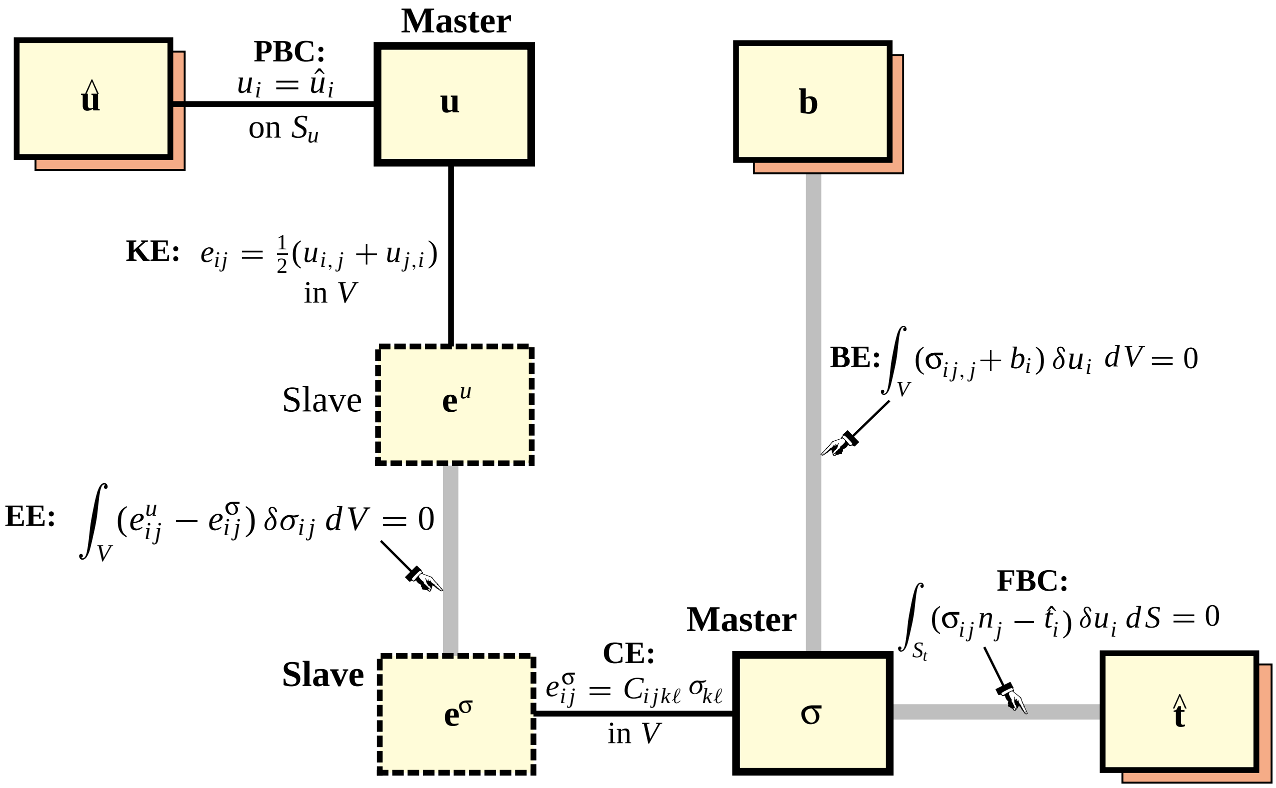
#	Type	Master fields	Name
(I)	Single-field	Displacements	Total Potential Energy (TPE)
(II)	Single-field	Stresses	Total Complementary Potential Energy (TCPE)
(III)	Single-field	Strains	No name
(IV)	Mixed 2 field	Displacements & stresses	Hellinger-Reissner (HR)
(V)	Mixed 2-field	Displacements & strains	No agreed upon name
(VI)	Mixed 2-field	Strains & stresses	No name
(VII)	Mixed 3-field	Displacements, stresses & strains	Veubeke-Hu-Washizu (VHW)

The Strong Form of the Tonti Diagram for Elastostatics

(Equations written in indicial form)



The Weak Form for Derivation of the Hellinger-Reissner (HR) Mixed Functional



Derivation of the HR Mixed Functional

Multiply weak link residuals by weight functions & integrate

$$\int_V (e^u_{ij} - e^\sigma_{ij}) w_{ij}\, dV + \int_V (\sigma_{ij,j} + b_i) w^*_i\, dV + \int_S (\sigma_{ij} n_j - \hat{t}_i) w^{**}_i\, dS = 0$$

