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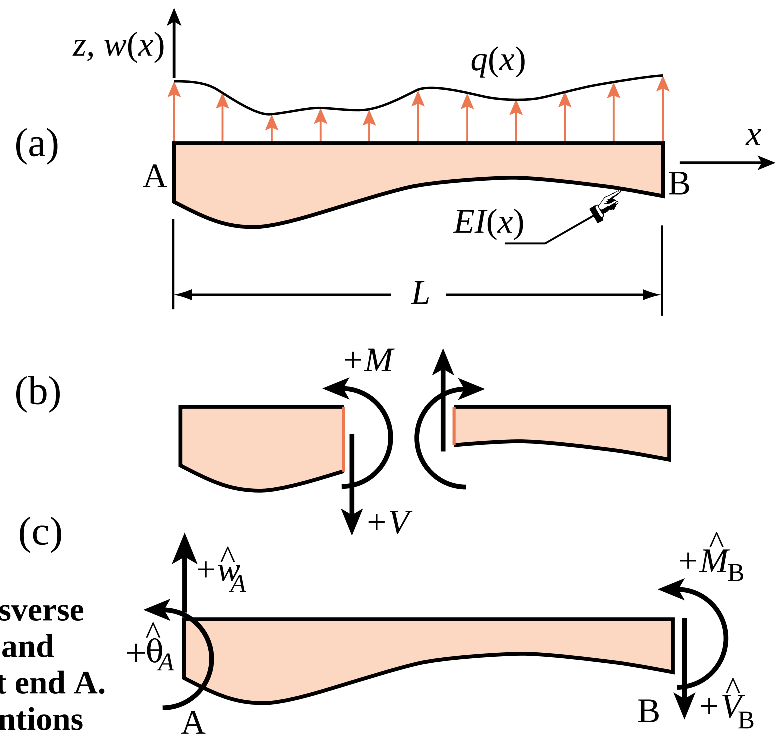
The Bernoulli-Euler Beam

The Bernoulli-Euler Beam Model



Beam and applied load

Internal forces. Note sign conventions

Prescribed transverse displacement *w* and rotation θ at left end A. Note sign conventions

Prescribed bending moment *M* and transverse shear *V* at right end B. Note sign conventions

Governing Equations

Field Equations

KE: $$\theta = \frac{dw}{dx} = w' \qquad \kappa = \frac{d^2 w}{dx^2} = w'' = \theta'$$

CE: $$M = EI\kappa$$

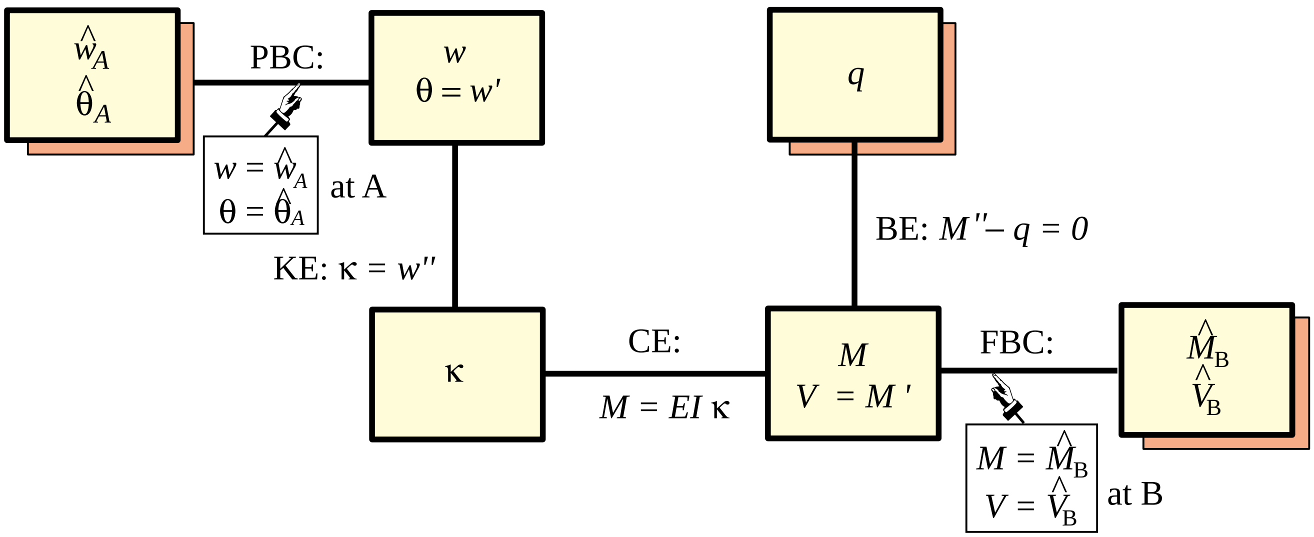
BE: $$V = \frac{dM}{dx} = M' \qquad \frac{dV}{dx} - q = V' - q = M'' - q = 0$$

