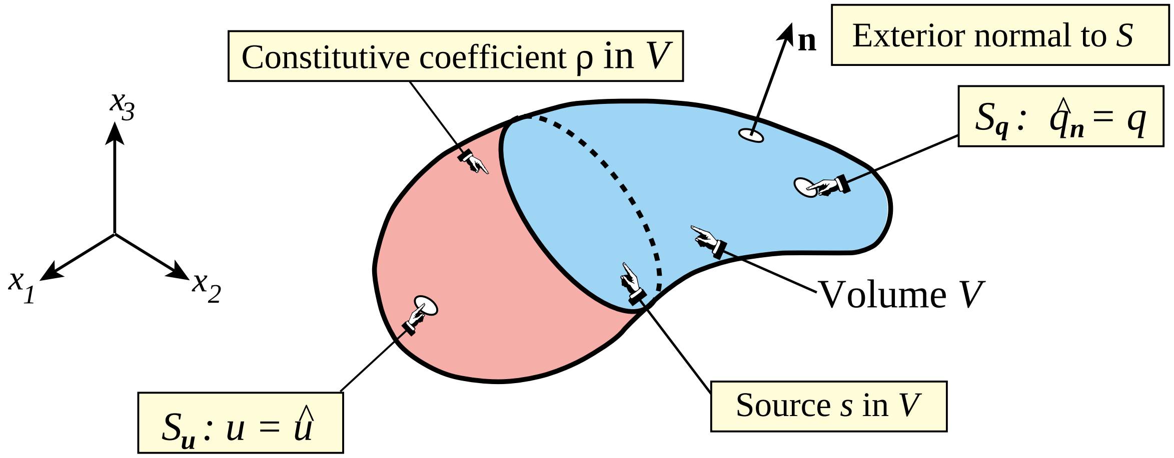
3

Weak and Variational Forms of the Poisson's Equation

The Poisson Problem in Generic Notation



Summary of Governing Equations

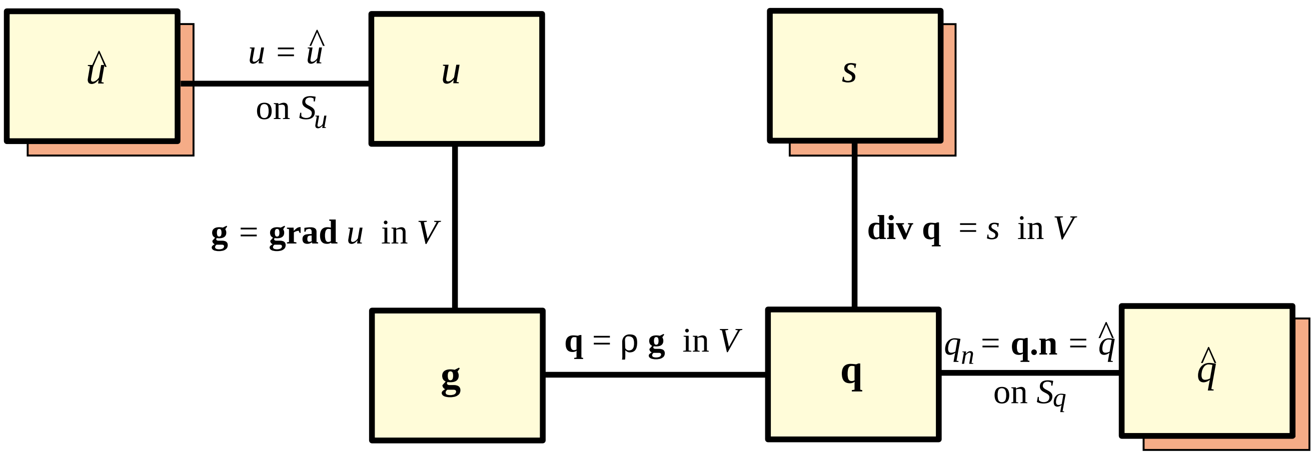
Field equations:

KE:	$\nabla u = \mathbf{g}$	in V,
CE:	$\rho\,\mathbf{g} = \mathbf{q}$	in V,
BE:	$\nabla \cdot \mathbf{q} = s$	in V.

