

# 3

## Weak and Variational Forms of the Poisson Equation

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§3.1. Introduction

The chapter explains the construction of Weak and Variational Forms for the Poisson equation introduced in the previous Chapter. The Tonti diagram is used to visualize links that are weakened in the construction process. Two examples are worked out. One leads to the *primal functional*, the other to one of several possible *mixed functionals*.

§3.2. The Poisson Equation

For convenience we repeat here the split governing equations for the Strong Form of the Poisson equation studied in §2.2. The problem domain is depicted in Figure 3.1. We employ a generic notation of  $u$  for the primary variable,  $k$  for the constitutive coefficient,  $s$  for the source,  $\mathbf{g} = \nabla u$  for the gradient of  $u$ , and  $\mathbf{q} = \rho \mathbf{g}$  for the flux function.<sup>1</sup>

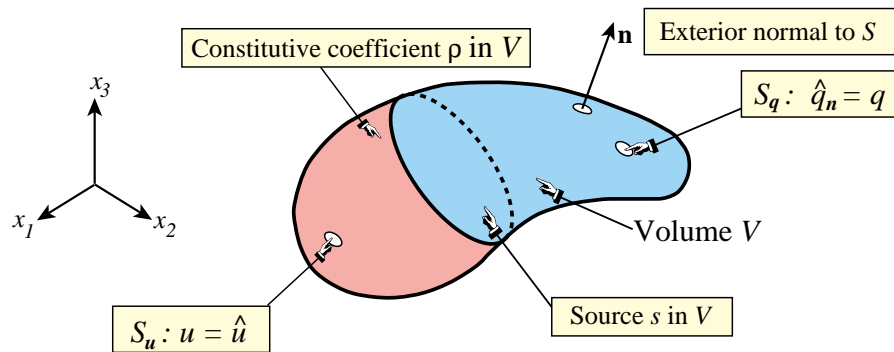


FIGURE 3.1. The problem domain for the Poisson equation, using generic notation.

Field equations:

KE:	$\nabla u = \mathbf{g}$	in $V$ ,	
CE:	$\rho \mathbf{g} = \mathbf{q}$	in $V$ ,	
BE:	$\nabla \cdot \mathbf{q} = s$	in $V$ .	(3.1)

Classical boundary conditions:

PBC:	$u = \hat{u}$	on $S_u$ ,	
FBC:	$\mathbf{q} \cdot \mathbf{n} = \mathbf{q}^T \mathbf{n} = q_n = \hat{q}_n$ ,	on $S_q$ .	(3.2)

Elimination of  $\mathbf{g}$  and  $\mathbf{q}$  yields the standard Poisson partial differential equation  $\nabla \cdot (\rho \nabla u) = s$ . If  $\rho$  is constant over  $V$ , this reduces to  $\rho \nabla^2 u = s$ , and if  $s = 0$ , to the Laplace equation  $\nabla^2 u = 0$ .

The Tonti diagram of the Strong Form is shown in Figure 3.2.

From the Strong Form one can proceed to several Weak Forms (WF) by selectively “weakening” strong connections. Two choices are worked out in the following sections: the Primal functional and a Hellinger-Reissner-like mixed functional.

<sup>1</sup> For the thermal conduction problem,  $u$ ,  $\rho$  and  $s$  become  $T$ ,  $-k$  and  $-h$ , respectively. In fluid problems  $\rho$  is the mass density.

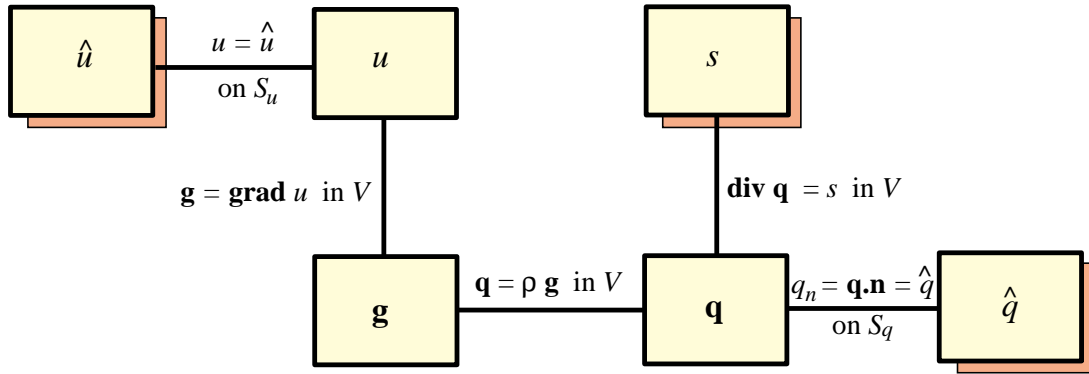


FIGURE 3.2. The Tonti diagram for the Strong Form of the generic Poisson equation.

### §3.3. The Primal Functional

The *Primal functional* is the most practically important one as a basis for FEM models of the Poisson equation. It is the analog of the Total Potential Energy of elasticity.

#### §3.3.1. Weighted Residual Form

Two of the strong links of Figure 3.2 are weakened, as shown in Figure 3.3:

1. The Balance Equation (BE)  $\nabla \cdot \mathbf{q} = s$  in  $V$ . The residual is  $r_{BE} = \nabla \cdot \mathbf{q} - s$ .
2. The Flux Boundary Condition (FBC)  $\mathbf{q} \cdot \mathbf{n} = q_n = \hat{q}$  on  $S_q$ . The residual is  $r_{FBC} = \mathbf{q} \cdot \mathbf{n} - \hat{q}$ .

The weighted residual statement is

$$R_{BE} = \int_V (\nabla \cdot \mathbf{q} - s) w_{BE} dV = 0, \quad R_{FBC} = \int_{S_q} (\mathbf{q} \cdot \mathbf{n} - \hat{q}) w_{FBC} dS = 0. \quad (3.3)$$

Here  $w_{BE}$  and  $w_{FBC}$  denote the weighting functions applied to the residuals  $r_{BE}$  and  $r_{FBC}$ , respectively. Both are scalar functions.

