5

Three-Dimensional Linear Elastostatics

TABLE OF CONTENTS

§5.1. Introduction

We move now from the easy ride of Poisson problems and Bernoulli-Euler beams to the tougher road of elasticity in three dimensions. This Chapter summarizes the governing equations of linear elastostatics. Various notational systems are covered in sufficient detail to help readers with the literature of the subject, which is enormous and spans over two centuries. The governing equations are displayed in a Strong Form Tonti diagram.

The classical single-field variational principle of Total Potential Energy is derived in this Chapter as prelude to mixed and hybrid variational principles, which are presented in the next two Chapters.[1]

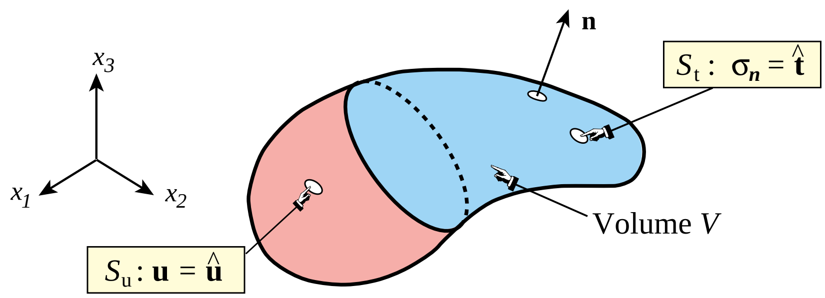

FIGURE 5.1. A linear-elastic body of volume V in static equilibrium. The body surface $S : S_t \cup S_u$ is split into S_t, on which surface tractions are prescribed, and S_u, on which surface displacements are prescribed.

§5.2. The Governing Equations

Consider a linearly elastic body of volume V, which is bounded by surface S, as shown in Figure 5.1. The body is referred to a three dimensional, rectangular, right-handed Cartesian coordinate system $x_i \equiv \{x_1, x_2, x_3\}$. The body is in *static* equilibrium under the action of body forces b_i in V, prescribed surface tractions $\hat{t}_i$ on S_t and prescribed displacements $\hat{u}_i$ on S_u, where $S_t \cup S_u \equiv S$ are two complementary portions of the boundary S. This separation of boundary conditions and source data is displayed in more detail in Figure 5.2.

The three unknown internal fields are *displacements* u_i, *strains* $e_{ij} = e_{ji}$ and *stresses* $\sigma_{ij} = \sigma_{ji}$. All of them are defined in V. In the absence of internal interfaces the three fields may be assumed to be continuous and piecewise differentiable.[2] At internal interfaces (for example a change in material) certain strain and stress components may jump, but such “jump conditions” are ignored in the present treatment.

The three known or data fields are the body forces b_i, prescribed surface tractions $\hat{t}_i$ and prescribed displacements $\hat{u}_i$. These are given in V, on S_t, and on S_u, respectively.

The equations that link the various volume fields are called the *field equations* of elasticity. Those linking volume fields (evaluated at the surface) and prescribed surface fields are called *boundary*

[1] The material in this and next two chapters is mostly taken from the Variational Methods in Mechanics course complemented with additional material on problem-solving.

[2] See, e.g., M. Gurtin, The Linear Theory of Elasticity, in *Encyclopedia of Physics* VIa, Vol II, ed. by C. Truesdell, Springer-Verlag, Berlin, 1972, pp. 1–295; reprinted as *Mechanics of Solids* Vol II, Springer-Verlag, 1984.

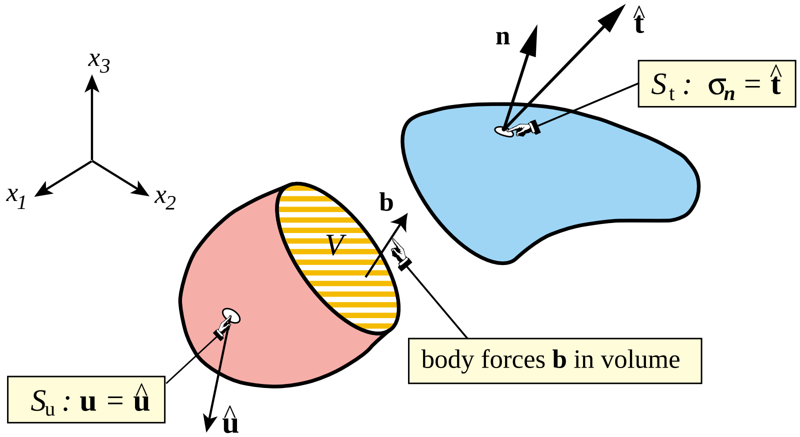

FIGURE 5.2. Showing in more detail the separation of the surface S into two complementary regions S_t and S_u.

conditions. The ensemble of field equations and boundary conditions represent the *governing equations* of elastostatics.

Remark 5.1. The field equations are generally partial differential equations (PDEs) although for elasticity the constitutive equations become algebraic. The classical boundary conditions are algebraic relations.

Remark 5.2. The separation of S into traction-specified S_t and displacement-specified S_u may be more complex than the simple surface partition of the Poisson problem. This is because $\hat{t}_i$ and $\hat{u}_i$ are now vectors with several components. These may be specified at the same surface point in various combinations. This happens in many practical problems. For example, one may consider a portion of S where a pressure force is applied whereas the tangential displacement components are zero. Or a bridge roller support: the displacement normal to the rollers is precluded (a displacement condition) but the tangential displacements are free (a traction condition). This mixture of force and displacement conditions over the same surface element would complicate the notation considerably. We shall use the "union of" notation $S \equiv S_t \cup S_u$ for notational simplicity but the presence of such complications should be kept in mind.

§5.2.1. Direct Tensor Notation

In the foregoing description we have used the so-called *component notation* or *indicial notation* for fields. More precisely, the notation appropriate to rectangular Cartesian coordinates. In this notation, writing u_i is equivalent to writing the three components u_1, u_2, u_3 of the displacement field **u**. We now review the so-called *direct tensor notation* or *compact tensor notation*.

Scalars, which are zero-dimensional tensors, are represented by non-boldface Roman or Greek symbols. Example: ρ for mass density and g for the acceleration of gravity.

Vectors, which are one-dimensional tensors, are represented by **boldface** symbols. These will be usually lowercase letters unless common usage dictates the use of uppercase symbols.[3] For

[3] This happens in electromagnetics: tradition since Maxwell has kept field vectors such as **E** (electric field) and **B** (magnetic field) in uppercase.

example:

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \qquad \mathbf{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}, \tag{5.1}$$

identify the vectors of displacements, body forces and surface tractions, respectively.

Two-dimensional tensors are represented by **underlined boldface** lowercase symbols. These will usually be lowercase Roman or Greek letters. For example

$$\underline{\mathbf{e}} = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ & e_{22} & e_{23} \\ \text{symm} & & e_{33} \end{bmatrix} \equiv e_{ij}, \qquad \underline{\boldsymbol{\sigma}} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ & \sigma_{22} & \sigma_{23} \\ \text{symm} & & \sigma_{33} \end{bmatrix} \equiv \sigma_{ij}, \tag{5.2}$$

denote the strain and stress tensors, respectively. The *transpose* of a second order tensor, denoted by $(.)^T$ is obtained by switching the two indices. A tensor is *symmetric* if it equates its transpose. Both the stress and strain tensors are symmetric: $\underline{\boldsymbol{\sigma}} = \underline{\boldsymbol{\sigma}}^T$ or $\sigma_{ij} = \sigma_{ji}$. Likewise $\underline{\mathbf{e}} = \underline{\mathbf{e}}^T$ or $e_{ij} = e_{ji}$.