Boundary Conditions (Fixed-Free)

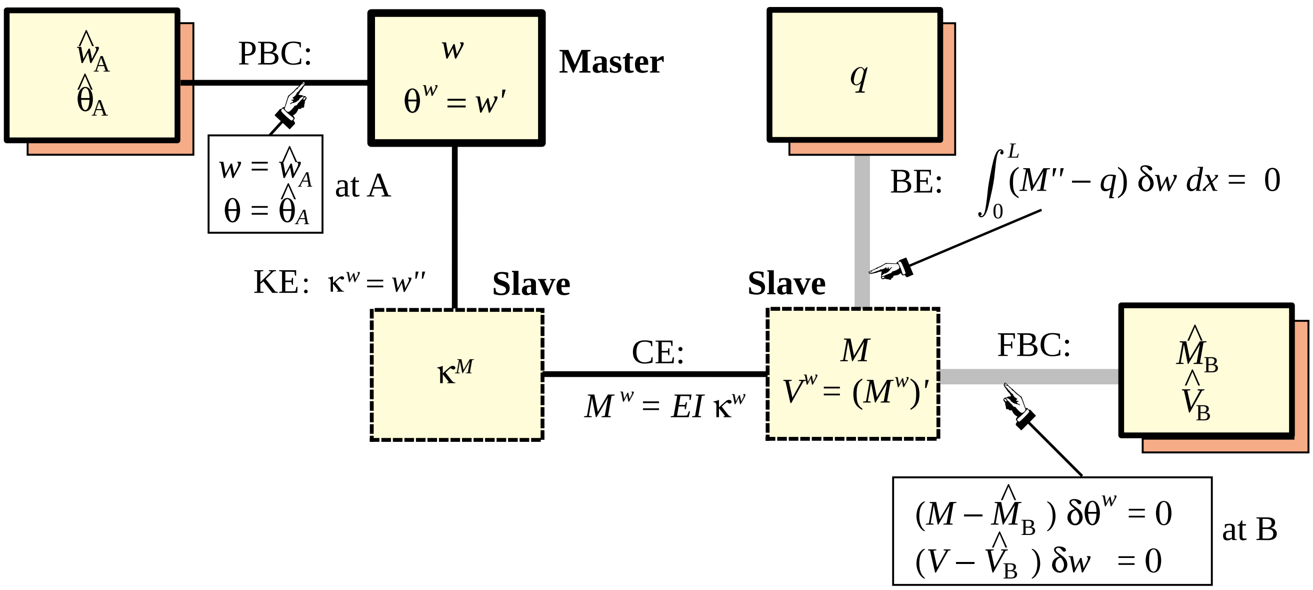
PBC: $$\text{At } A\,(x = 0): \quad w = \hat{w}_A, \quad \theta = \hat{\theta}_A$$

FBC: $$\text{At } B\,(x = L): \quad M = \hat{M}_B, \quad V = \hat{V}_B$$

The Strong Form Tonti Diagram for the B-E Beam



Weak Form Used as Departure Point for Deriving the TPE (Primal) Functional



Derivation of TPE Functional

Add contributions of weak links

$$\delta\Pi = \int_0^L \left[(M^w)'' - q \right] \delta w \, dx + (M^w - \hat{M})\, \delta\theta^w \Big|_B - (V^w - \hat{V})\, \delta w \Big|_B = 0$$

