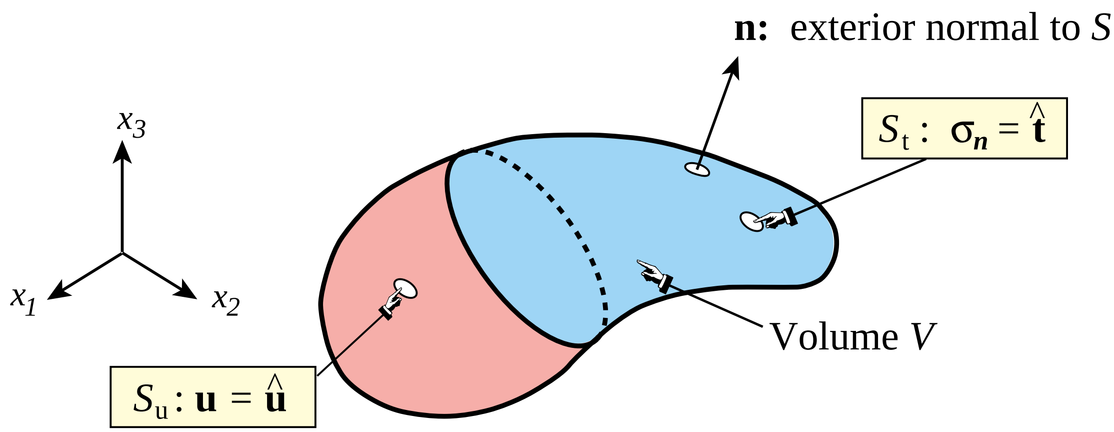
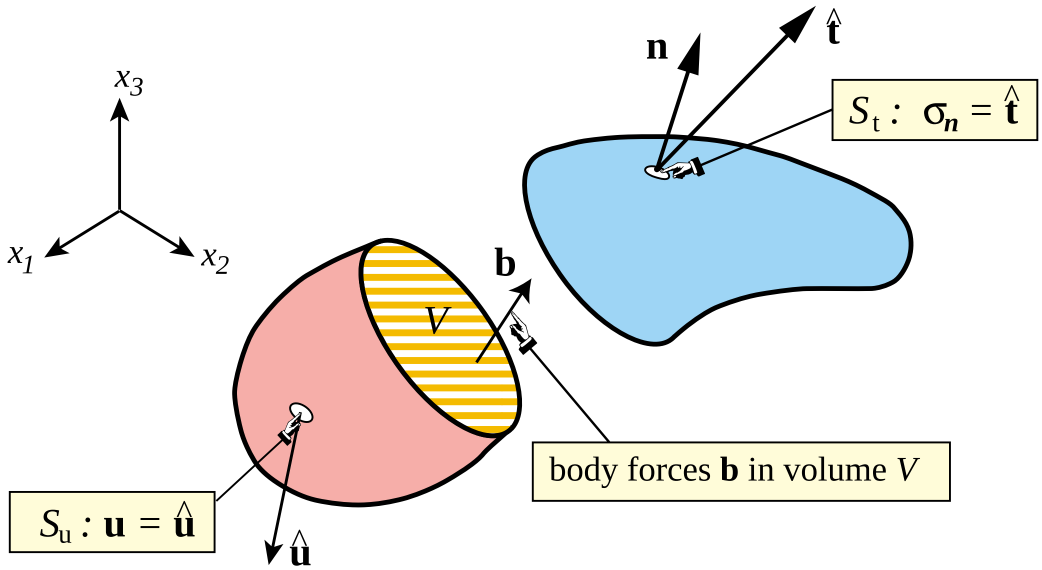
5