Two product operations may be defined between second-order tensors. The *scalar product* or *inner product* is a scalar, which in terms of components is defined as[4]

$$\underline{\boldsymbol{\sigma}} : \underline{\mathbf{e}} = \sum_{i=1}^{3} \sum_{j=1}^{3} \sigma_{ij} e_{ij} = \sigma_{ij} e_{ij}. \tag{5.3}$$

With $\underline{\boldsymbol{\sigma}}$ and $\underline{\mathbf{e}}$ as stress and strain tensors, respectively, $\underline{\boldsymbol{\sigma}} : \underline{\mathbf{e}}$ is twice the strain energy density $\mathcal{U}$.

The *tensor product* or *open product* of two second order tensors is a second-order tensor defined by the composition rule:

$$\text{if} \quad \underline{\mathbf{p}} = \underline{\boldsymbol{\sigma}} \cdot \underline{\mathbf{e}}, \quad \text{then} \quad p_{ij} = \sum_{k=1}^{3} \sigma_{ik} e_{kj} = \sigma_{ik} e_{kj}. \tag{5.4}$$

This is exactly the same rule as the matrix product. For matrices the dot is omitted. Some authors also omit the dot for tensors.

Four-dimensional tensors are represented by **underlined boldface uppercase** symbols. In elasticity the tensor of elastic moduli provides the most important example:

$$\underline{\mathbf{E}} \equiv E_{ijk\ell}, \tag{5.5}$$

The components of $\underline{\mathbf{E}}$ form a $3 \times 3 \times 3 \times 3$ hypercube with $3^4 = 81$ components, so the whole thing cannot be displayed so compactly as (5.2).

Operators that map vectors to vectors are usually represented by boldface uppercase symbols. An ubiquitous operator is nabla: ∇, which should be boldface except that the symbol is not available in bold. Applied to a scalar function, say ϕ, it produces its gradient:

$$\nabla\phi = \mathbf{grad}\,\phi \equiv \phi_{,i} = \frac{\partial \phi}{\partial x_i} = \begin{bmatrix} \frac{\partial \phi}{\partial x_1} \\ \frac{\partial \phi}{\partial x_2} \\ \frac{\partial \phi}{\partial x_3} \end{bmatrix}. \tag{5.6}$$

---

[4] Some textbooks use the notation $\underline{\boldsymbol{\sigma}}..\underline{\mathbf{e}}$ for the scalar $\sigma_{ij} e_{ji}$, but this is unnecessary as it is easily expressed in terms of : by transposing the second tensor.

Applying nabla to a vector via the dot product yields the divergence of the vector:

$$\nabla \cdot \mathbf{u} = \mathbf{div}\,\mathbf{u} \equiv u_{i,i} = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}. \tag{5.7}$$

Applying nabla to a second order tensor yields the divergence of a tensor, which is a vector. For example:

$$\nabla \cdot \underline{\boldsymbol{\sigma}} \equiv \mathbf{div}\ \boldsymbol{\sigma} = \sigma_{ij,j} = \sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j} = \begin{bmatrix} \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} \\ \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} \\ \frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} \end{bmatrix} \tag{5.8}$$

Applying ∇ to a vector via the cross product yields the curl or spin operator. This operator is not needed in classical elasticity but it appears in applications that deal with rotational fields such as fluid dynamics with vorticity, or corotational structural dynamics.

§5.2.2. Matrix Notation

Matrix notation is a modification of direct tensor notation in which everything is placed in matrix form, with some trickery used if need be. The main advantages of matrix notation are historical compatibility with finite element formulations, and ready computer implementation in symbolic or numeric form.[5]

The representation of scalars, which may be viewed as 1×1 matrices, does not change. Neither does the representation of vectors because vectors are column (or row) matrices.

Two-dimensional *symmetric* tensors are converted to one-dimensional arrays that list only the independent components (six in three dimensions, three in two dimensions). Component order is a matter of convention, but usually the diagonal components are listed first followed by the off-diagonal components. A factor of 2 may be applied to the latter, as the strain vector example below shows. The tensor is then represented as if were an actual vector, that is by non-underlined **boldface** lowercase Roman or Greek letters.

For the strain and stress tensors this "vectorization" process produces the 6-vectors

$$\underline{\mathbf{e}} \equiv \mathbf{e} = \begin{bmatrix} e_{11} \\ e_{22} \\ e_{33} \\ 2e_{23} \\ 2e_{31} \\ 2e_{12} \end{bmatrix}, \qquad \underline{\boldsymbol{\sigma}} \equiv \boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix}, \tag{5.9}$$

Note that off-diagonal (shearing) components of the strain vector are scaled by 2, but that no such scaling applies to the off-diagonal (shear) stress components. The idea behind the scaling is to

[5] Particularly in high level languages such as *Matlab*, *Mathematica* or *Maple*, which directly support matrix operators.

maintain inner product equivalence so that for example, the strain energy density is simply

$$\begin{aligned}\mathcal{U} = \tfrac{1}{2}\underline{\boldsymbol{\sigma}} : \underline{\mathbf{e}} = \tfrac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\sigma_{ij}e_{ij} = \tfrac{1}{2}\sigma_{ij}e_{ij} = \tfrac{1}{2}\boldsymbol{\sigma}^T\mathbf{e} \\ = \tfrac{1}{2}\big(\sigma_{11}e_{11} + \sigma_{22}e_{22} + \sigma_{33}e_{33} + 2\sigma_{31}e_{31} + 2\sigma_{23}e_{23} + 2\sigma_{12}e_{12}\big).\end{aligned} \tag{5.10}$$

Four-dimensional tensors are mapped to square matrices and denoted by matrix symbols, that is, **non-underlined boldface uppercase** Roman or Greek letters. Indices are appropriately collapsed to reflect symmetries and maintain product equivalence. Rather than stating boring rules, the example of the elastic moduli tensor is given to illustrate the mapping technique.

The stress-strain relation for linear elasticity in component notation is $\sigma_{ij} = E_{ijk\ell}\, e_{k\ell}$, and in compact tensor form $\underline{\boldsymbol{\sigma}} = \underline{\mathbf{E}} \cdot \underline{\mathbf{e}}$. We would of course like to have $\boldsymbol{\sigma} = \mathbf{E}\,\mathbf{e}$ in matrix notation. This can be done by defining the 6×6 elastic modulus matrix

$$\mathbf{E} = \begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} & E_{15} & E_{16} \\ & E_{22} & E_{23} & E_{24} & E_{25} & E_{26} \\ & & E_{33} & E_{34} & E_{35} & E_{36} \\ & & & E_{44} & E_{45} & E_{46} \\ & & & & E_{55} & E_{56} \\ & \text{symm} & & & & E_{66} \end{bmatrix} \tag{5.11}$$

The components E_{pq} of $\mathbf{E}$ are related to the components $E_{ijk\ell}$ of $\underline{\mathbf{E}}$ through an appropriate mapping that preserves the product relation. For example: $\sigma_{11} = E_{1111}e_{11} + E_{1122}e_{22} + E_{1133}e_{33} + E_{1112}e_{12} + E_{1121}e_{21} + E_{1113}e_{13} + E_{1131}e_{31} + E_{1123}e_{23} + E_{1132}e_{32}$ maps to $\sigma_{11} = E_{11}e_{11} + E_{12}e_{22} + E_{13}e_{33} + E_{14}\,2e_{23} + E_{15}\,2e_{31} + E_{16}\,2e_{12}$, whence $E_{11} = E_{1111}$, $E_{14} = E_{1123} + E_{1132}$, etc.