#### §3.3.2. Master and Slave Fields

The statement (3.3) is not very useful by itself. The field  $\mathbf{q}$  is “floating.” The weights are unknown. Looks like trying to find a black cat in a dark cellar at midnight.

We circumvent the first uncertainty by linking  $\mathbf{q}$  to the primary variable  $u$ . How? By traversing the strong links:  $u \rightarrow \mathbf{g} \rightarrow \mathbf{q}$ . A special notation, illustrated in Figure 3.4, is introduced to remind us that the auxiliary fields  $\mathbf{g}$  and  $\mathbf{q}$  are “strongly anchored” to  $u$ :

$$\mathbf{g}'' = \nabla u, \quad \mathbf{q}'' = \rho \mathbf{g}'' = k \nabla u. \quad (3.4)$$

The notation separates the unknown fields  $u$ ,  $\mathbf{g}$  and  $\mathbf{q}$  into two groups.  $u$  is the *master field*, also called *primary*, *varied* or *parent* field. It is the only one to be varied in the sense of Variational Calculus. The other two:  $\mathbf{g}''$  and  $\mathbf{q}''$  are the *slave* fields, also called *secondary*, *derived* or *sibling*

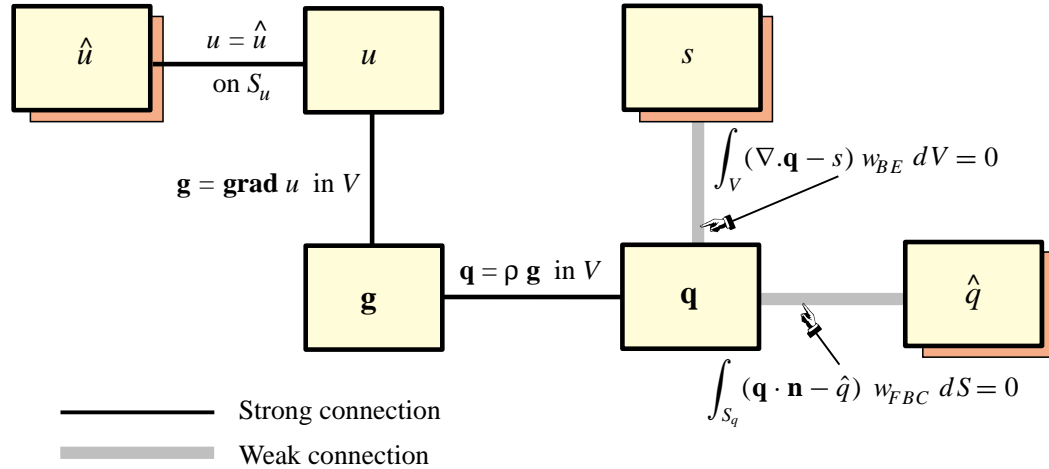


FIGURE 3.3. A Weak Form of the Poisson problem, in which the BE and FBC links have been weakened.

fields. The superscript, in this case  $(\cdot)^u$ , indicates ownership or “line of descent.” Slave fields *must be connected to their master through strong links*.

We now rewrite the integrated residuals (3.3) using the master field:

$$\begin{aligned} R_{BE} &= \int_V (\nabla \cdot \mathbf{q}^u - s) w_{BE} dV = \int_V (\nabla \cdot \rho \nabla u - s) w_{BE} dV \\ R_{FBC} &= \int_{S_q} (\mathbf{q}^u \cdot \mathbf{n} - \hat{q}) w_{FBC} dS = \int_{S_q} (\rho \nabla u) \cdot \mathbf{n} - \hat{q}) w_{FBC} dS. \end{aligned} \quad (3.5)$$

Now we have a better handle on the residuals since those depend only on  $u$ , whereas the weights are still at our disposal. These equations can be used to generate numerical methods upon selection of the weight functions, using the Method of Weighted Residuals. For example if  $w_{BE} = w_{FBC} = 1$  one obtains the so-called subdomain method, one of whose variants (in fluid applications) is the Fluid Volume Method. Other possibility is to take weights to be the same as the corresponding residuals, which leads to the two-century-old method of Least Squares.

But for the Poisson equation a Variational Form exists. Why not try for the best?

### §3.3.3. Going for the Gold

Start from (3.5) as the point of departure. Replace  $w_{BE}$  and  $w_{FBC}$  by the variations  $-\delta u$  and  $\delta u$  of the primary variable  $u$ .<sup>2</sup> Also rename residual  $R$  as  $\delta \Pi$  to emphasize that this will be hopefully the variation of a functional  $\Pi$  as yet unknown:

$$\delta \Pi_{BE} = \int_V (-\nabla \cdot \rho \nabla u + s) \delta u dV, \quad \delta \Pi_{FBC} = \int_{S_q} (\rho \nabla u \cdot \mathbf{n} - \hat{q}) \delta u dS. \quad (3.6)$$

<sup>2</sup> The minus sign in the substitution  $w_{BE} \rightarrow -\delta u$  is inconsequential; it just gives a nicer fit with the divergence theorem transformation derived in §3.5.