Three-Dimensional Linear Elastostatics

The 3D Elasticity Problem



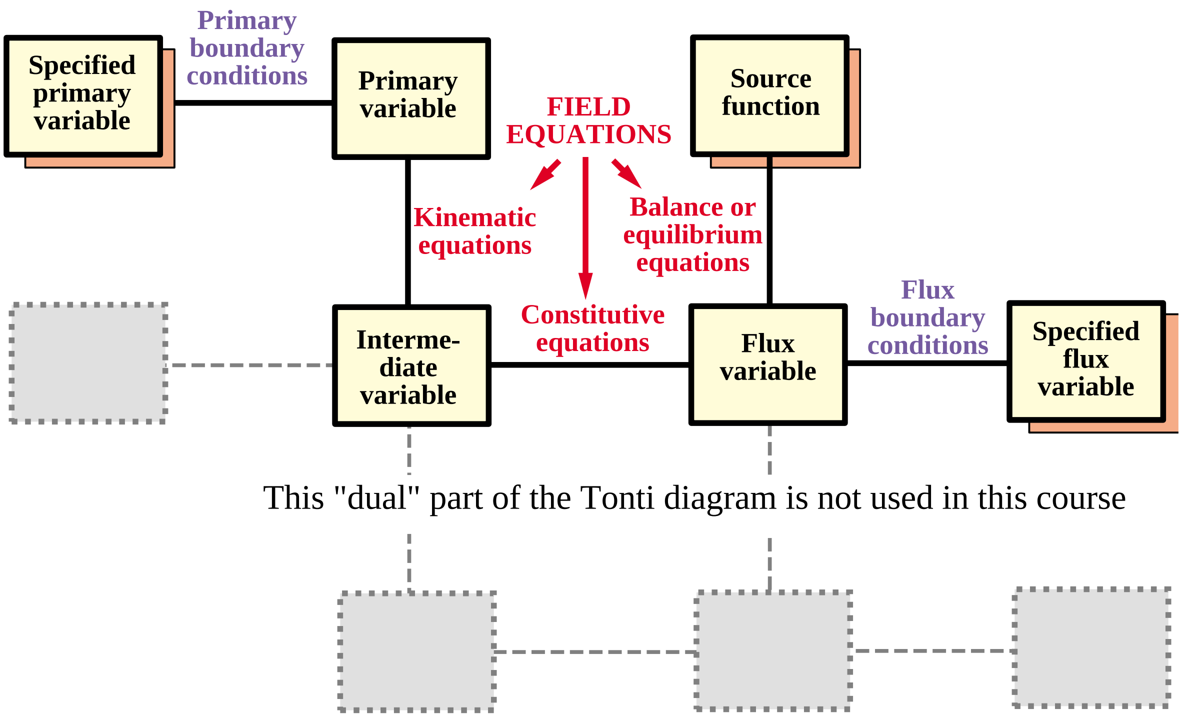
Data Fields in Detail



Governing Equations of Elastostatics

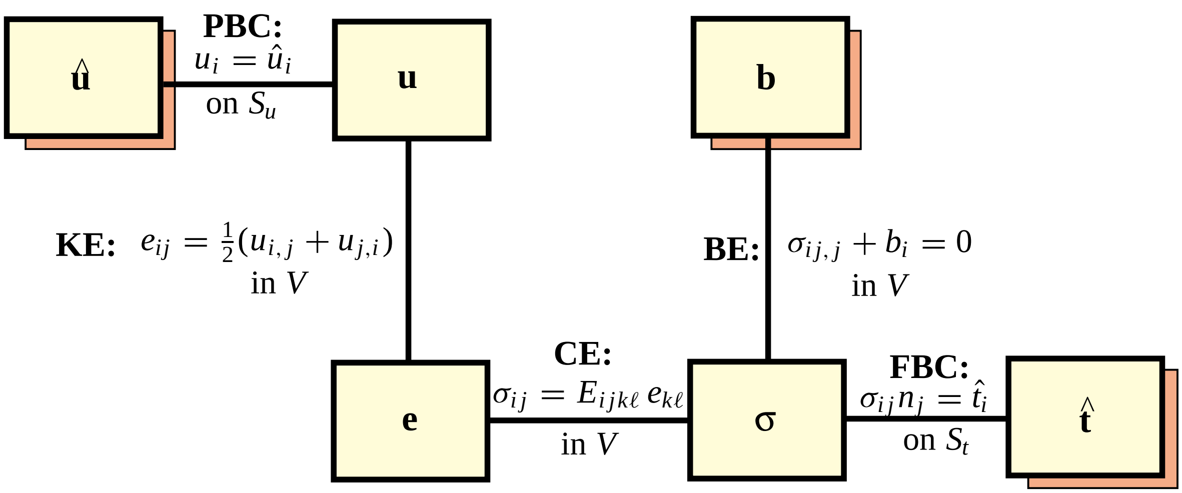
Acr	*Valid*	*Compact tensor form*	*Matrix form*	*Component (indicial) form*
KE	in V	$\underline{\mathbf{e}} = \frac{1}{2}(\nabla + \nabla^T) \cdot \mathbf{u} = \mathbf{D} \cdot \mathbf{u}$	$\mathbf{e} = \mathbf{D}\,\mathbf{u}$	$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$
CE	in V	$\underline{\boldsymbol{\sigma}} = \underline{\mathbf{E}} \cdot \underline{\mathbf{e}}$	$\boldsymbol{\sigma} = \mathbf{E}\,\mathbf{e}$	$\sigma_{ij} = E_{ijk\ell}\, e_{k\ell}$
BE	in V	$\nabla \cdot \underline{\boldsymbol{\sigma}} + \mathbf{b} = \mathbf{0}$	$\mathbf{D}^T \boldsymbol{\sigma} + \mathbf{b} = \mathbf{0}$	$\sigma_{ij,j} + b_i = 0$
PBC	on S_u	$\mathbf{u} = \hat{\mathbf{u}}$	$\mathbf{u} = \hat{\mathbf{u}}$	$u_i = \hat{u}_i$
FBC	on S_t	$\underline{\boldsymbol{\sigma}} \cdot \mathbf{n} = \boldsymbol{\sigma}_n = \mathbf{t} = \hat{\mathbf{t}}$	$\mathbf{P}_n\, \boldsymbol{\sigma} = \boldsymbol{\sigma}_n = \mathbf{t} = \hat{\mathbf{t}}$	$\sigma_{ij} n_j = \sigma_{ni} = t_i = \hat{t}_i$