Finally, operators that can be put in vector form are usually represented by a vector symbol **boldface lowercase** whereas operators that can be put in matrix form are usually represented as matrices. Here is an example:

$$\mathbf{e} = \begin{bmatrix} e_{11} \\ e_{22} \\ e_{33} \\ 2e_{12} \\ 2e_{23} \\ 2e_{31} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u_1}{\partial x_1} \\ \dfrac{\partial u_2}{\partial x_2} \\ \dfrac{\partial u_3}{\partial x_3} \\ \dfrac{\partial u_3}{\partial x_2} + \dfrac{\partial u_2}{\partial x_3} \\ \dfrac{\partial u_1}{\partial x_3} + \dfrac{\partial u_3}{\partial x_1} \\ \dfrac{\partial u_2}{\partial x_1} + \dfrac{\partial u_1}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial}{\partial x_1} & 0 & 0 \\ 0 & \dfrac{\partial}{\partial x_2} & 0 \\ 0 & 0 & \dfrac{\partial}{\partial x_3} \\ 0 & \dfrac{\partial}{\partial x_3} & \dfrac{\partial}{\partial x_2} \\ \dfrac{\partial}{\partial x_3} & 0 & \dfrac{\partial}{\partial x_1} \\ \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_1} & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \mathbf{D}\,\mathbf{u}. \tag{5.12}$$

Operator $\mathbf{D}$ is called the *symmetric gradient* in the continuum mechanics literature.[6] In the matrix notation defined above it is written as a 6×3 matrix. In direct tensor notation $\underline{\mathbf{D}} = \frac{1}{2}(\nabla + \nabla^T)$ is the tensor that maps $\mathbf{u}$ to $\underline{\mathbf{e}}$, and we write $\underline{\mathbf{e}} = \underline{\mathbf{D}} \cdot \mathbf{u}$. For the indicial form see below.

[6] Some books use variants of ∇, such as $\bar{\nabla}$, $\hat{\nabla}$, ∇_S or ∇^S, for this operator.

§5.3. The Field Equations

§5.3.1. The Strain-Displacement Equations

The strain-displacement equations, also called the kinematic equations (KE) or deformation equations, yield the strain field given the displacement field. For linear elasticity the infinitesimal strain tensor e_{ij} is given by

$$\boxed{e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),} \tag{5.13}$$

where a comma denotes differentiation with respect to the space variable whose index follows.

In compact tensor notation, with $\underline{\mathbf{D}}$ as the symmetric gradient operator,

$$\boxed{\underline{\mathbf{e}} = \tfrac{1}{2}(\nabla + \nabla^T)\cdot\mathbf{u} = \underline{\mathbf{D}}\cdot\mathbf{u}.} \tag{5.14}$$

The matrix form is $\mathbf{e} = \mathbf{D}\,\mathbf{u}$. The full form is given in (5.12).

The inverse problem: given a strain field find the displacements, is not generally soluble unless the strain components satisfy the *strain compatibility conditions*. These are complicated second-order partial differential equations given in any book on elasticity. This inverse problem will not be considered here.

§5.3.2. Constitutive Equations

The constitutive equations connect the stress and strain fields in V. These equations are intended to model the behavior of materials as continuum media. Generally they are partial differential equations (PDEs) or even integrodifferential equations in space and time. For linear elasticity, however, a considerable simplification occurs because the relation becomes algebraic, linear and homogeneous. For this case the stress-strain relations may be written in component notation as

$$\boxed{\sigma_{ij} = E_{ijk\ell}\, e_{k\ell} \quad \text{in } V.} \tag{5.15}$$

The $E_{ijk\ell}$ are called *elastic moduli*. They are the components of a fourth order tensor **E** called the *elasticity tensor*. The elastic moduli satisfy generally the following symmetries

$$E_{ijk\ell} = E_{jik\ell} = E_{ij\ell k}, \tag{5.16}$$

which reduce their number from $3^4 = 729$ to $6^2 = 36$. Furthermore, if the body admits a strain energy (that is, the material is not only elastic but hyperelastic) the elastic moduli satisfy additional symmetries:

$$E_{ijk\ell} = E_{k\ell ij}, \tag{5.17}$$

which reduce their number to 21. Further symmetries occur if the material is orthotropic or isotropic. In the latter case the elastic moduli may be expressed as function of only two independent material constants, for example Young's modulus E and Poisson ratio ν.

In compact tensor notation:

$$\boxed{\underline{\boldsymbol{\sigma}} = \underline{\mathbf{E}} \cdot \underline{\mathbf{e}}.} \tag{5.18}$$

In matrix form:

$$\boxed{\boldsymbol{\sigma} = \mathbf{E}\,\mathbf{e},} \tag{5.19}$$

where $\mathbf{E}$ is the 6×6 matrix given in (5.11).

If the elasticity tensor is invertible, the relation that connects strains to stresses is written

$$e_{ij} = C_{ijk\ell}\sigma_{k\ell} \quad \text{in } V. \tag{5.20}$$

The $C_{ijk\ell}$ are called *elastic compliances*. They are also the components of a fourth order tensor called the *compliance tensor*, which satisfies the same symmetries as $\mathbf{E}$. In compact tensor notation

$$\underline{\mathbf{e}} = \underline{\mathbf{C}} \cdot \underline{\boldsymbol{\sigma}} = \underline{\mathbf{E}}^{-1} \cdot \underline{\boldsymbol{\sigma}}, \tag{5.21}$$

and in matrix form: $\mathbf{e} = \mathbf{C}\ \boldsymbol{\sigma} = \mathbf{E}^{-1}\boldsymbol{\sigma}$.

§5.3.3. Internal Equilibrium Equations

The internal equilibrium equations of elastostatics are

$$\boxed{\sigma_{ij,j} + b_i = \frac{\partial \sigma_{ij}}{\partial x_j} + b_i = 0 \quad \text{in } V.} \tag{5.22}$$

These follow from the linear momentum balance equations derived in books on continuum mechanics.

The compact tensor notation is

$$\boxed{\nabla \cdot \underline{\boldsymbol{\sigma}} + \mathbf{b} = \mathbf{0} \quad \text{in } V.} \tag{5.23}$$

The matrix form is

$$\boxed{\mathbf{D}^T\boldsymbol{\sigma} + \mathbf{b} = \mathbf{0} \quad \text{in } V.} \tag{5.24}$$

Here $\mathbf{D}^T$ is the transpose of the symmetric gradient operator (5.12).