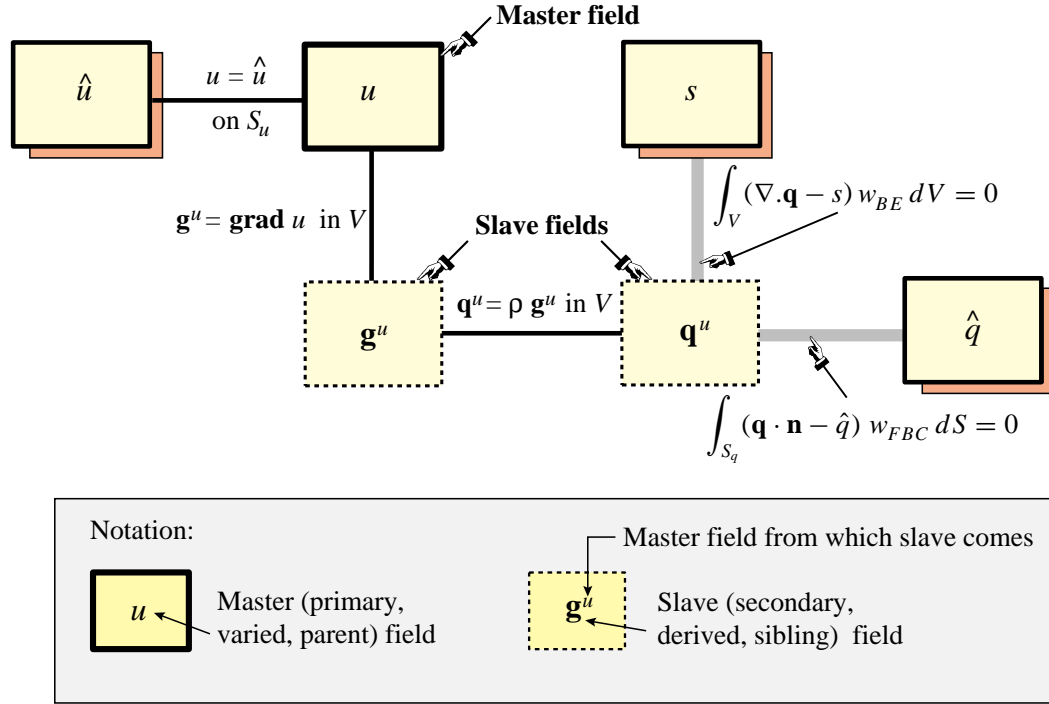


FIGURE 3.4. Figure 3.4. Refresh of the previous figure, in which the gradient  $\mathbf{g}$  and flux  $\mathbf{q}$ , relabeled  $\mathbf{g}^u$  and  $\mathbf{q}^u$ , are designated as *slave* fields. Both derive from the *master* field  $u$ .

The variation  $\delta\Pi$  will be a combination of these two, for example  $\delta\Pi_{BE} + \delta\Pi_{FBC}$ . Signs could be adjusted so that  $\delta\Pi$  is an exact variation, as worked out below.

The next operation is technical. We must reduce the order of the derivatives appearing in the volume integral of  $\delta\Pi_{BE}$  from two to one using the form of the Gauss divergence theorem worked out in detail in §3.5.<sup>3</sup> The useful transformation is:

$$-\int_V \nabla \cdot (\rho \nabla u) \delta u dV = \int_V \rho \nabla u \cdot \delta \nabla u dV - \int_S \rho \nabla u \cdot \mathbf{n} \delta u dS. \quad (3.7)$$

By definition  $u = \hat{u}$  over  $S_u$  so  $\delta u = 0$  there, and the surface integral over  $S$  reduces to one over  $S_q$ :

$$-\int_V \nabla \cdot (\rho \nabla u) \delta u dV = \int_V \rho \nabla u \cdot \delta \nabla u dV - \int_{S_q} \rho \nabla u \cdot \mathbf{n} \delta u dS. \quad (3.8)$$

Inserting this into the first of (3.6) yields

$$\delta\Pi_{BE} = \int_V (\rho \nabla u \cdot \delta \nabla u + s \delta u) dV - \int_{S_q} \rho \nabla u \cdot \mathbf{n} \delta u dS. \quad (3.9)$$

The combination  $\delta\Pi_{BE} + \delta\Pi_{FBC}$  conveniently cancels out the integral of  $k \nabla u \cdot \mathbf{n} \delta u$  over  $S_q$ . The

<sup>3</sup> Presented there for convenience. The theorem can be found in any Advanced Calculus textbook.

variation symbol  $\delta$  can be then pulled in front of the integrals:

$$\begin{aligned}\delta\Pi &= \delta\Pi_{BE} + \delta\Pi_{FBC} = \int_V (\rho\nabla u \cdot \delta\nabla u + s\delta u) dV - \int_{S_q} \hat{q}\delta u dS \\ &= \delta \int_V \frac{1}{2}\rho\nabla u \cdot \nabla u dV + \delta \int_V s u dV - \delta \int_{S_q} \hat{q} u dS.\end{aligned}\quad (3.10)$$

Consequently the required Primal functional  $\Pi$ , subscripted by “TPE” to emphasize its similarity to the Total Potential Energy functional of elasticity, is

$$\begin{aligned}\Pi_{\text{TPE}}[u] &= \frac{1}{2} \int_V k \nabla u \cdot \nabla u dV + \int_V s u dV - \int_{S_q} \hat{q} u dS \\ &= \frac{1}{2} \int_V (\mathbf{q}^u)^T \mathbf{g}^u dV + \int_V s u dV - \int_{S_q} \hat{q} u dS\end{aligned}\quad (3.11)$$

The bracket argument of  $\Pi$  is often used in Variational Calculus books to denote what is the primary variable. Using full component notation,

$$\Pi_{\text{TPE}}[u] = \frac{1}{2} \int_V \rho \left[ \left( \frac{\partial u}{\partial x_1} \right)^2 + \left( \frac{\partial u}{\partial x_2} \right)^2 + \left( \frac{\partial u}{\partial x_3} \right)^2 \right] dV + \int_V s u dV - \int_{S_q} \hat{q} u dS. \quad (3.12)$$

The variational principle is

$$\delta\Pi_{\text{TPE}} = 0. \quad (3.13)$$

where the variation is taken with respect to  $u$ . Equations (3.12) and (3.13) represent the Primal Variational Form of the Poisson equation.