Integrate $\int_0^L (M^w)'' \delta w \, dx$ **twice by parts**

$$\begin{aligned} \int_0^L (M^w)'' \, \delta w \, dx &= -\int_0^L (M^w)' \, \delta w' \, dx + \Big[(M^w)' \, \delta w \Big]_A^B \\ &= \int_0^L M^w \, \delta w'' \, dx + \Big[(M^w)' \, \delta w \Big]_A^B - \Big[M^w \, \delta w' \Big]_A^B \\ &= \int_0^L M^w \, \delta\kappa^w \, dx + V^w \, \delta w \Big|_B - M^w \, \delta\theta^w \Big|_B \end{aligned}$$

Derivation of the Primal Functional (cont'd)

Replacing result of integration by parts

$$\delta\Pi = \int_0^L (M^w\,\delta\kappa^w - q\,\delta w)\,dx \quad - \hat{M}\,\delta\theta^w\Big|_B + \hat{V}\,\delta w\Big|_B$$
$$= \int_0^L (EI\,w''\,\delta w'' - q\,\delta w)\,dx - \hat{M}\,\delta w'\Big|_B + \hat{V}\,\delta w\Big|_B$$

This is the variation of

$$\Pi_{\rm TPE}[w] = \tfrac{1}{2}\int_0^L EI(w'')^2\,dx - \int_0^L q w\,dx - \hat{M}_B w'_B + \hat{V}_B w_B$$

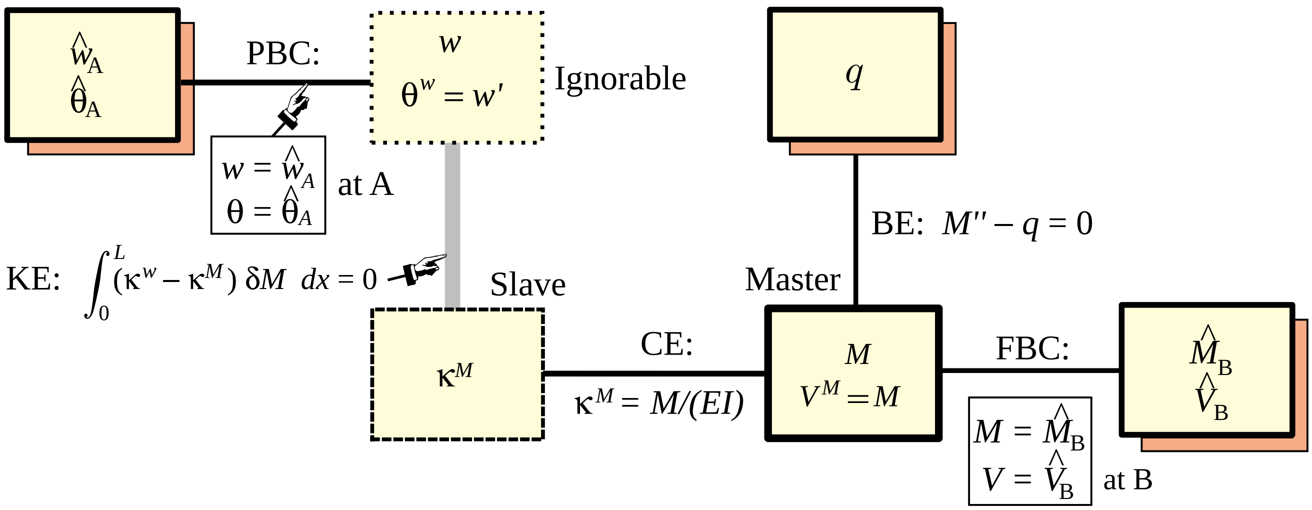
Often split as

$$\Pi[w] = U[w] - W[w]$$

$$U[w] = \tfrac{1}{2}\int_0^L EI(w'')^2\,dx, \qquad W[w] = \int_0^L q w\,dx + \hat{M}_B w'_B - \hat{V}_B w_B$$

Internal (stored, strain) energy – External work potential

Weak Form Used as Departure Point for Derivation of the Total Complementary Energy Functional (TCPE)