Apply work pairing $\quad w_{ij} = \delta\sigma_{ij},\ \ w^*_i = -\delta u_i,\ \ w^{**}_i = \delta u_i,$

$$\int_V (e^u_{ij} - e^\sigma_{ij})\, \delta\sigma_{ij}\, dV - \int_V (\sigma_{ij,j} + b_i)\, \delta u_i\, dV + \int_S (\sigma_{ij} n_j - \hat{t}_i)\, \delta u_i\, dS = 0.$$

Use Divergence Theorem to get rid of stress derivatives

$$\begin{aligned} -\int_V \sigma_{ij,j}\, \delta u_i\, dV &\overset{\text{DT}}{=} \int_V \sigma_{ij}\, \delta e^u_{ij}\, dV - \int_S \sigma_{ij} n_j\, \delta u_i\, dS \\ &= \int_V \sigma_{ij}\, \delta e^u_{ij}\, dV - \int_{S_u} \sigma_{ij} n_j\, \cancelto{0}{\delta u_i}\, dS - \int_{S_t} \sigma_{ij} n_j\, \delta u_i\, dS \\ &= \int_V \sigma_{ij}\, \delta e^u_{ij}\, dV - \int_{S_t} \sigma_{ij} n_j\, \delta u_i\, dS. \end{aligned}$$

Derivation of the HR Mixed Functional (cont'd)

Replace DT result to get the variational statement

$$\delta\Pi_{\rm HR} = \int_V \left[(e^u_{ij} - e^\sigma_{ij})\,\delta\sigma_{ij} + \sigma_{ij}\,\delta e^u_{ij} - b_i\,\delta u_i \right] dV - \int_{S_t} \hat{t}_i\,\delta u_i\,dS$$

This is the exact first variation of

$$\Pi_{\rm HR} = \int_V \left(\sigma_{ij} e^u_{ij} - \tfrac{1}{2}\sigma_{ij} C_{ijk\ell}\sigma_{k\ell} - b_i u_i\right) dV - \int_{S_t} \hat{t}_i u_i\,dS$$

Often written in literature as

$$\Pi_{\rm HR} = \int_V [-\mathcal{U}^*(\sigma_{ij}) + \sigma_{ij}\tfrac{1}{2}(u_{i,j} + u_{j,i}) - b_i u_i]\,dV - \int_{S_t} \hat{t}_i u_i\,dS$$

where

$$\mathcal{U}^*(\sigma_{ij}) = \tfrac{1}{2}\sigma_{ij} C_{ijk\ell}\sigma_{k\ell} = \tfrac{1}{2}\sigma_{ij} e^\sigma_{ij}$$

the *complementary energy density* in terms of the master stresses

The HR Functional (cont'd)

In FEM work it is often split into internal + external energies

$$\Pi_{\rm HR} = U_{\rm HR} - W_{\rm HR}$$

$$U_{\rm HR} = \int_V \left(\sigma_{ij} e^u_{ij} - \tfrac{1}{2}\sigma_{ij} C_{ijk\ell}\sigma_{k\ell}\right) dV$$

$$W_{\rm HR} = \int_V b_i u_i \, dV + \int_{S_t} \hat{t}_i u_i \, dS$$

Variational principle:

$$\delta\Pi_{\rm HR} = 0$$

This principle yields the weak internal connections as Euler-Lagrange equations, and the weak BCs as natural BCs

Variational Indices of Master Fields of HR Functional

Displacement variational index: 1

Stress variational index: 0

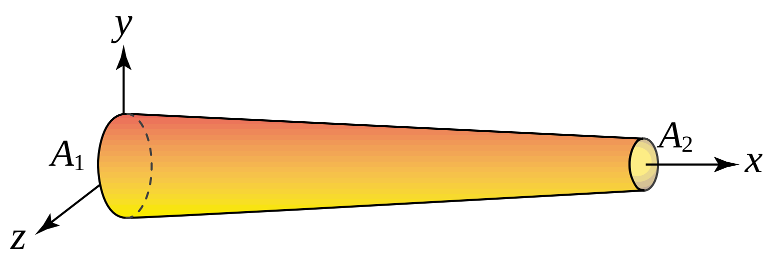
Consequence for FEM model development:

Displacements must be interelement continuous, that is, class C^0, but derivatives need not be

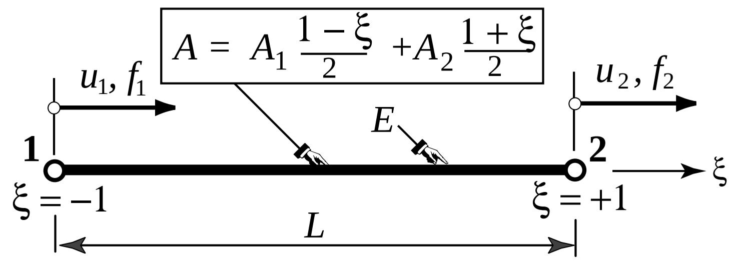
Stresses may be discontinuous between elements, that is, class C^{-1}