**Remark 3.1.** If we work out the Euler-Lagrange (EL) equations of (3.12)-(3.13) using the rules of Variational Calculus, we obtain  $\nabla \cdot \rho\nabla u = s$  in  $V$  as EL equation, and  $\rho\nabla u = \hat{q}$  on  $S_q$  as natural boundary condition. These are precisely the equations that were weakened in §3.3.1. See Figure 3.4. The remaining equations, which pertain to the strong links, are assumed to hold *a priori*.

This happening is not accidental, but can be presented as general rule:

*The variational principle only reproduces the weak links*

as EL equations and natural boundary conditions, respectively. The rule is discussed in more detail in chapters dealing with hybrid principles.

### §3.3.4. Work Pairings

The primal functional (3.11) or (3.12) displays the following “variable pairings” in the volume integrals:  $s u$  and  $\mathbf{q} \cdot \mathbf{g}$ . In the surface integral over  $S_q$  we find  $\hat{q} u$ . These products can be interpreted physically as work or energy of the kind determined by the application being modeled.<sup>4</sup>

<sup>4</sup> For example, thermal energy in the thermal conduction problem.

The physical units of the variables and data must reflect that fact. The groupings are known as *work pairings* or *energy conjugates*.<sup>5</sup> For volume integrals, the rule is: work pairings relate the *left and right boxes drawn at the same level* in the field-equations portion of the Tonti diagram.

In variational statements, such as  $\delta\Pi = 0$ , we find pairings such as  $s \delta u$  and  $(\nabla \cdot \mathbf{q}) \delta u$  in  $V$  and  $q_n \delta u$  on  $S_q$ . These pairings provide guidelines on how to replace weight functions by variations with the correct physical dimensions. The rules are particularly important when constructing multifield and mixed functionals, as done next.

### §3.4. A Mixed Functional of HR Type

Some definitions to start. A *multifield functional* is one that has more than one master field. It is *single-field* otherwise; for example the functional (3.12) derived above. A multifield functional is called *mixed* when the multiple master fields are internal.

There are several motivations for constructing mixed functionals. One reason related to numerical methods is to try for *balanced* approximations. Generally the master field is well approximated by a FEM discretization, whereas associated slave fields, such as gradients and fluxes in the  $\Pi_{\text{TPE}}[u]$  functional, may be comparatively inaccurate. This is because differentiation, represented here by operations such as  $\mathbf{g}^u = \nabla u$ , amplifies errors. A functional in which  $u$  and  $\mathbf{q}$  are master fields would be more balanced in that regard. A more general motivation for mixed functionals is the development of hybrid functionals, a topic covered latered in this course.<sup>6</sup>

The prototype of mixed functionals is the famous Hellinger-Reissner (HR) functional of linear elasticity, in which both displacements and stresses are independently varied. We proceed to derive now the analogous functional for the Poisson equation.

#### §3.4.1. The Weak Form

The development of the single-field functional in §3.3 started from the identification of weak links, followed by picking a master field. In the case of a multifield functional, the process should be reversed because “multiple slaves” appear, which forces us to draw more boxes.

The first decision is: *select the masters*. Here we pick  $u$  and  $\mathbf{q}$ . Then weaken selected links. The appropriate choices for an HR-like principle are shown in Figure 3.5. Three links are weakened. They are BE, FBC, and the connection between the slave gradients  $\mathbf{g}^u$  and  $\mathbf{g}^q$ , denoted as GG. Links PBC, KE and CE are kept strong.

The weak link residuals are  $r_{BE} = \nabla \cdot k \nabla u - s$ ,  $r_{FBC} = \rho \nabla u - \hat{q}$ , and  $r_{GG} = \mathbf{g}^u - \mathbf{g}^q = \nabla u - \rho^{-1} \mathbf{q}$ , respectively. The weighted residuals, expressed in terms of the masters, are

$$\begin{aligned} R_{BE} &= \int_V (\nabla \cdot \mathbf{q} - s) w_{BE} dV, & R_{FBC} &= \int_{S_q} (\mathbf{q} \cdot \mathbf{n} - \hat{q}) w_{FBC} dS \\ R_{GG} &= \int_V (\mathbf{g}^u - \mathbf{g}^q) w_{GG} dV = \int_V (\nabla u - \rho^{-1} \mathbf{q}) w_{GG} dV. \end{aligned} \quad (3.14)$$

<sup>5</sup> In mathematical-oriented treatments they receive names such as *bilinear pairs*, *inner product pairs* or *bilinear concomitants*.

<sup>6</sup> Still another motivation in FEM applications is the reduction of interelement continuity requirements in problems of beams, plates and shells.

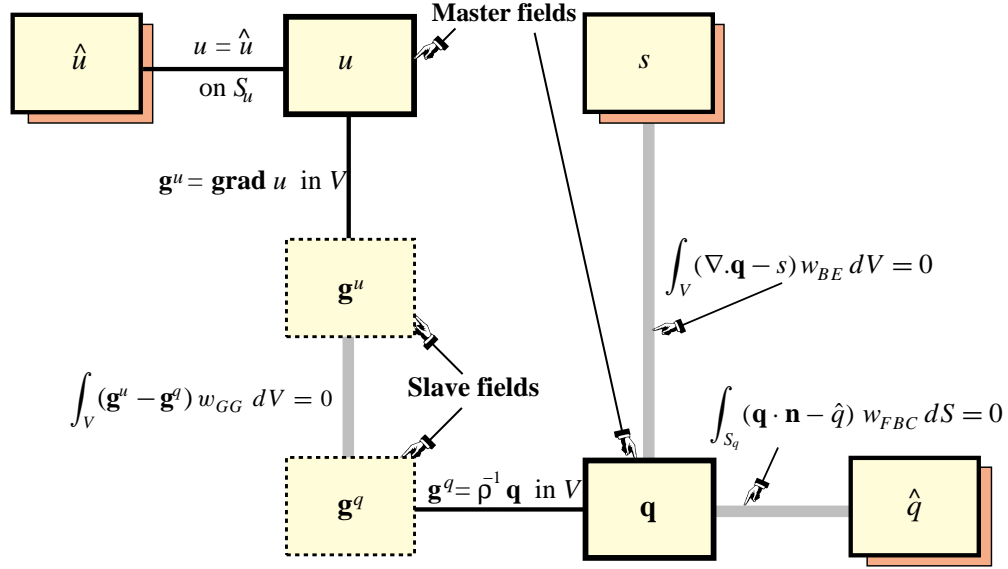


FIGURE 3.5. The Weak Form that leads to a mixed VF of HR type.