Derivation of the Total Complementary Energy (Dual) Functional TCPE

Not to be covered in class, see Notes

Total Complementary Energy Functional (TCPE) - Final Result

The TCPE (dual) functional is

$$\Pi_{\rm TCPE}[M] = \tfrac{1}{2}\int_0^L \frac{M^2}{EI}\,dx + M\hat{\theta}_A - V^M \hat{w}_A$$

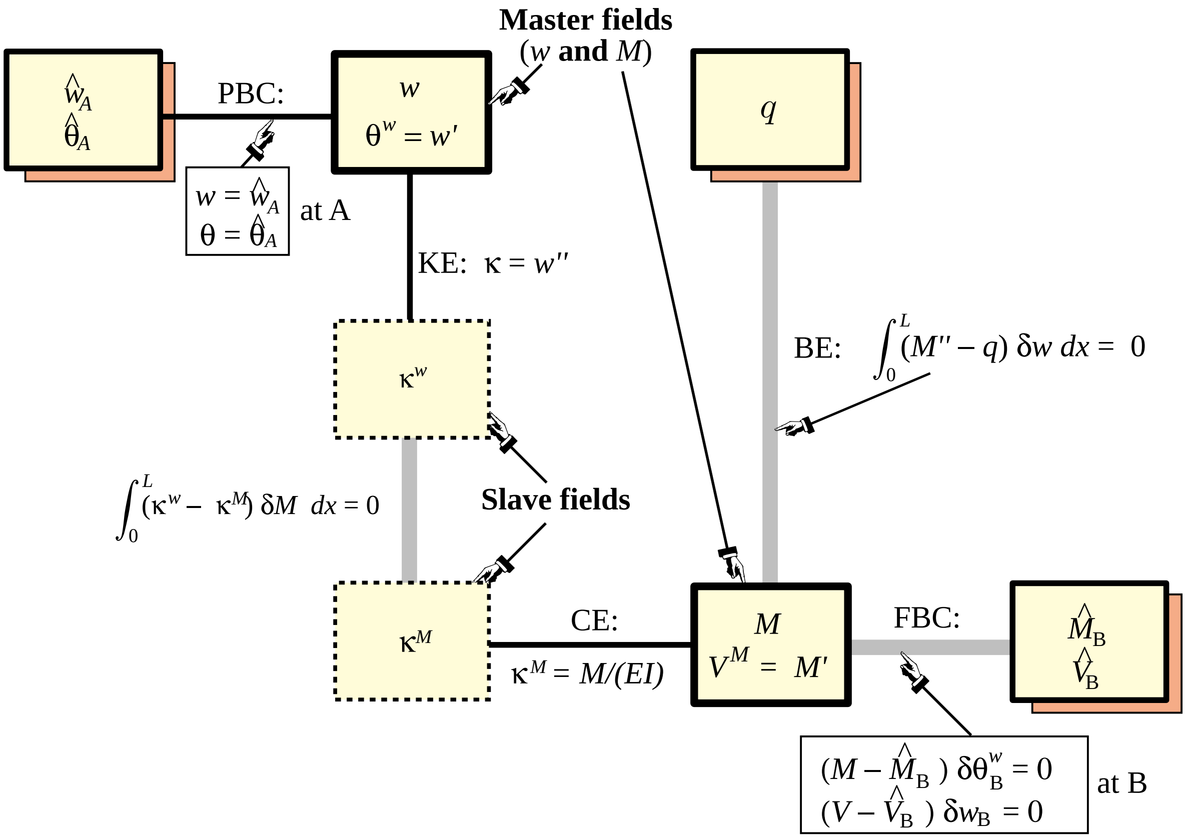
Often split as

$$\Pi[M] = U^*[M] - W^*[M]$$

$$U^* = \tfrac{1}{2}\int_0^L \frac{M^2}{EI}\,dx, \qquad W^* = -M\hat{\theta}_A + V^M \hat{w}_A$$

Complementary internal energy **Complementary external work**

Weak Form Used as Departure Point to Derive the Hellinger-Reissner (HR) Mixed Functional