§5.4. The Boundary Conditions

§5.4.1. Surface Compatibility Equations

The surface compatibility equations, also called *displacement boundary conditions*, are

$$\boxed{u_i = \hat{u}_i \quad \text{on } \ S_u,} \tag{5.25}$$

or in direct notation (both tensor and matrix)

$$\boxed{\mathbf{u} = \hat{\mathbf{u}} \quad \text{on } \ S_u.} \tag{5.26}$$

The physical meaning is that the displacement components at points of S_t must match the prescribed values.

§5.4.2. Surface Equilibrium Equations

The surface equilibrium equations, also called *stress boundary conditions*, or *traction boundary conditions*, are

$$\boxed{\sigma_{ij}\, n_j = \hat{t}_i \quad \text{on } \ S_t,} \tag{5.27}$$

where n_j are the components of the external unit normal $\mathbf{n}$ at points of S_t where tractions are specified; see Figure 5.2. Note that

$$\sigma_{ni} = \sigma_{ij} n_j = t_i \tag{5.28}$$

are the components of the *internal traction vector* $\mathbf{t} \equiv \boldsymbol{\sigma}_n$. The physical interpretation of the stress boundary condition is that the internal traction vector must equal the prescribed traction vector. Or: the net flux $t_i - \hat{t}_i$ on S_t vanishes, component by component. In compact tensor form

$$\boxed{\mathbf{t} = \boldsymbol{\sigma}_n = \underline{\boldsymbol{\sigma}} \cdot \mathbf{n} = \hat{\mathbf{t}}.} \tag{5.29}$$

Stating (5.27) in a matrix form that uses the stress vector $\boldsymbol{\sigma}$ defined in (5.9) requires some care. It would be incorrect to write either $\mathbf{t} = \boldsymbol{\sigma}^T \mathbf{n}$ or $\mathbf{t} = \mathbf{n}^T \boldsymbol{\sigma}$ because $\boldsymbol{\sigma}$ is 6×1 and $\mathbf{n}$ is 3×1. Not only are these vectors non-conforming but their inner product is a scalar. The proper matrix form is a bit contrived:

$$\mathbf{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} n_1 & 0 & 0 & 0 & n_3 & n_2 \\ 0 & n_2 & 0 & n_3 & 0 & n_1 \\ 0 & 0 & n_3 & n_2 & n_1 & 0 \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix} = \mathbf{P}_n\, \boldsymbol{\sigma}, \tag{5.30}$$

where $\mathbf{P}_n$ is the 3×6 normal-projection matrix shown above.

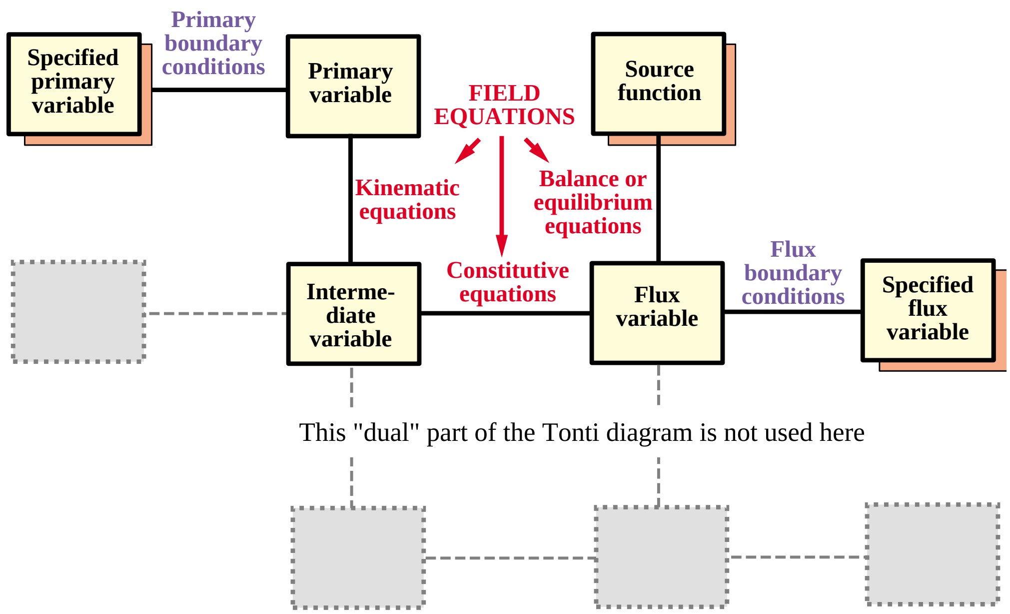

FIGURE 5.3. The general configuration of the Tonti diagram. Upper portion reproduced from Chapter 2. The diagram portion shown in dashed lines, which represents the so-called dual or potential equations, is not used in this book.

§5.5. Tonti Diagrams

The Tonti diagram was introduced in Chapter 2 to represent the field equations of a mathematical model in graphical form. The general configuration of the expanded form of that diagram ("expanded" means that it shows boundary conditions) is repeated in Figure 5.3 for convenience. This diagram lists generic names for the "box occupants" and the connecting links.

Boxes and box-connectors drawn in solid lines are said to constitute the *primal* formulation of the governing equations. Dashed-lines boxes and connectors shown in the bottom pertain to the so-called *dual* formulation in terms of potentials, which will not be used in this book.[7]

Figure 5.4 shows the primal formulation of the linear elasticity problem represented as a Tonti diagram. For this particular problem the displacements are the primary (or primal) variables, the strains the intermediate variables, and the stresses the flux variables. The source variables are the body forces. The prescribed configuration variables are prescribed displacements on S_t and the prescribed flux variables are the surface tractions on S_t.

Tables 5.1 and 5.2 lists the generic names for the components of the Tonti diagram, as well as those specific for the elasticity problem. Table 5.3 summarizes the governing equations of linear elastostatics written down in three notational schemes.

[7] In the dual formulation the intermediate and flux variable exchange roles, so that boundary conditions of flux type are linked to the intermediate variable of the primal formulation. In this way it is possible, for instance, to specify strain boundary conditions: just to for the dual formulation.

Table 5.1 Abbreviations for Tonti Diagram Box Contents

Acronym	*Meaning*	*Alternate names in literature*
PV	Primary variable	Primal variable, configuration variable, "across" variable
IV	Intermediate variable	First intermediate variable, auxiliary variable
FV	Flux variable	Second intermediate variable, "through" variable
SV	Source variable	Internal force variable, production variable
PPV	Prescribed primary variable	
PFV	Prescribed flux variable	

Table 5.2 Abbreviations for Tonti Diagram Box Connectors

Acronym	*Generic name*	*Name(s) given in the elasticity problem*
KE	Kinematic equations	Strain-displacement equations
CE	Constitutive equations	Stress-strain equations, material equations
BE	Balance equations	Internal equilibrium equations
PBC	Primary boundary conditions	Displacement BCs
FBC	Flux boundary conditions	Stress BCs, traction BCs