### §3.4.2. Variational Statement

Equations (3.14) can be converted into a variational statement by replacing

$$w_{BE} \rightarrow -\delta u, \quad w_{FBC} \rightarrow \delta u, \quad w_{GG} \rightarrow \delta \mathbf{q}. \quad (3.15)$$

Why these choices? The key rule is: work pairing. As explained in §3.3.4, fields such as  $\nabla \cdot \mathbf{q}$  and  $s$  should be paired with  $\delta u$  in  $V$  so that their product, once integrated to build a functional, represents work or energy. Consequently,  $w_{BE}$  must be  $\pm \delta u$ , and similarly for the other weight functions. The choice of the right sign is not that crucial, since signs can be tweaked to get cancellations on total-variations later.

Upon substitution, the  $R$ 's are renamed as variation components  $\delta \Pi_{BE}$ ,  $\delta \Pi_{FBC}$  and  $\delta \Pi_{GG}$  of an alleged functional  $\Pi[u, \mathbf{q}]$ :

$$\begin{aligned} \delta \Pi_{BE} &= \int_V (-\nabla \cdot \mathbf{q} + s) \delta u dV = \int_V (\mathbf{q} \delta \nabla u + s \delta u) dV - \int_{S_q} \mathbf{q} \cdot \mathbf{n} \delta u dS, \\ \delta \Pi_{FBC} &= \int_{S_q} (\mathbf{q} \cdot \mathbf{n} - \hat{q}) \delta u dS, \quad \delta \Pi_{GG} = \int_V (\nabla u - \rho^{-1} \mathbf{q}) \delta \mathbf{q} dV. \end{aligned} \quad (3.16)$$

The foregoing transformation of  $\delta \Pi_{BE}$  comes from applying the divergence theorem to  $\nabla \cdot \mathbf{q} \delta u$ , as worked out at the end of §3.5:  $-\int_V \nabla \cdot \mathbf{q} \delta u dV = \int_V \mathbf{q} \cdot \delta \nabla u dV - \int_{S_q} \mathbf{q} \cdot \mathbf{n} \delta u dS$ .

### §3.4.3. The Variational Form

Adding the three weak link contributions gives

$$\begin{aligned} \delta \Pi &= \delta \Pi_{BE} + \delta \Pi_{FBC} + \delta \Pi_{GG} \\ &= \int_V [(\mathbf{q} \delta \nabla u + (\nabla u - \rho^{-1} \mathbf{q}) \delta \mathbf{q} + s \delta u)] dV - \int_{S_q} \hat{q} \delta u dS. \end{aligned} \quad (3.17)$$

This is the variation of the mixed functional

$$\begin{aligned}
 \Pi_{\text{HR}}[u, \mathbf{q}] &= \int_V (\mathbf{q} \cdot \nabla u - \frac{1}{2} \rho^{-1} \mathbf{q} \cdot \mathbf{q}) dV + \int_V s u dV - \int_{S_q} \hat{q} u dS \\
 &= \int_V \left[ q_1 \frac{\partial u}{\partial x_1} + q_2 \frac{\partial u}{\partial x_2} + q_3 \frac{\partial u}{\partial x_3} - \frac{1}{2k} (q_1^2 + q_2^2 + q_3^2) \right] dV \\
 &\quad + \int_V s u dV - \int_{S_q} \hat{q} u dS.
 \end{aligned} \tag{3.18}$$

in which the first term, which characterizes internal energy, is rewritten in full component notation in the second line of (3.18).

The variational principle is

$$\delta \Pi_{\text{HR}} = 0. \tag{3.19}$$

where the variations are taken with respect to  $u$  and  $\mathbf{q}$ .

### §3.5. Overview of the Divergence Theorem

Specialized forms of Gauss' divergence theorem have been used on the way to the VF. In this section we summarize some useful forms in 3D. Assume that  $\mathbf{a}$  is a differentiable 3-vector field in  $V$ . Begin from the canonical form of the theorem, which says that the vector divergence over a volume is equal to the vector flux over the surface:

$$\int_V \nabla \cdot \mathbf{a} dV = \int_S \mathbf{a} \cdot \mathbf{n} dS \tag{3.20}$$

Here  $\nabla \cdot \mathbf{a}$  as usual denotes the vector divergence  $\mathbf{div} \mathbf{a} = \partial a_1 / \partial x_1 + \partial a_2 / \partial x_2 + \partial a_3 / \partial x_3$ .

Plug in  $\mathbf{a} = \phi \mathbf{b}$ , where  $\phi$  is a scalar function and  $\mathbf{b}$  a 3-vector field, both being differentiable:

$$\int_V (\phi \nabla \cdot \mathbf{b} + \nabla \phi \cdot \mathbf{b}) dV = \int_S \phi \mathbf{b} \cdot \mathbf{n} dS. \tag{3.21}$$

Next, if  $\mathbf{b}$  is a gradient vector of the form  $\mathbf{b} = \alpha \nabla \psi$ , where  $\alpha$  and  $\psi$  are scalar functions, the second being twice differentiable, (3.21) becomes

$$\int_V (\phi \nabla \cdot (\alpha \nabla \psi) + \nabla \phi \cdot (\alpha \nabla \psi)) dV = \int_S \phi \alpha \nabla \psi \cdot \mathbf{n} dS. \tag{3.22}$$