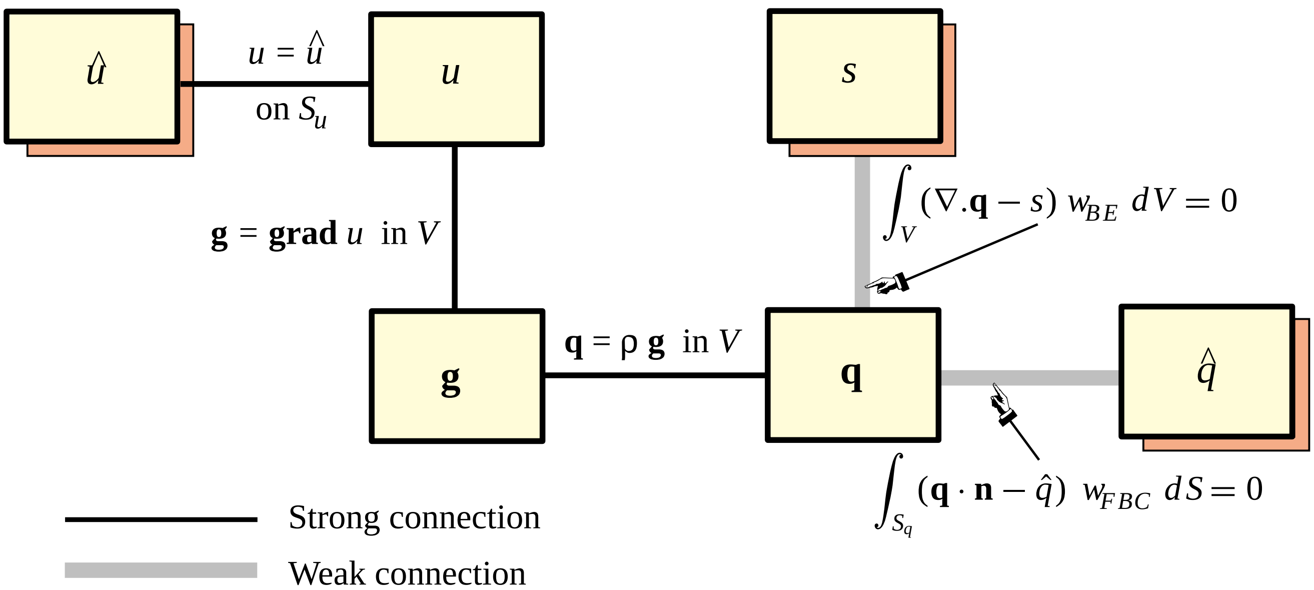
Classical BCs:

PBC:	$u = \hat{u}$	on S_u,
FBC:	$\mathbf{q}.\mathbf{n} = q_n = \hat{q}_n$,	on S_q.