HR Application Example: Tapered Bar Element

(a)



(b)



Tapered Bar Element HR Formulation

HR Functional in terms of displacement and axial force

$$\Pi_{\rm HR}[u, N] = \int_L \left(N u' - \frac{N^2}{2EA} \right) dx - f_1 u_1 - f_2 u_2$$

Axial displacement assumption: **piecewise linear** over element

$$u\,(x) \approx u_1^{(e)} \frac{1-\xi}{2} + u_2^{(e)} \frac{1+\xi}{2}$$

Axial force assumption: **constant** over element

$$N\,(x) \approx \bar{N}^{(e)}$$

Insert in functional to get algebraic form with 3 DOFs:

$$\Pi_{\rm HR}^{(e)} = \tfrac{1}{2} \begin{bmatrix} \bar{N}^{(e)} \\ u_1^{(e)} \\ u_2^{(e)} \end{bmatrix}^T \begin{bmatrix} -\frac{\gamma L}{E A_m} & -1 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{N}^{(e)} \\ u_1^{(e)} \\ u_2^{(e)} \end{bmatrix} - \begin{bmatrix} 0 \\ f_1^{(e)} \\ f_2^{(e)} \end{bmatrix}^T \begin{bmatrix} \bar{N}^{(e)} \\ u_1^{(e)} \\ u_2^{(e)} \end{bmatrix}$$

where $A_m = \tfrac{1}{2}(A_1 + A_2)$ and $\gamma = \dfrac{A_m}{A_2 - A_1} \log \dfrac{A_2}{A_1}$

Tapered Bar Element HR Formulation (cont'd)

Apply stationarity conditions wrt element DOFs:

$$\frac{\partial \Pi_{\rm HR}}{\partial \bar{N}^{(e)}} = \frac{\partial \Pi^{(e)}_{\rm HR}}{\partial u^{(e)}_1} = \frac{\partial \Pi^{(e)}_{\rm HR}}{\partial u^{(e)}_2} = 0$$

to get the finite element equations $^{(e)}$

$$\begin{bmatrix} -\dfrac{\gamma L}{E A_m} & -1 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{N}^{(e)} \\ u^{(e)}_1 \\ u^{(e)}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ f^{(e)}_1 \\ f^{(e)}_2 \end{bmatrix}$$

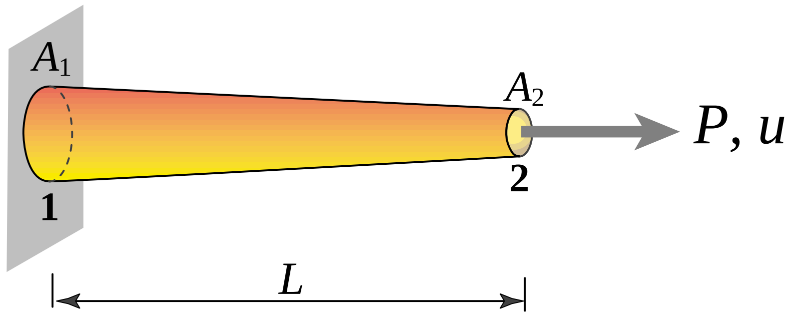
The axial force DOF may be statically condensed because it does not couple with other elements:

$$\frac{E A_m}{\gamma L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u^{(e)}_1 \\ u^{(e)}_2 \end{bmatrix} = \begin{bmatrix} f^{(e)}_1 \\ f^{(e)}_2 \end{bmatrix}$$