This form can be applied to the Poisson equation  $\nabla \cdot (k \nabla u) = s$  by substituting  $\phi \rightarrow -\delta u$ , (the minus sign gives a nicer formula below)  $\psi \rightarrow u$  and  $\alpha \rightarrow \rho$  to get

$$- \int_V (\delta u \nabla \cdot (\rho \nabla u) + \nabla \delta u \cdot (\rho \nabla u)) dV = - \int_S \delta u \rho \nabla u \cdot \mathbf{n} dS. \tag{3.23}$$

Rearranging terms, separating the surface integral in two portions, and noting that  $\delta u = 0$  on  $S_u$  (because  $u = \hat{u}$  there) gives

$$\begin{aligned}
 - \int_V \nabla \cdot (\rho \nabla u) \delta u dV &= \int_V \rho \nabla u \cdot \nabla \delta u dV - \int_S \rho \nabla u \cdot \mathbf{n} \delta u dS \\
 &= \int_V \rho \nabla u \cdot \nabla \delta u dV - \int_{S_u} \rho \nabla u \cdot \mathbf{n} \delta u dS - \int_{S_q} \rho \nabla u \cdot \mathbf{n} \delta u dS \\
 &= \int_V \rho \nabla u \cdot \delta \nabla u dV - \int_{S_q} \rho \nabla u \cdot \mathbf{n} \delta u dS.
 \end{aligned} \tag{3.24}$$

Note that  $\delta$  and  $\nabla$  commute:  $\nabla \delta u = \delta \nabla u$ , a fact used in the last equation. This relation is used for the development of the primal functional in §3.3.

Another useful formula for the mixed functional development in §3.4 is obtained by applying the divergence theorem to  $\nabla \cdot \mathbf{q} \delta u$ :

$$\int_V \nabla \cdot \mathbf{q} \delta u \, dV + \int_V \mathbf{q} \cdot \nabla \delta u \, dV = \int_S \mathbf{q} \cdot \mathbf{n} \delta u \, dS = \int_{S_q} \mathbf{q} \cdot \mathbf{n} \delta u \, dS. \quad (3.25)$$

Again  $\delta$  and  $\nabla$  can be switched in the second term.

### §3.6. Textbooks & Monographs on Variational Methods

There is a very large number of books that focus on variational methods and their applications to engineering and physics. Such material may be also found in supplementary form in books with primary focus on more general subjects, such as mathematical physics, or special ones such as finite elements.

The following list was prepared in 1993 for the Variational Methods in Mechanics (VMM) course, but is also relevant to AFEM. It was updated in 1999 and 2003 with references to Internet bookstores.<sup>7</sup> It collects only books that the writer has examined, at least superficially.

#### §3.6.1. Textbooks Used in VMM

I. M. Gelfand and S. V. Fomin, *Calculus of Variations*, Prentice-Hall, 1963. Used in 1991-93 as textbook for the standard variational calculus portion of VMM. Well written (Silverman's translation from the Russian is excellent), compact, modern, rigorous, notation occasionally fuzzy, many exercises of varying difficulty, good general-reference book to keep. Ages well. Technically still the best book on classical variational calculus, by a mile. Recently (2000) reprinted by Dover and thus inexpensive (\$9.95 new at Amazon).

B. D. Vujanovic and S. E. Jones, *Variational Methods in Nonconservative Phenomena*, Academic Press, 1989. As of this writing the only textbook that treats new, nonstandard techniques for the title subject, such as the method of vanishing parameters, time-dependent Lagrangians, and noncommutative variations. Exposition uneven, with flashes of brilliance followed by unending pedestrian examples. Out of print. Worth buying on the Internet despite cost (over \$120) if material covered can help your doctoral work.

C. Lanczos, *The Variational Principles of Mechanics*, Dover, 4th edition, 1970. (First edition 1949). Used in VMM courses 1991-93 for reading assignments. A classic. Beautifully written, respectful of history, sometimes down to a Scientific American style but never Popular Mechanics. Strength is in classical and relativistic particle and field mechanics. The continuum mechanics part (added in latter editions) is weak. Not good as literature guide. Inexpensive (about \$15); can be found in new-book bookstores since Dover periodically reprints it. Also easily purchased on the Internet as used book.

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<sup>7</sup> Many of the books listed here are out of print. The advent of the Internet has meant that it is easier to surf for used books across the world without moving from your desk. There is a fast search engine for comparing prices at URL <http://www3.addall.com>: go to the "search for used books" link. Amazon.com has also a search engine, which is badly organized, confusing and full of unnecessary hype, but links to online reviews. [Since about 2008, old scanned books posted online on Google are an additional potential source; free of charge if the useful pages happen to be displayed.]

### §3.6.2. A Potpourri of References

The following are listed by (first) author's alphabetic order.

M. Becker, *The Principles and Applications of Variational Methods*, MIT Press, Cambridge, 1964. A reprinted thesis. Focus on least-squares weighted residual methods applied to nuclear fuel problems.

C. Caratheodory, *Calculus of Variations and Partial Differential Equations of the First Order*, Chelsea Pub. Co., 1982 (reprint of the original German edition, 1935). Still in print. A historically important work by a renowned mathematician. Good source for original topical articles in the XIX and early-XX Century. Hopelessly outdated for any other use. Written in a flat, boring style.

R. Courant and D. Hilbert, *Methods of Mathematical Physics*, 2 vols, Interscience Pubs, 1962. Periodically reprinted. Although a universally touted classical reference, it is now antiquated in style (first editions were written in the 1920s). Still useful as reference material and source to developments in the early half of the XX century.

H. T. Davis, *Introduction to Nonlinear Differential and Integral Equations*, Dover, 1962. Periodically reprinted but also easy to find in the Internet as used book. Contains one chapter (14) on standard variational calculus, which gives a nice and quick introduction to the subject. If you have only a couple of hours to learn SVC in 20 pages, this is the book. Although certainly old, it feels more modern than some overhyped "classics".