Table 5.3 Summary of Elastostatic Governing Equations

Acr	*Valid*	*Compact or direct tensor form*	*Matrix form*	*Component (indicial) form*
KE	in V	$\underline{\mathbf{e}} = \frac{1}{2}(\nabla + \nabla^T) \cdot \mathbf{u} = \underline{\mathbf{D}} \cdot \mathbf{u}$	$\mathbf{e} = \mathbf{D}\,\mathbf{u}$	$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$
CE	in V	$\underline{\boldsymbol{\sigma}} = \underline{\mathbf{E}} \cdot \underline{\mathbf{e}}$	$\boldsymbol{\sigma} = \mathbf{E}\,\mathbf{e}$	$\sigma_{ij} = E_{ijk\ell}\, e_{k\ell}$
BE	in V	$\nabla \cdot \underline{\boldsymbol{\sigma}} + \mathbf{b} = \mathbf{0}$	$\mathbf{D}^T \boldsymbol{\sigma} + \mathbf{b} = \mathbf{0}$	$\sigma_{ij,j} + b_i = 0$
PBC	on S_u	$\mathbf{u} = \hat{\mathbf{u}}$	$\mathbf{u} = \hat{\mathbf{u}}$	$u_i = \hat{u}_i$
FBC	on S_t	$\underline{\boldsymbol{\sigma}} \cdot \mathbf{n} = \boldsymbol{\sigma}_n = \mathbf{t} = \hat{\mathbf{t}}$	$\mathbf{P}_n\, \boldsymbol{\sigma} = \boldsymbol{\sigma}_n = \mathbf{t} = \hat{\mathbf{t}}$	$\sigma_{ij} n_j = \sigma_{ni} = t_i = \hat{t}_i$

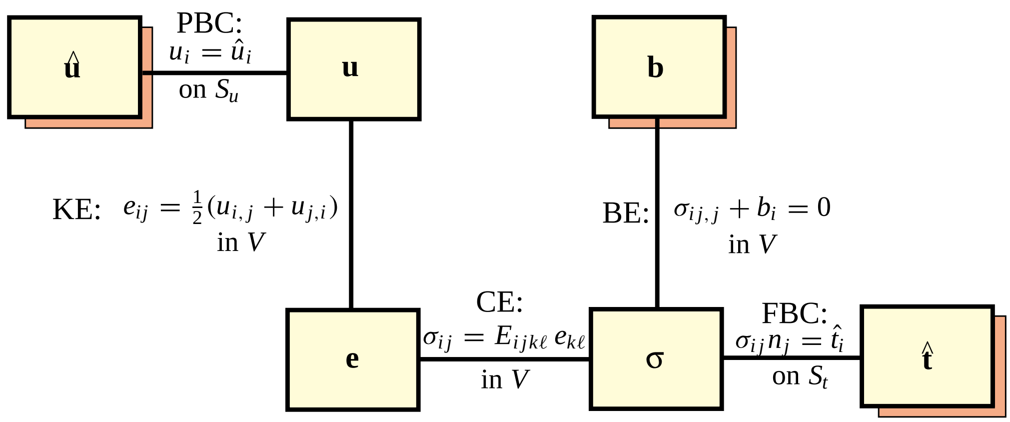

FIGURE 5.4. Tonti diagram for linear elastostatics. Governing equations are expressed along links in indicial form.

§5.6. Other Notational Conventions

To facilitate comparison with older textbooks and papers, the governing equations are restated below in two more alternative forms: in "grad/div" notation, and in full form.

§5.6.1. Grad-Div Direct Tensor Notation

This is a variation of the "nabla" direct tensor notation. Symbols **grad** and **div** are used instead of ∇ and $\nabla\cdot$ forgradient and divergence, respectively, and **symm grad** means the symmetric gradient operator $\mathbf{D} = \frac{1}{2}(\nabla + \nabla^T)$. The notation is slightly mode readable but takes more room.

$$
\begin{array}{|lcl|}
\hline
\text{KE:} & \underline{\mathbf{e}} = \textbf{symm grad}\,\mathbf{u} & \text{in } V, \\
\text{CE:} & \underline{\boldsymbol{\sigma}} = \underline{\mathbf{E}}\ \underline{\mathbf{e}} & \text{in } V, \\
\text{BE:} & \textbf{div}\,\underline{\boldsymbol{\sigma}} + \mathbf{b} = \mathbf{0} & \text{in } V, \\
\text{PBC:} & \mathbf{u} = \hat{\mathbf{u}} & \text{on } S_u, \\
\text{FBC:} & \underline{\boldsymbol{\sigma}} \cdot \mathbf{n} = \boldsymbol{\sigma}_n = \mathbf{t} = \hat{\mathbf{t}}, & \text{on } S_t. \\
\hline
\end{array}
\tag{5.31}
$$

§5.6.2. Full Notation

In the full-form notation everything is spelled out. No ambiguities of interpretation can arise; consequently this works well as a notation of last resort, and also as a "comparison template" against one can check out the meaning of more compact notations. It is also useful for programming in low-order languages.

The full form has, however, two major problems. First, it can become quite voluminous when higher order tensors are involved. Notice that most of the equations below are truncated because there is no space to state them fully. Second, compactness encourages visualization of essentials: long-windedness can obscure the forest with too many trees.

$$
\boxed{
\begin{array}{lcr}
\text{KE:} & e_{11} = \dfrac{\partial u_1}{\partial x_1}, \quad e_{12} = e_{21} = \tfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1}\right), \quad \ldots & \text{in } V, \\
\text{CE:} & \sigma_{11} = E_{1111} e_{11} + E_{1112} e_{12} + \ \ldots \text{ (7 more terms)}, \quad \sigma_{12} = \ \ \ldots & \text{in } V, \\
\text{BE:} & \dfrac{\partial \sigma_{11}}{\partial x_1} + \dfrac{\partial \sigma_{12}}{\partial x_2} + \dfrac{\partial \sigma_{13}}{\partial x_3} + b_1 = 0, \ \ldots & \text{in } V, \\
\text{PBC:} & u_1 = \hat{u}_1, \quad u_2 = \hat{u}_2, \quad u_3 = \hat{u}_3 & \text{on } S_u, \\
\text{FBC:} & \sigma_{11}\, n_1 + \sigma_{12}\, n_2 + \sigma_{13}\, n_3 = \hat{t}_1, \ \ \ldots & \text{on } S_t.
\end{array}
}
\tag{5.32}
$$

§5.7. Solving Elastostatic Problems

By *solving an elastostatic problem* it is meant to find the displacement, strain and stress fields that satisfy all governing equations; that is, the field equations and the boundary conditions.

Under mild assumptions of primary interest to mathematicians, the elastostatic problem has one and only one solution. There are, however, practical problems where the solution is not unique. Two instances:

1. *"Free Floating" Structures*. The displacement field is not unique but strains and stresses are. Example are aircraft structures in flight and space structures in orbit.

2. *Incompressible Materials*. The mean (hydrostatic) stress field is not determined from the displacements and strains. Determination of the hydrostatic stress field depends on the stress boundary conditions, and these may be insufficient in some cases.

An analytical solution of the elastostatic problem is only possible for very simple cases. Most practical problems demand a numerical solution. Numerical methods require a *discretization* process through which an approximate solution with a finite number of degrees of freedom is constructed.

§5.7.1. Discretization Methods in Computational Mechanics

Discretization methods of highest importance in mechanics can be grouped into three classes: finite difference, finite element, and boundary methods.

Finite Difference Method (FDM). The governing differential equations are replaced by difference expressions based on the field values at nodes of a finite difference grid. Although FDM remains important in fluid mechanics and in dynamic problems for the time dimension, it has been largely superseded by the finite element method in a structural mechanics in general and elastostatics in particular.