or

$$\mathbf{K}^{(e)}\,\mathbf{u}^{(e)} = \mathbf{f}^{(e)}$$

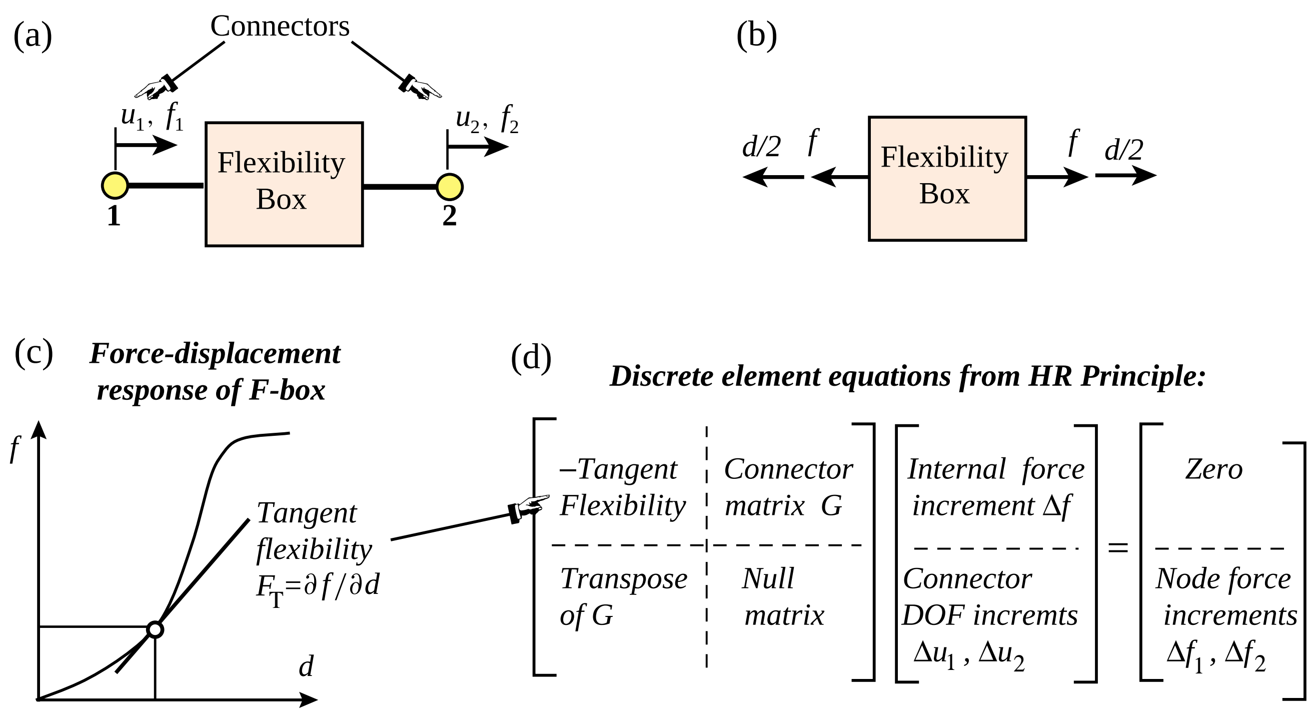
Results for one-element analysis of fixed-free bar



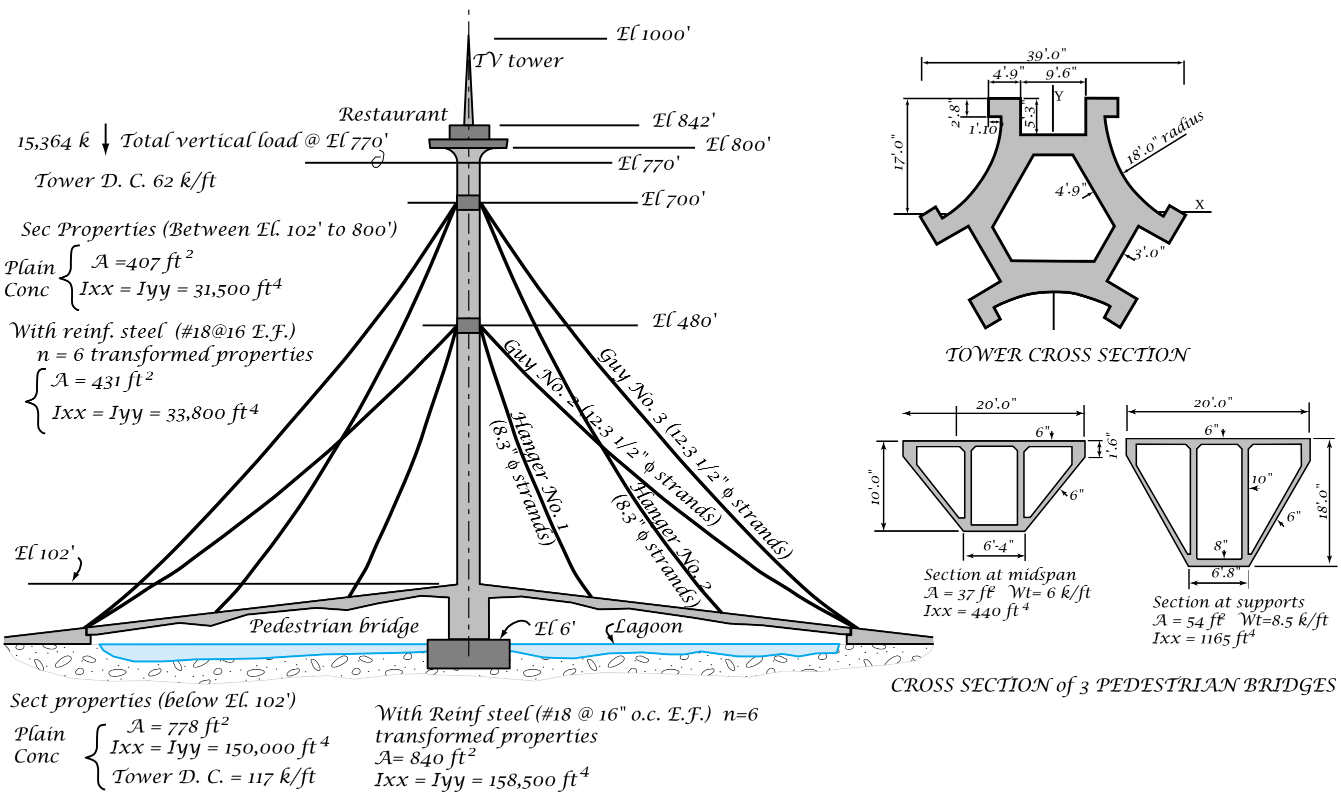
Area ratio	u_2 from HR	u_2 from TPE	Exact u_2
$A_1/A_2 = 1$	$PL/(EA_m)$	$PL/(EA_m)$	$PL/(EA_m)$
$A_1/A_2 = 2$	$1.0397PL/(EA_m)$	$PL/(EA_m)$	$1.0397PL/(EA_m)$
$A_1/A_2 = 5$	$1.2071PL/(EA_m)$	$PL/(EA_m)$	$1.2071PL/(EA_m)$