B. M. Finlayson, *The Methods of Weighted Residuals and Variational Principles*, Academic Press, 1972. Poorly written, disorganized and unfocused but contains material not available elsewhere in book form, especially in Chapters 9 and 10. Focus on chemical engineering problems hinders those interested in other applications. Good guide to literature before 1970. Out of print, not easily found as used book.

C. A. J. Fletcher, *Computational Galerkin Methods*, Springer-Verlag, 1984. A good exposition of the applications of Galerkin techniques to certain classes of problems in fluid dynamics. Has little on variational methods *per se*.

A. R. Forsyth, *Calculus of Variations*, Dover, 1960. Another oldie (Euler would feel at home with it) but more advanced than Fox and Weinstock. Out of print.

C. Fox, *An Introduction to the Calculus of Variations*, Oxford, 1963, in Dover since 1987. Periodically reprinted, easily found as used book, inexpensive. Readable but uneven and not well organized. Inexplicable omissions (for example, natural boundary conditions) and antiquated terminology. Good coverage of maxmin conditions, conjugate points and transversality conditions, but old-fashioned terminology hinders value.

P. Hammond, *Energy Methods in Electromagnetics*, Clarendon Press, Oxford, 1981. Best textbook coverage of the title subject.

H. L. Langhaar, *Energy Methods in Applied Mechanics*, McGraw-Hill, 1960. The most readable "old fashioned" explanation of the classical variational principles of structural mechanics. Beautiful treatment of virtual work. Out of print. Can be bought on the Internet for \$25 to \$60, depending on condition.

C. W. Misner, K. Thorne and J. A. Wheeler, *Gravitation*, W. H. Freeman, San Francisco, 1973. Although devoted to general relativity and cosmology, a fun book to peruse through. At 8" x 10" x 2.5" and 1279 pages, it can be used to improve your upper body strength too. Still in print, lists for \$84 new. Has nice chapters on use of Hamilton-like variational principles. Here is an online review posted in Amazon.com:

"Yes, it's so massive you can measure its gravitational field. Yes, people refer to it as "the phone book." But all joking aside, as an undergraduate who is very curious about general relativity, I must say that this textbook has done more for me than any other. I've gotten occasional help from other books (Wald, Weinberg, etc.) but this is the one that I really LEARN from. There's more physical insight in this book than any I've yet seen, and the reading is truly enjoyable. One great thing is the treatment of tensors. I knew next to nothing about tensors coming into the book, but the book assumes very little initial knowledge and teaches you the needed math as you go along. This book is truly a model for anyone who wants to write a textbook. Nothing I've seen even comes close."

P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, 2 vols, McGraw-Hill, 1953. Same comments as for Courant-Hilbert. As the title says, it is oriented to physics, not engineering. Good treatment of adjoint and “mirror” systems. Out of print; can be bought on the Internet but the set is very expensive.

J. T. Oden and J. N. Reddy, *Variational Methods in Theoretical Mechanics*, Springer-Verlag, 1982. An advanced monograph that contains material not readily available elsewhere in book form, such as Tonti diagrams and canonical functionals. Heavily theoretical, with abundant abstract math flourishes. No worked out problems or exercises. Selection of applications follows authors’ interest. Good but not exhaustive reference source. Fairly inexpensive (about \$25) when it came out in paperback. Out of print, difficult to find as used book.

J. N. Reddy, *Energy and Variational Methods in Applied Mechanics*, Wiley, 1986. In print. This was selected as textbook for the first offering of the VMM course in 1987.

R. Santilli, *Foundations of Theoretical Mechanics I*, Springer-Verlag, Berlin, 1978. One of the few books that concentrate on the Inverse Problem of Lagrangian Mechanics. Unfortunately it is poorly written, disorganized, egocentric, wordy and repetitious, with many typos. For the patient specialist only.

M. J. Sewell, *Maximum and Minimum Principles*, Cambridge, 1987. Well written (author is British), fun to read, with a very good selection of examples and exercises in mechanics. [Sewell is an “applied mathematician” in the best British tradition of natural philosophy, a disciple of the great plasticity “guru” Rodney Hill]. Good and up-to-date (till 1986) bibliography. Main drawback is an inexplicable reluctance to use the standard variational calculus, which leaves many parts “dangling” and the reader wondering. Recommended despite that deficiency. Fairly inexpensive in paperback. Out of print.

M. M. Vainberg, *Variational Methods for the Study of Nonlinear Operators*, Holden-Day, 1964. An advanced monograph highly touted when it appeared since it was one of the first books covering nonlinear variational operators and the Newton-Kantorovich solution method. Requires good command of functional analysis, else forget it. Translated from Russian, but the job was poorly done. Out of print.

K. Washizu, *Variational Methods in Elasticity and Plasticity*, Pergamon Press, 1972 (2nd expanded edition 1981). With an encyclopedic coverage of the title subject, this is a good reference monograph. Drawbacks are the flat exposition style (there is no unifying, lifting theme a la Lanczos) and the cost (since it is out of print — Washizu passed away in 1985 — used copies, if found on the Internet, go for over \$150).

R. Weinstock, *Calculus of Variations, with Applications to Physics and Engineering*, McGraw-Hill, 1952. In Dover edition since 1974. Inexpensive and very old fashioned.

W. Yourgrau and S. Mandelstam, *Variational Principles in Dynamics and Quantum Theory*, Dover, 1968. A potpourri of history, philosophy and mathematical physics brewed in a small teacup (only 200 pages). In depth treatment of Hamilton’s principle, which is fundamental in quantum physics. Two chapters on quantum mechanics and one on fluid mechanics including He superconductivity. Occasionally sloppy and effusive but worth the modest admission price (\$7 to \$10 used; can be bought new at Amazon for \$15.).