The Tonti Diagram - Generic Form

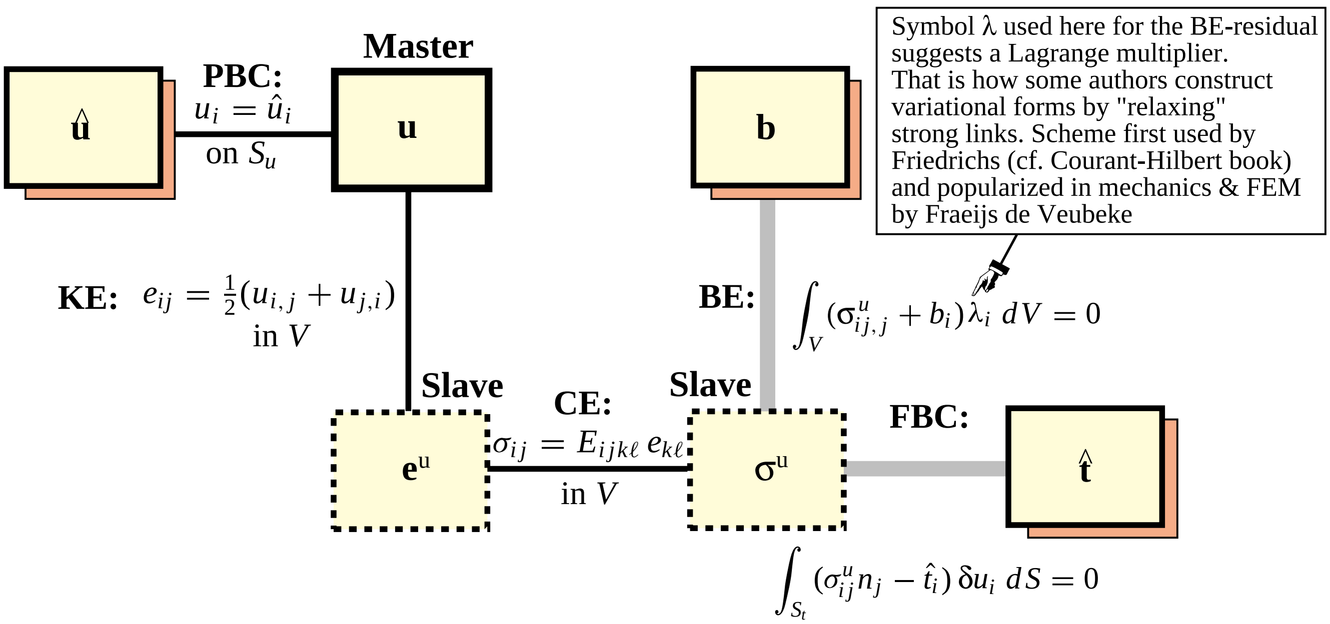


The Strong Form of the Tonti Diagram for Elastostatics

(Equations written in indicial form)



Weak Form Departure Point to Derive the Primal (TPE) Functional of Elastostatics



Derivation of the Primal (TPE) Functional of Elastostatics: Step 1

Pick master field: displacements u_i

Pick strong and weak links

Strong : $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ in V, $\sigma_{ij} = E_{ijk\ell}\, e_{k\ell}$ in V, $u_i = \hat{u}_i$ on S_u.