Finite Element Method (FEM). This is the most important "volume integral" method. One or more of the governing equations are recast to hold in some *average* sense over subdomains of simple geometry. This recasting is often done in terms of *variational forms* if variational principles can be readily constructed, as is the case in elastostatics. The procedure for constructing the simplest class of these principles is outlined in the next section.

Boundary Methods. Under certain conditions the field equations between volume fields can be eliminated in favor of *boundary unknowns*. This dimensionality reduction process leads to integro-differential equations taken over the boundary S. Discretization of these equations through finite element or collocation techniques leads to the so-called *boundary element methods* (BEM).

Further discussion on the role of these methods within the process of simulating of structural systems was offered in Chapter 1.

§5.8. Constructing Variational Forms

Finite element methods for the elasticity problem are based on *Variational Forms*, or VFs, of the foregoing Strong Form (SF) equations. Although the SF is unique, there are many VFs.[8] As explained in Chapters 2–4, the search for a VF begins by selecting one or more master fields, and weakening one or more links. This process produces a set of equations called the Weak Form, or WF, which may be viewed as an midway stop between the SF and the VF.

The end result of the process is the construction of a *functional* Π that contains integrals of the known and unknown fields. Associated with the functional is a *variational principle:* setting the first variation $\delta\Pi$ to zero recovers the strong form of the weakened field equations as Euler-Lagrange equations, and the strong form of the weakened BCs as natural boundary conditions.

Here is a summary of the VF construction steps: (1) choose the master(s), (2) weaken selected links, (3) work out the (total) variation of the alleged functional, (4) construct the functional. These four steps are elaborated below keeping the elasticity equations in mind. There are then illustrated with the construction of the single-field primal functional, called the Total Potential Energy.

§5.8.1. Step 1: Choose Master Field(s)

One or more of the unknown internal fields

$$u_i, \quad e_{ij}, \quad \sigma_{ij}, \tag{5.33}$$

are chosen as masters. A master (also called *primary*, *varied* or *parent*) field is one that is subjected to the δ-variation process of the calculus of variations. Fields that are not masters, *i.e.* not subject to variation, are called *slave*, *secondary* or *derived*. The *owner* (also called *parent* or *source*) of a slave field is the master from which it comes from.

If only one master field is chosen, the resulting variational principle (obtained after going through Steps 2, 3 and 4) is called *single-field*, and *multifield* otherwise.

A known or data field (for example: body forces or surface tractions in elastostatics) cannot be a master field because it is not subject to variation, and is not a secondary field because it does not derive from others. Hence we see that *fields can only be of three types*: master, slave, or data.

§5.8.2. Step 2: Choose Weak Connections

Given a master field, consider the equations that link it to other known and unknown fields. These are called the *connections* of that field. Classify these connections into two types:

Strong connection. The connecting relation is enforced *point by point* in its original form. For example if the connection is a PDE or an algebraic equation we use it as such. Also called *a priori* enforcement. When applied to a boundary condition, a strong connection is also referred to as an *essential constraint* or *essential B.C.*

[8] There is in fact an infinite number, parametrizable by a finite number of parameters, as shown in: C. A. Felippa, A survey of parametrized variational principles and applications to computational mechanics, *Comp. Meths. Appl. Mech. Engrg.*, **113**, 109–139, 1994. Most books give the impression, however, that there is only a finite number.

Weak connection. The connection relationship is enforced only in an *average* or *mean* sense through the use of a weight or test function, or of a distributed Lagrange multiplier. Also called *a-posteriori* enforcement. When applied to a boundary condition, a weak connection is also referred to as a *natural constraint* or *natural B.C.*

A general rule to keep in mind is that *a slave field must be reachable from its owner through strong connections*.

If there is more than one master field (*i.e.* we are constructing a multifield principle), the foregoing definitions *must be applied to each master field in turn*. In other words, we must consider the connections that "emanate" from each of the master fields. The end result is that the same field may appear more than once. For example in elasticity the strain field **e** may appear up to three times: (1) as a master field, (2) as a slave field derived from displacements, and (3) as a slave field derived from stresses. These complications cannot occur with single-field principles.

Remark 5.3. There is usually limited freedom as regards the choice of strong vs. weak connections. The key test comes when one tries to form the total variation in Step 3. If this happens to be the exact variation of a functional, the choice is admissible. Else is back to the drawing board.

§5.8.3. Step 3: Construct a First Variation

Once all choices of Steps 1 and 2 have been made, the remaining manipulations are technical in nature, and essentially consist of applying the tools and techniques of vector, tensor and variational calculus: Lagrange multipliers, integration by parts, homogenization of variations, surface integral splitting, and so on. Since the number of operational combinations is huge, the techniques are best illustrated through specific examples.

The end result of these gyrations should be a variational statement

$$\boxed{\delta\Pi = 0,} \tag{5.34}$$

where the symbol δ here embodies variations with respect to *all master fields*.

§5.8.4. Step 4: Functionalize

With luck, the variational statement (5.34) will be recognized as the *exact variation* of a functional Π, whence the variational statement becomes a true variational principle. If so, Π represents the Variational Form we were looking for, and the search is successful.

We now illustrate the foregoing steps with the detailed derivation of the most important single-field VF in elastostatics: the principle of Total Potential Energy or TPE.

§5.9. Derivation of Total Potential Energy Principle

§5.9.1. A Long Journey Starts with the First Step

The departure point for deriving the classical TPE principle is the WF diagrammed in Figure 5.5. Such modifications are briefly explained in the figure label and in the text below. The displacement field u_i is the only master. The strain and stress fields are slaves. The slave-provenance notation

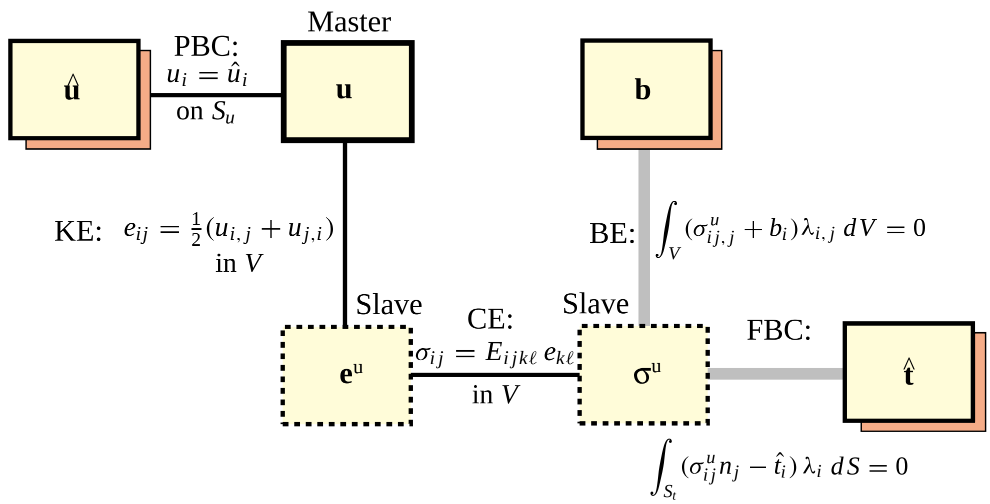

FIGURE 5.5. The WF used as departure point for deriving the TPE functional of linear elastostatics.

introduced in Chapter 3 is used: the owner of a slave field is marked by a superscript. For example, $\mathbf{e}^u = \mathbf{D}\,\mathbf{u}$ means "$\mathbf{e}^u$ is owned by $\mathbf{u}$" through the strong KE link.