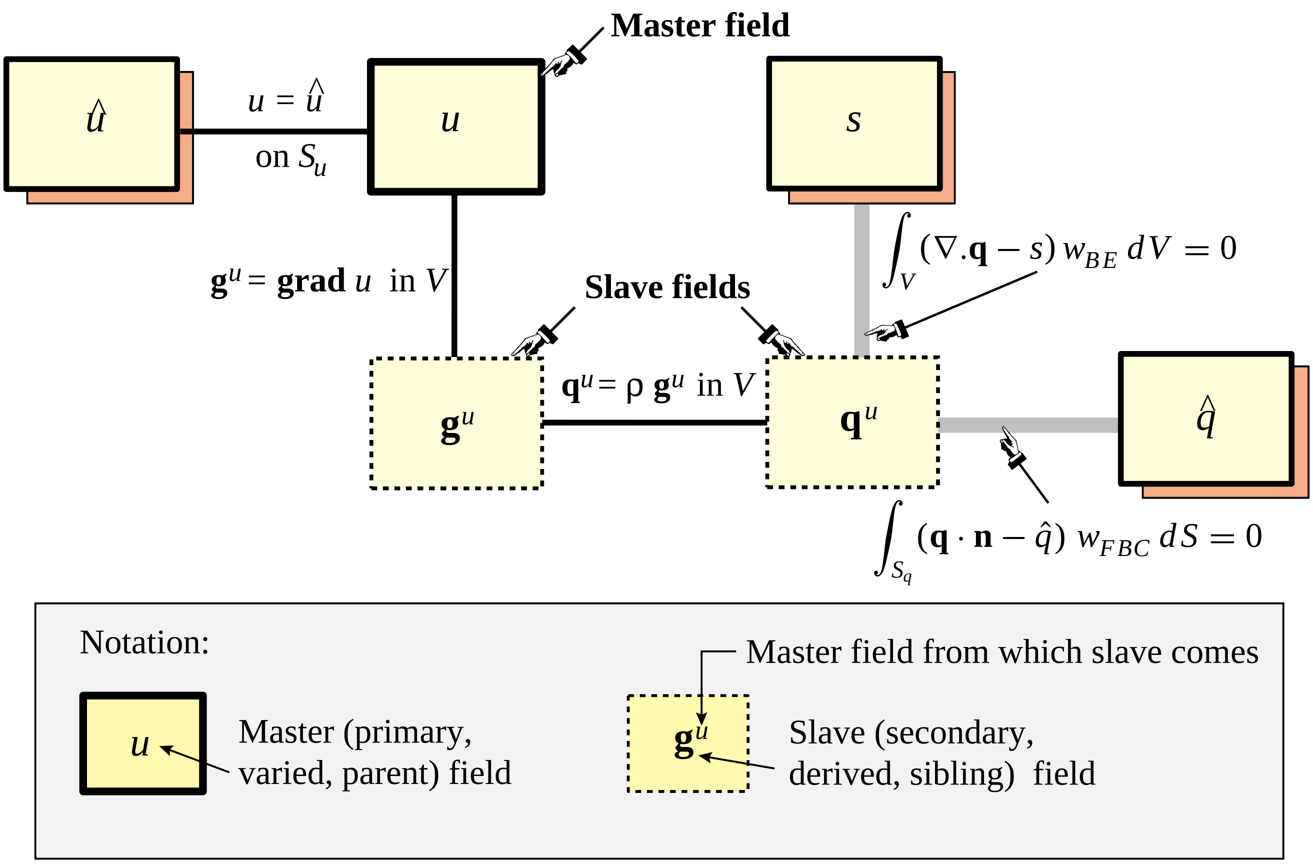
The Strong Form Tonti Diagram for the Poisson Equation



Weak Form Used as Departure Point for Deriving a Primal (TPE-Like) Functional



Master/Slave Field Notation



Derivation of the Primal Functional

Form weighted residuals of weak links:

$$R_{BE} = \int_V (\nabla \,.\, \mathbf{q}^u - s)\; w_{BE}\; dV = \int_V (\nabla \,.\, \rho\, \nabla\, u - s)\; w_{BE}\; dV$$

$$R_{FBC} = \int_{S_q} (\mathbf{q}^u - \hat{q})\; w_{FBC}\; dS = \int_{S_q} (\rho\, \nabla\, u \,.\, \mathbf{n} - \hat{q})\; w_{FBC}\; dS$$

Rewrite them as variations of an alleged functional:

$$\delta\Pi_{BE} = \int_V (-\nabla \,.\, \rho\, \nabla u + s)\; \delta u\; dV$$

$$\delta\Pi_{FBC} = \int_{S_q} (\rho\, \nabla u \,.\, \mathbf{n} - \hat{q})\; \delta u\; dS$$

if we are lucky

Derivation of the Primal Functional (cont'd)

Apply divergence theorem to $\delta\Pi_{BE}$

$$\delta\Pi_{BE} \overset{\textbf{DT}}{=} \int_V (\rho\ \nabla u\ .\ \delta\nabla u\ \ + s\ \delta u)\ \ dV - \int_{S_q} \rho\ \nabla u\ .\ \mathbf{n}\ \ \ \delta u\ \ dS$$

why S_q and not S?

Add the contributions of the two weak links and express each term as variation of something (if we are lucky):

$$\delta\Pi \overset{\text{alleged}}{=} \delta\Pi_{BE} + \delta\Pi_{FBC} = \int_V (\rho\ \nabla u\ .\ \delta\nabla u\ \ + s\ \delta u)\ \ dV - \int_{S_q} \hat{q}\ \ \ \delta u\ \ dS$$

$$= \delta \int_V \tfrac{1}{2}\rho\ \nabla u\ .\nabla u\ \ dV\ +\ \delta \int_V s\ u\ \ dV\ -\ \delta \int_{S_q} q\ u\ dS$$

Yes, we were lucky!

