Homework Exercises for Chapter 4

Solutions

EXERCISE 4.1 Transforming the weak-BE integral by parts as before:

$$\int (M^\kappa)''\,\delta w\,dx = \int_0^L M^\kappa\,\delta w''\,dx - M^\kappa\,\delta\theta^w\Big|_B - V^\kappa\,\delta w\Big|_B, \tag{E4.5}$$

and adding to the other weak relations we get

$$\int_0^L (M^w\delta\kappa + M^\kappa\delta w'' - M^\kappa\,\delta\kappa)\,dx - \int_0^L q\,\delta w\,dx - \hat{M}\,\delta\theta^w\Big|_B - \hat{V}\,\delta w\Big|_B, \tag{E4.6}$$

The integral $\int_0^L (M^w\delta\kappa + M^\kappa\delta w'')\,dx$ is the variation of [7] $\int_0^L M^w\,\kappa\,dx$ whereas the integral $\int_0^L -M^\kappa\,\delta\kappa\,dx$ is the variation of $-\frac{1}{2}\int_0^L M^\kappa\,\kappa\,dx$. Therefore the required functional is

$$\Pi[w,\kappa] = \int_0^L (M^w - \tfrac{1}{2}M^\kappa)\,\kappa\,dx - \int_0^L q\,w\,dx - \hat{M}\,\theta^w\Big|_B - \hat{V}\,w\Big|_B. \tag{E4.7}$$

to which the one given in the Exercise statement can be contracted. If the PBC is made weak the following additional boundary term appears:

$$(\hat{w} - w)\,V^\kappa\Big|_A + (\hat{\theta} - \theta^w)\,M^\kappa\Big|_A. \tag{E4.8}$$

EXERCISE 4.2 Integrating U by parts:

$$U = \tfrac{1}{2}\int_0^L EI(w'')^2\,dx = \tfrac{1}{2}\int_0^L Mw''\,dx = \tfrac{1}{2}\int_0^L M''w\,dx + \tfrac{1}{2}Mw'\Big|_B - \tfrac{1}{2}M'w\Big|_B \tag{E4.9}$$

At equilibrium $M'' = q$, $\hat{M} = M|_B$ and $\hat{V} = -M'|_B$, which substituted in the above gives

$$U = \tfrac{1}{2}\int_0^L qw\,dx + \tfrac{1}{2}\hat{M}w'\Big|_B + \tfrac{1}{2}\hat{V}w\Big|_B = \tfrac{1}{2}W. \tag{E4.10}$$

EXERCISE 4.3

To be done. (not assigned so far)

---

[7] Proof: $\delta\int_0^L M^w\,\kappa\,dx = \delta\int_0^L (M^w\delta\kappa + \delta M^w\,\kappa)dx = \int_0^L (M^w\delta\kappa + M^\kappa\,\delta\kappa^w)dx = \int_0^L (M^w\delta\kappa + M^\kappa\,\delta w'')dx$. The tricky part is $M^w\,\kappa = EIw''\kappa = EI\kappa w'' = M^\kappa\,\kappa^w$.

EXERCISE 4.4 Transforming the weak-BE integral by parts:

$$\int (M^\kappa)''\,\delta w\,dx = \int_0^L M^\kappa\,\delta w''\,dx - M^\kappa\,\delta\theta^w\Big|_B - V^\kappa\,\delta w\Big|_B, \tag{E4.11}$$

and adding to the other weak relations we get

$$\int_0^L (M^w - M^\kappa)\,\delta\kappa\,dx + \int_0^L M^\kappa\,\delta w''\,dx - \int_0^L q\,\delta w\,dx - \hat{M}\,\delta\theta^w\Big|_B - \hat{V}\,\delta w\Big|_B, \tag{E4.12}$$

The first integral is the variation of

$$\int_0^L (M^\kappa\kappa - \tfrac{1}{2}M^\kappa\kappa)\,dx, \tag{E4.13}$$

whereas the second is the variation of

$$\int_0^L M^\kappa\,w''\,dx = \int_0^L EI\kappa\,w''\,dx = \int_0^L EI\,w''\,\kappa\,dx = \int_0^L M^w\,\kappa\,dx. \tag{E4.14}$$

Therefore the required functional is

$$\Pi[w,\kappa] = \tfrac{1}{2}\int_0^L M^\kappa\kappa\,dx + \int_0^L (M^w - M^\kappa)\,\kappa\,dx - \int_0^L q\,w\,dx - \hat{M}\,\theta^w\Big|_B - \hat{V}\,w\Big|_B, \tag{E4.15}$$

as in the Exercise statement. This may be further contracted to

$$\Pi[w,\kappa] = \int_0^L (M^w - \tfrac{1}{2}M^\kappa)\,\kappa\,dx - \int_0^L q\,w\,dx - \hat{M}\,\theta^w\Big|_B - \hat{V}\,w\Big|_B. \tag{E4.16}$$

If the PBC is made weak the following additional boundary term appears:

$$(\hat{w} - w)\,V^\kappa\Big|_A + (\hat{\theta} - \theta^w)\,M^\kappa\Big|_A. \tag{E4.17}$$

The derivation is very similar to that of Exercise 4.1 and gives

$$\Pi[w,\kappa,M] = \int_0^L \left[\tfrac{1}{2}M^\kappa\kappa + M(\kappa^w - \kappa)\right]dx - \int_0^L q w\,dx - \hat{M}\theta^w\Big|_B - \hat{V}w\Big|_B. \tag{E4.18}$$

Note: there is an infinite number of three-field functionals and associated weak forms. The one shown above leads to the historically important Hu-Washizu functional. If you start from a different diagram you may arrive at another three-field functional.