The strong connections are the kinematic equations KE (in elasticity the strain-displacement equations), the constitutive equations CE, and the primary boundary conditions PBC (in elasticity the displacement boundary conditions). These are depicted in Figure 5.5 as solid box-connecting lines:

$$\boxed{\text{Strong}: \quad e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \ \text{ in } V, \quad \sigma_{ij} = E_{ijk\ell}\, e_{k\ell} \ \text{ in } V, \quad u_i = \hat{u}_i \ \text{ on } S_u.} \tag{5.35}$$

The *weak* connections are the balance equations BE (in elasticity the stress equilibrium equations), and the flux boundary conditions FBC (in elasticity the traction boundary conditions), These are shown in Figure 5.5 as shaded lines:

$$\boxed{\text{Weak}: \quad \sigma_{ij,j} + b_i = 0 \ \text{ in } V, \quad \sigma_{ij} n_j = \hat{t}_i \ \text{ on } S_t.} \tag{5.36}$$

§5.9.2. Lagrangian Glue

Now we get down to the business of variational calculus. A slight notational variation of the *residual weighting* technique of previous Chapters is used. The notation has certain interpretation advantages that will become apparent later when dealing with hybrid principles.

To treat BE as a weak connection, take the first of (5.36), replace σ_{ij} by the slave σ^u_{ij}, multiply by a piecewise differentiable 3-vector field λ_i and integrate over V:

$$\int_V (\sigma^u_{ij,j} + b_i)\, \lambda_i\, dV = 0. \tag{5.37}$$

Apply the divergence theorem to the first term in (5.37):

$$\int_V \sigma^u_{ij,j}\, \lambda_i\, dV = -\int_V \sigma^u_{ij} \lambda_{i,j}\, dV + \int_S \sigma^u_{ij}\, n_j \lambda_i\, dS. \tag{5.38}$$

For a symmetric stress tensor $\sigma^u_{ij} = \sigma^u_{ji}$ this formula may be transformed[9] to

$$\int_V \sigma^u_{ij,j} \lambda_i \, dV = -\int_V \sigma^u_{ij} \, \tfrac{1}{2}(\lambda_{i,j} + \lambda_{j,i}) \, dV + \int_S \sigma^u_{ij} \, n_j \lambda_i \, dS. \tag{5.39}$$

Assignation of meaning of internal energy to the second term in (5.39) suggests identifying λ_i with the variation of the displacement field u_i (a "lucky guess" that can be proved rigorously *a posteriori*):

$$\int_V \sigma^u_{ij,j} \, \delta u_i \, dV = -\int_V \sigma^u_{ij} \;\, \delta e^u_{ij} \, dV + \int_S \sigma^u_{ij} \, n_j \, \delta u_i \, dS, \tag{5.40}$$

in which the strain-variation symbol means

$$\delta e^u_{ij} = \tfrac{1}{2}(\delta u_{i,j} + \delta u_{j,i}) \;\text{ in } V, \tag{5.41}$$

because of the strong connection $e^u_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$, which if varied with respect to u_i yields (5.41).

Remark 5.4. Although the essence of the treatment of weak conditions is ultimately the same, there is far from universal agreement on terminology in the literature. The foregoing scheme is known as the *Lagrange multiplier* treatment. It closely follows Fraeijs de Veubeke (a major contributor to variational mechanics). The technique was originally introduced by Friedrichs (a disciple of Courant and Hilbert) in a mathematical context.

Other authors, primarily in fluid mechanics, favor *weight functions* (as in previous Chapters) or *test functions*. If the WF is directly discretized, as often done in fluid mechanics, the former technique leads to weighted-residual subdomain methods (for example the Fluid Volume Method) whereas test functions lead to Galerkin and Petrov-Galerkin methods. Some authors, such as Lanczos,[10] multiply directly equilibrium residuals by displacement variations, which are called then *virtual displacements*. Some music but different lyrics.

§5.9.3. Constructing the First-Variation Pieces

Substituting (5.40) into (5.37), with $\lambda_i \to \delta u_i$, we obtain

$$\int_V \sigma^u_{ij} \, \delta e^u_{ij} \, dV - \int_V b_i \, \delta u_i \, dV - \int_S \sigma^u_{ij} \, n_j \, \delta u_i \, dS = 0. \tag{5.42}$$

The surface integral may be split as follows:

$$\int_S \sigma^u_{ij} \, n_j \, \delta u_i \, dS = \int_{S_t} \sigma^u_{ij} \, n_j \, \delta u_i \, dS + \int_{S_u} \sigma^u_{ij} \, n_j \, \cancelto{0}{\delta \hat{u}_i} \, dS = \int_{S_t} \sigma^u_{ij} \, n_j \, \delta u_i \, dS. \tag{5.43}$$

where the substitution $\delta u_i = \delta \hat{u}_i$ on S_u results from the strong connection $u_i = \hat{u}_i$ on S_u. But $\delta \hat{u}_i = 0$ because prescribed (data) fields are not subject to variation, and the S_u integral drops out. Treating the FBC weak connection with δu_i as 3-vector weight function we obtain

$$\int_{S_t} (\sigma^u_{ij} n_j - \hat{t}_i) \, \delta u_i \, dS = 0, \qquad \text{whence} \qquad \int_{S_t} \sigma^u_{ij} n_j \, \delta u_i \, dS = \int_{S_t} \hat{t}_i \, \delta u_i \, dS. \tag{5.44}$$

---

[9] This transformation is stated in §5.5 of Sewell's book: M. J. Sewell, *Maximum and Minimum Principles*, Cambridge, 1987. It may also be verified directly using indicial calculus, as in Exercise 5.4.

[10] C. Lanczos, *The Variational Principles of Mechanics*, Dover, 4th edition, 1970.

§5.9.4. A Happy Ending

Substituting (5.43) and the second of (5.44) into (5.42), we obtain the final form of the variation in the master field u_i, which we write (hopefully) as the variation of a functional $\Pi_{\rm TPE}$:

$$\delta\Pi_{\rm TPE} = \int_V \sigma^u_{ij}\ \delta e^u_{ij}\, dV - \int_V b_i\ \delta u_i\, dV - \int_{S_t} \hat{t}_i\ \delta u_i\, dS = 0. \tag{5.45}$$

And indeed (5.45) can be recognized[11] as the exact variation, with respect to u_i, of

$$\boxed{\Pi_{\rm TPE}[u_i] = \tfrac{1}{2}\int_V \sigma^u_{ij}\ e^u_{ij}\, dV - \int_V b_i\ u_i\, dV - \int_{S_t} \hat{t}_i\ u_i\, dS.} \tag{5.46}$$

This $\Pi_{\rm TPE}$ is called the *total potential energy* functional. It is often written as the difference of the strain energy and the external work functionals:

$$\boxed{\begin{aligned} &\Pi_{\rm TPE} = U_{\rm TPE} - W_{\rm TPE}, \qquad \text{in which} \\ &U_{\rm TPE} = \tfrac{1}{2}\int_V \sigma^u_{ij} e^u_{ij}\, dV, \qquad W_{\rm TPE} = \int_V b_i u_i\, dV + \int_{S_t} \hat{t}_i u_i\, dS. \end{aligned}} \tag{5.47}$$

Consequently (5.45) is a true variational principle and not just a variational statement.