And We Arrive at the Primal Functional

δΠ is the variation of the TPE (Total Potential Energy Like) functional, often called the Primal Functional

alleged

$$\Pi_{\mathrm{TPE}} = \tfrac{1}{2}\int_V \rho\, \nabla u \,.\nabla u \;dV + \int_V s\, u\; dV - \int_{S_q} \hat{q}\, u\, dS$$

$$= \tfrac{1}{2}\int_V (\mathbf{q}^u).\,\mathbf{g}^u\, dV + \int_V s\, u\; dV - \int_{S_q} \hat{q}\, u\, dS$$

In full component notation,

$$\tfrac{1}{2}\int_V \rho \left[\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2 + \left(\frac{\partial u}{\partial x_3}\right)^2 \right] dV + \int_V s\, u\, dV - \int_{S_q} \hat{q} u\, dS$$

The variational principle is

$$\delta\Pi_{\mathrm{TPE}} = 0$$

Multifield and Mixed Functionals

Definitions

Multifield functional: involves more than one master field

Mixed functional: all master fields are internal (volume) fields

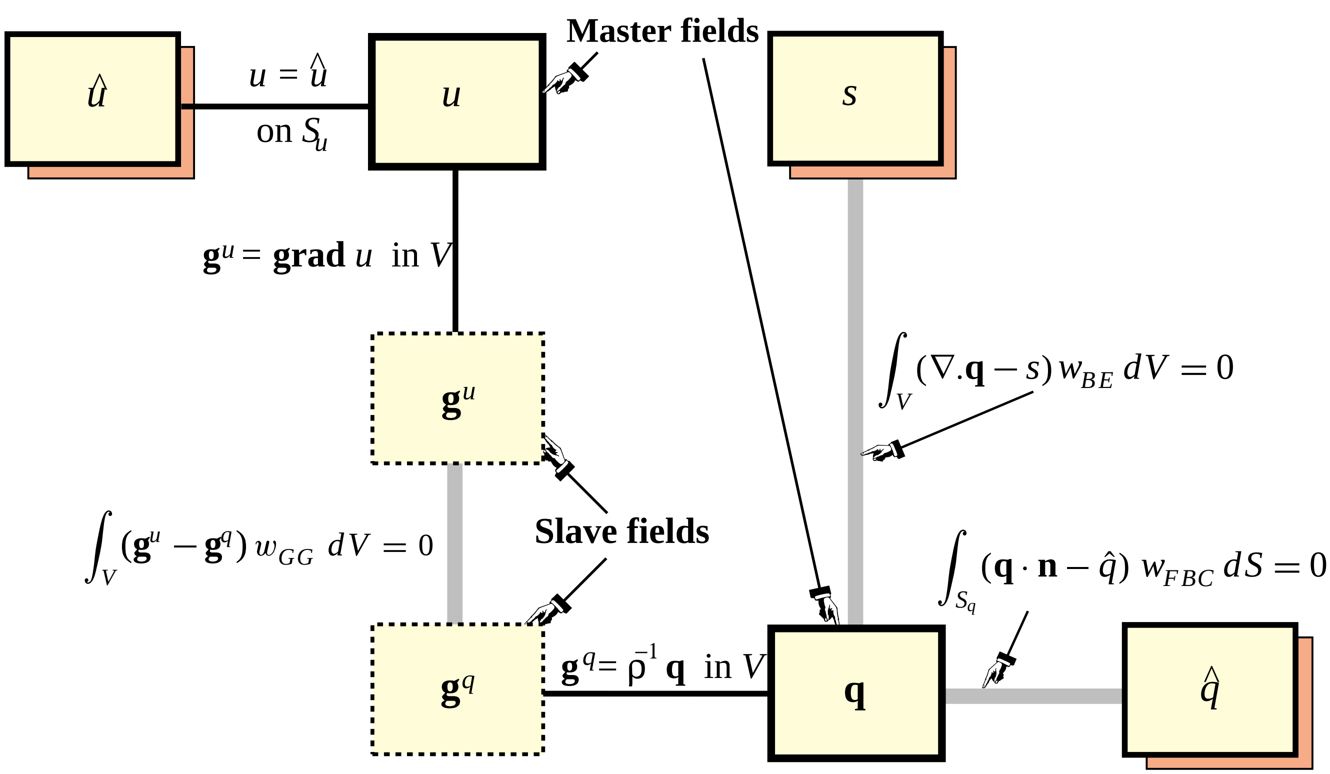
Motivation for Developing Mixed Functionals in FEM