### §3.6.3. Variational Methods as Supplementary Material

In addition to the foregoing, many books in mechanics give “recipe” introductions to variational methods. One of the best is the excellent textbook, unfortunately out of print,<sup>8</sup> by Fung:

Y. C. Fung, *Foundations of Solid Mechanics*, Prentice-Hall, 1965, Chapters 10ff.

Gurtin’s article in the Encyclopedia of Physics gives a terse but rigorous coverage of the classical principles of linear elasticity, including the famous Gurtin convolution-type principles for initial-value problems in dynamics:

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<sup>8</sup> Much of Fung’s material was recently “modernized” as a reasonably priced new book co-authored by Y. C. Fung and P. Tong, *Classic and Computational Solid Mechanics*, World Scientific Pub. Co., 2001. This has been used as text for Mechanics of Aerospace Structures (in Aero) and Mechanics of Solids (in ME). According to the instructors, student reaction has been strongly negative. Shows how easy is to mung a good oldie.

M. Gurtin, The Linear Theory of Elasticity, in *Encyclopedia of Physics* VIa, Vol II, ed. by C. Truesdell, Springer-Verlag, Berlin, 1972, pp. 1–295; reprinted as *Mechanics of Solids* Vol II, Springer-Verlag, 1984.

Finite element books always provide some coverage ranging from superficial to adequate through pedantic to insufferable. The 4th edition of Zienkiewicz-Taylor is substantially improved, going from superficial (with noticeable mistakes in the previous 3 editions) to adequate, thanks to Bob Taylor's contribution:

O. C. Zienkiewicz and R. E. Taylor, *The Finite Element Method*, Vol. I, 4th ed. McGraw-Hill, New York, 1989.<sup>9</sup>

The coverage in the popular Cook-Malkus-Plesha's textbook is elementary but appropriate for the intended audience:

R. D. Cook, D. S. Malkus and M. E. Plesha, *Concepts and Application of Finite Element Methods*, 3rd ed., Wiley, New York, 1989. (More recent editions have appeared but have not improved on the 3rd.)

The most readable FEM-oriented treatment from a mathematical standpoint is still the excellent monograph by Strang and Fix:

G. Strang and G. Fix, *An Analysis of the Finite Element Method*. Prentice-Hall, 1973. Out of print. Available on the Internet as used book in range \$100-\$250. [As can be expected some material is dated.]

Engineers should avoid any "FEM math" book other than Strang-Fix unless they are comfortable with advanced functional analysis.

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<sup>9</sup> A 5th edition in several volumes has appeared in the late 1990s.

**Homework Exercises for Chapter 3**  
**Weak and Variational Forms of the Poisson Equation**

For an explanation of the Exercise ratings (given in brackets at the start of each one) see page 2 of the Homework Guidelines posted on the course web site.

**EXERCISE 3.1** [A:10] Convert the functional  $\Pi_{\text{TPE}}$  derived in §3.3 into the corresponding functional of the heat conduction equation treated in §2.3. Hint: cf. footnote 1 of this Chapter, and be careful with signs.

**EXERCISE 3.2** [A:20] Consider the generic Poisson problem of §3.2. Suppose that the boundary  $S$  splits into three parts:  $S_u$ ,  $S_q$  and  $S_r$  so  $S : S_u \cup S_q \cup S_r$ . The BCs on  $S_u$  and  $S_q$  are the classical PBC and FBC:  $u = \hat{u}$  and  $q_n = \hat{q}$ , respectively. On  $S_r$  the boundary condition is

$$q_n = \chi(u - u_0) \quad \text{on } S_r, \quad (\text{E3.1})$$

where  $u_0$  and  $\chi$  are given; both may be functions of position on  $S_r$ . This is called a *Robin boundary condition* or RBC. For the heat conduction problem, (E3.1) models a *convection* boundary condition: a moving fluid contacting the body on  $S_r$  dissipates or conveys heat.<sup>10</sup>

- (a) Show that expanding  $\Pi_{\text{TPE}}$  with a surface term  $\pm \frac{1}{2} \int_{S_r} \chi(u - u_0)^2 dS$  accounts for this boundary condition. Find out which sign fits.
- (b) Account for the RBC (E3.1) in the extended SF Tonti diagram.<sup>11</sup>

**EXERCISE 3.3** [A:20] Consider the thermal conduction problem with *absolute temperature*  $T$  (in Kelvin) as primary variable. Suppose that the boundary  $S$  splits into three parts:  $S_T$ ,  $S_q$  and  $S_r$  so  $S : S_T \cup S_q \cup S_r$ . The BCs on  $S_T$  and  $S_q$  are the classical PBC and FBC:  $T = \hat{T}$  and  $q_n = \hat{q}$ , respectively. On  $S_r$  we have a *radiation* boundary condition:

$$q_n = \sigma(T^4 - T_r^4) \quad \text{on } S_r, \quad (\text{E3.2})$$

where  $T_r$  is a given reference temperature (for orbiting space structures,  $T_r \approx 3^\circ\text{K}$ ) and  $\sigma$  is a given coefficient that characterizes radiational heat emission per unit of area.

Show that (E3.2) can be exactly linearized as  $q_n = h_r(T - T_r)$ , where  $h_r$  is a function of  $\sigma$ ,  $T$  and  $T_r$ . Find  $h_r$  and explain how this trick can make (E3.2) fit into the Robin BC treated in the previous Exercise.<sup>12</sup>

**EXERCISE 3.4** [A:20] Show that the  $\Pi_{\text{HR}}$  mixed functional derived in §3.4 can be expanded with a term  $\pm \int_{S_u} q_n(u - \hat{u}) dS$  to account for weakening the PBC link:  $u = \hat{u}$  on  $S_u$  (find out which sign fits).

<sup>10</sup> More precisely, (E3.1) is a linearization of an actual convection condition. If the flow is turbulent (e.g. the Earth atmosphere) the actual condition is nonlinear.

<sup>11</sup> The answer is not unique; try something. Whatever you come up with, it will mess up the neat form of the diagram.

<sup>12</sup> The resulting functional is called a *Restricted Variational Form* because  $h_r$  must be kept fixed during variation. This form can be used as a basis for numerically solving this highly nonlinear problem, which is important in Aerospace, Environmental and Mechanical Engineering.