The physical interpretation is well known: $\frac{1}{2}\sigma^u_{ij} e^u_{ij}$ is the *strain energy density* $\mathcal{U}$ in terms of displacements. Integrating this density over the volume V gives the total strain energy stored in the body. In elasticity this is the only stored energy, and consequently it is also the *internal energy* U. Likewise, $b_i u_i$ is the *external work density* of the body forces, whereas $\hat{t}_i u_i$ is the *external work density* of the applied surface tractions. Integrating these densities over V and S_t, respectively, and adding gives the total *external work* potential W.

Remark 5.5. What we have just gone through is called the Inverse Problem of Variational Calculus: given the governing equations (field equations and boundary conditions), find the functional(s) that have those governing equations as Euler-Lagrange equations and natural boundary conditions, respectively.

The Direct Problem of Variational Calculus is the reverse one: given a functional such as (5.46), show that the vanishing of its variation is equivalent to the governing equations. This problem is normally the first one tackled in Variational Calculus instruction in math support courses.[12] The Direct Problem is done by carrying out the foregoing steps in reverse order: get the variation (5.45), integrate by parts as appropriate to homogenize variations, and use the strong connections to finally arrive at

$$\delta\Pi_{\rm TPE} = \int_V (\sigma^u_{ij,j} + b_i)\,\delta u_i\, dV + \int_{S_t} (\sigma_{ij} n_j - \hat{t}_i)\,\delta u_i\, dS. \tag{5.48}$$

Using the fundamental lemma of variational calculus[13] one then shows that $\delta\Pi_{\rm TPE} = 0$ yields the weak connections (5.36) as Euler-Lagrange equations and natural boundary conditions, respectively.

---

[11] See Exercise 5.5 for the variation of the strain energy term.

[12] For example, Aerospace Math.

[13] Ch. 1, §3 of I. M. Gelfand and S. V. Fomin, *Calculus of Variations*, Prentice-Hall, 1963, reprinted by Dover, 2000.

§5.10. The Tensor Divergence Theorem and the PVW

Recall from §3.6 the canonical form of the theorem, which says that the vector divergence of a vector **a** over a volume is equal to the vector flux over the surface:

$$\int_V \nabla \cdot \mathbf{a}\, dV = \int_S \mathbf{a} \cdot \mathbf{n}\, dS. \tag{5.49}$$

Take $\mathbf{a} = \boldsymbol{\sigma} \cdot \mathbf{u}$, where $\boldsymbol{\sigma} = [\sigma_{ij}]$ is a symmetric stress tensor and $\mathbf{u} = [u_i]$ a displacement vector:

$$\int_V (\boldsymbol{\sigma} : \nabla\mathbf{u} + \nabla\boldsymbol{\sigma} \cdot \mathbf{u})\, dV = \int_S \boldsymbol{\sigma} \cdot \mathbf{u} \cdot \mathbf{n}\, dS. \tag{5.50}$$

Here $\nabla\mathbf{u} = [\partial u_i/\partial x_j]$ is an unsymmetric tensor called the deformation gradient. Its transpose is $\mathbf{u}^T\nabla^T = [\partial u_j/\partial x_i]$. Now $\boldsymbol{\sigma} : \nabla\mathbf{u} = (\boldsymbol{\sigma} : \nabla\mathbf{u})^T = \boldsymbol{\sigma} : \mathbf{u}^T\nabla^T = \boldsymbol{\sigma} : \frac{1}{2}(\nabla + \nabla^T) \cdot \mathbf{u} = \boldsymbol{\sigma} : \mathbf{D} \cdot \mathbf{u}$, where $\mathbf{D} = \frac{1}{2}(\nabla + \nabla^T)$. Hence

$$\int_V \boldsymbol{\sigma} : \mathbf{D} \cdot \mathbf{u}\, dV = -\int_V \nabla\boldsymbol{\sigma} \cdot \mathbf{u}\, dV + \int_S \boldsymbol{\sigma} \cdot \mathbf{u} \cdot \mathbf{n}\, dS. \tag{5.51}$$

In indicial notation this is

$$\int_V \sigma_{ij}\, \tfrac{1}{2}\Big(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\Big)\, dV = -\int_V \frac{\partial \sigma_{ij}}{\partial x_i} u_j\, dV + \int_S \boldsymbol{\sigma}_{ij} u_j n_i\, dS. \tag{5.52}$$

Recognizing that $e^u_{ij} = \frac{1}{2}(\partial u_j/\partial x_i + \partial u_i/\partial x_j)$ we finally arrive at

$$\int_V \sigma_{ij}\, e^u_{ij}\, dV = -\int_V \frac{\partial \sigma_{ij}}{\partial x_i} u_j\, dV + \int_S \boldsymbol{\sigma}_{ij} u_j n_i\, dS. \tag{5.53}$$

Taking the variation of this equation with respect to the displacements while keeping σ_{ij} fixed yields the Principle of Virtual Work (PVW):

$$\int_V \sigma_{ij}\, \delta e^u_{ij}\, dV = -\int_V \frac{\partial \sigma_{ij}}{\partial x_i}\, \delta u_j\, dV + \int_S \boldsymbol{\sigma}_{ij}\, \delta u_j\, n_i\, dS. \tag{5.54}$$

So far σ_{ij} and e^u_{ij} are disconnected in (5.54) because no constitutive assumption has been stated in this derivation. Consequently the PVW is valid for arbitrary materials (for example, in plasticity), which underscores its generality. Setting $\sigma_{ij} = \sigma^u_{ij}$ provides the form used in §5.9.2.

Homework Exercises for Chapter 5

Three-Dimensional Linear Elastostatics

EXERCISE 5.1 [A:10] Specialize the elasticity problem to a bar directed along x_1. Write down the field equations in indicial, tensor and matrix form.

EXERCISE 5.2 [A:10] Justify the matrix form (5.30).

EXERCISE 5.3 [A:20] Suppose that the displacement $\hat{\mathbf{u}}_P$ at an *internal* point $P(\mathbf{x}_P)$ is known. How can that condition be accomodated as a boundary condition on S_u? Hint: draw a little sphere of radius ϵ about P, then . . . oops I almost told the story.

EXERCISE 5.4 [A:20] Justify passing from (5.38) to (5.39) by proving that if σ_{ij} is symmetric, that is, $\sigma_{ij} = \sigma_{ji}$, then $\sigma_{ij}\lambda_{i,j} = \sigma_{ij}\,\frac{1}{2}(\lambda_{i,j} + \lambda_{j,i})$. Hint: one (elegant) way is to split $\lambda_{i,j} + \lambda_{j,i}$ into symmetric and antisymmetric parts; other approaches are possible.

EXERCISE 5.5 [A:15] Prove that $\delta(\frac{1}{2}\sigma^u_{ij}e^u_{ij}) = \sigma^u_{ij}\,\delta e^u_{ij}$, where the variation δ is taken with respect to displacements u_i.