Errors in stress $\sigma = N/A$ can be much higher,
e.g. 200% for area ratio 5

HR for Connector Elements

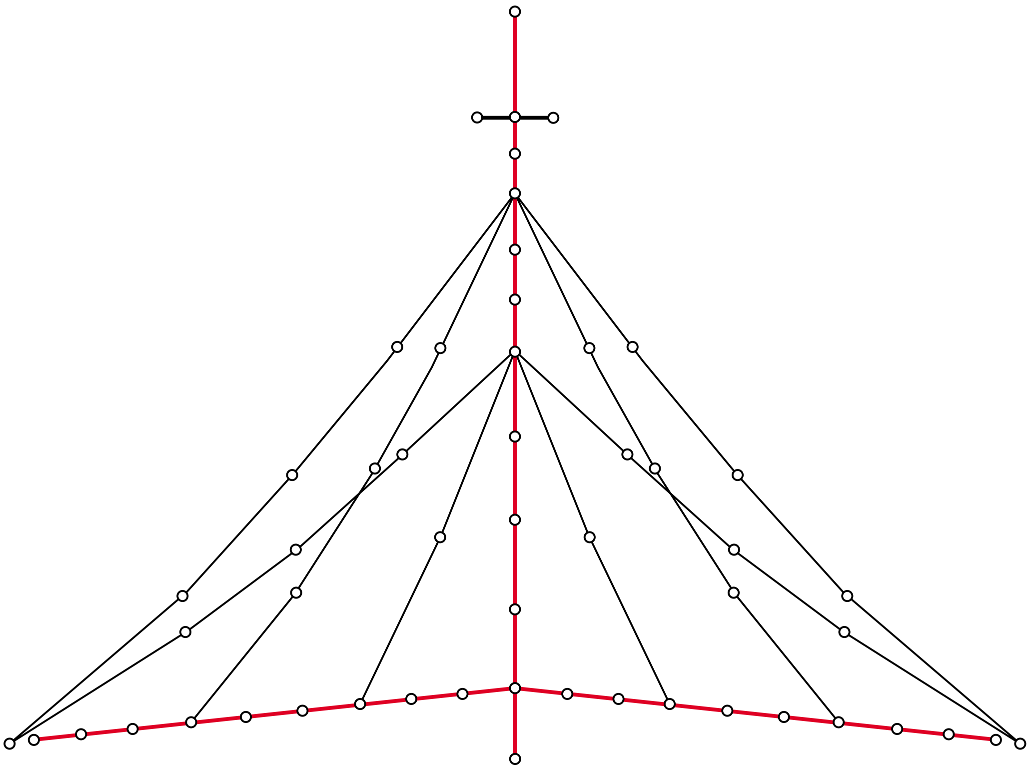


Cable Structure (Proposed 1967) Example



Cable Structure FEM Model

Advanced FEM



Curved Cable Element