Weak: $\sigma_{ij,j} + b_i = 0$ in V, $\sigma_{ij} n_j = \hat{t}_i$ on S_t.

Derivation of the Primal (TPE) Functional of Elastostatics: Step 2

Write BE as integral statement with 3-vector residual weight

$$\int_V (\sigma^u_{ij,j} + b_i)\,\lambda_i\,dV = 0$$

Apply (tensor form of) Divergence Theorem:

$$\int_V \sigma^u_{ij,j}\,\lambda_i\,dV \overset{\text{DT}}{=} -\int_V \sigma^u_{ij}\lambda_{i,j}\,dV + \int_S \sigma^u_{ij}\,n_j\lambda_i\,dS$$

$$\int_V \sigma^u_{ij,j}\lambda_i\,dV = -\int_V \sigma^u_{ij}\,\tfrac{1}{2}(\lambda_{i,j} + \lambda_{j,i})\,dV + \int_S \sigma^u_{ij}\,n_j\lambda_i\,dS$$

Replace λ_i by displacement variation

$$\int_V \sigma^u_{ij,j}\,\delta u_i\,dV = -\int_V \sigma^u_{ij}\;\delta e^u_{ij}\,dV + \int_S \sigma^u_{ij}\,n_j\,\delta u_i\,dS$$

in which

$$\delta e^u_{ij} = \tfrac{1}{2}(\delta u_{i,j} + \delta u_{j,i})$$

Derivation of the Primal (TPE) Functional of Elastostatics: Step 3

Replace previous results into BE integral statement

$$\int_V \sigma_{ij}^u \, \delta e_{ij}^u \, dV - \int_V b_i \, \delta u_i \, dV - \int_S \sigma_{ij}^u \, n_j \, \delta u_i \, dS = 0$$

Split surface integral and enforce strong connection on S$_u$

$$\int_S \sigma_{ij}^u \, n_j \, \delta u_i \, dS = \int_{S_t} \sigma_{ij}^u \, n_j \, \delta u_i \, dS + \int_{S_u} \sigma_{ij}^u \, n_j \, \cancelto{0}{\delta \hat{u}_i} \, dS = \int_{S_t} \sigma_{ij}^u \, n_j \, \delta u_i \, dS$$

Treat the FBC weak connection with δu_i as weight

$$\int_{S_t} (\sigma_{ij}^u n_j - \hat{t}_i) \, \delta u_i \, dS = 0 \qquad \textbf{whence} \qquad \int_{S_t} \sigma_{ij}^u n_j \, \delta u_i \, dS = \int_{S_t} \hat{t}_i \, \delta u_i \, dS$$

Derivation of the Primal (TPE) Functional of Elastostatics: Step 4

State as first variation of an (alleged) functional

$$\delta\Pi_{\text{TPE}} = \int_V \sigma_{ij}^u \;\delta e_{ij}^u \, dV - \int_V b_i \,\delta u_i \, dV - \int_S \hat{t}_i \;\delta u_i \, dS = 0$$

This is indeed the first variation of

$$\Pi_{\text{TPE}}[u_i] = \tfrac{1}{2}\int_V \sigma_{ij}^u \, e_{ij}^u \, dV - \int_V b_i \, u_i \, dV - \int_{S_t} \hat{t}_i \, u_i \, dS$$

which is the Total Potential Energy functional of elastostatics

Split Form of the TPE Functional

$$\Pi_{\rm TPE} = U_{\rm TPE} - W_{\rm TPE}$$

in which

$$U_{\rm TPE} = \tfrac{1}{2}\int_V \sigma^u_{ij} e^u_{ij}\, dV$$

$$W_{\rm TPE} = \int_V b_i u_i\, dV + \int_{S_t} \hat{t}_i u_i\, dS$$

Physical interpretation:

U **is the *strain energy* stored in the body, which for elasticity is the *internal energy***

W **is the *work of the applied external forces*; which is the same as the negated loads potential**: $W = -V$