Derivation of HR Mixed Functional for Bernoulli-Euler Beam

Not to be covered in class,   see Notes

Mixed HR Functional - Final Result

The HR mixed functional is

$$\Pi_{\mathrm{HR}}[w, M] = \int_0^L \left(M w'' - \tfrac{1}{2} \frac{M^2}{EI} - q w \right) dx + \hat{V}_B w_B - \hat{M}_B \theta_B^w$$

Often split as $\Pi_{\mathrm{HR}} = U - W$ **in which**

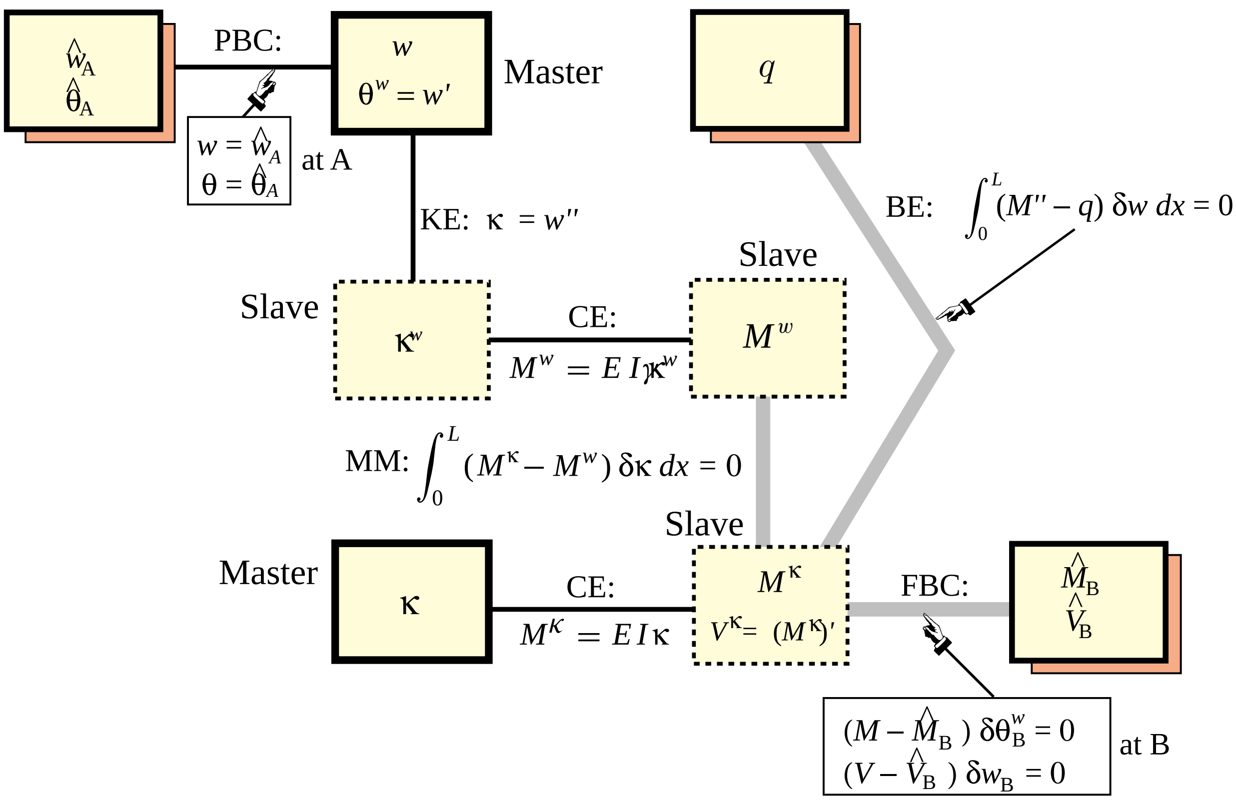
$$U[w, M] = \int_0^L \left(M w'' - \tfrac{1}{2} \frac{M^2}{EI} \right) dx$$

Internal energy

$$W[w] = \int_0^L q w \, dx - \hat{V}_B w_B + \hat{M}_B \theta_B^w$$

External work

In Exercises: A Curvature-Displacement Two-Master-Field Mixed Functional



In Exercises: A Moment-Curvature-Displacement (Veubeke-Hu-Washizu) Three-Master-Field Functional