Balanced approximation to multiple fields

Key ingredient of hybrid functionals

May relax interelement continuity requirements (important in plates and shells)

Weak Form Used as Departure Point to Derive a HR-like Mixed Functional



Derivation of Mixed Functional

Form weighted residuals of weak links

$$R_{BE} = \int_V (\nabla \cdot \mathbf{q} - s)\, w_{BE}\, dV \qquad R_{FBC} = \int_{S_q} (\mathbf{q} \cdot \mathbf{n} - \hat{q})\, w_{FBC}\; dS$$

$$R_{GG} = \int_V (\mathbf{g}^u - \mathbf{g}^q)\, w_{GG}\; dV = \int_V (\nabla u - \rho^{-1}\mathbf{q})\, w_{GG}\; dV$$

Rewrite as variations of an alleged functional

$$w_{BE} \to -\delta u, \quad w_{FBC} \to \delta u, \quad w_{FBC} \to \delta \mathbf{q}$$

$$\delta\Pi_{BE} = \int_V \big(-\nabla \cdot \mathbf{q} + s \big)\, \delta u\;\, dV \overset{\text{DT}}{=} \int_V \big(\mathbf{q}\; \delta\nabla u + s\, \delta u \big)\, dV - \int_{S_q} \mathbf{q} \cdot \mathbf{n}\, \delta u\, dS$$

$$\delta\Pi_{FBC} = \int_{S_q} \big(\mathbf{q} \cdot \mathbf{n} - \hat{q} \big)\, \delta u\, dS \qquad \delta\Pi_{GG} = \int_V \big(\nabla u - \rho^{-1}\mathbf{q} \big)\, \delta\mathbf{q}\, dV$$

Derivation of Mixed Functional (cont'd)

Adding the contributions of the three weak links:

$$\delta\Pi = \delta\Pi_{BE} + \delta\Pi_{FBC} + \delta\Pi_{GG}$$
$$= \int_V \left[(\mathbf{q}\ \delta\nabla u + (\nabla u - \rho^{-1}\mathbf{q})\,\delta\mathbf{q} + s\,\delta u \right] dV - \int_{S_q} \hat{q}\,\delta u\,dS$$

This is the variation of the functional

$$\Pi_{\mathrm{HR}}[u, \mathbf{q}] = \int_V (\mathbf{q}\cdot\nabla u - \tfrac{1}{2}\rho^{-1}\mathbf{q}\cdot\mathbf{q})\,dV + \int_V s u\,dV - \int_{S_q} \hat{q}\,u\,dS$$
$$= \int_V \left[q_1\frac{\partial u}{\partial x_1} + q_2\frac{\partial u}{\partial x_2} + q_3\frac{\partial u}{\partial x_3} - \frac{1}{2\rho}\left(q_1^2 + q_2^2 + q_3^2\right) \right] dV$$
$$+ \int_V s\,u\,dV - \int_{S_q} \hat{q}\,u\,dS$$

The variational principle is

$$\delta\Pi_{\mathrm{HR}} = 0$$

Gauss' Divergence Theorem (Review)

Departure point is classical form of theorem: divergence of a vector field **a** in *V* is equal to the vector flux over *S*, or

$$\int_V \nabla \cdot \mathbf{a}\, dV = \int_S \mathbf{a} \cdot \mathbf{n}\, dS$$

Take **a** = ϕ **b**

$$\int_V (\phi\, \nabla \cdot \mathbf{b} + \nabla\phi \cdot \mathbf{b})\, dV = \int_S \phi\mathbf{b} \cdot \mathbf{n}\, dS$$

Take b = α∇ψ

$$\int_V \big(\phi\, \nabla \cdot (\alpha \nabla \psi) + \nabla\phi \cdot (\alpha\nabla\psi)\big)\, dV = \int_S \phi\, \alpha\, \nabla\psi \cdot \mathbf{n}\, dS$$

Gauss' Divergence Theorem (Review, cont'd)

Can be applied to Poisson's equation $\nabla \cdot (\rho \nabla u) = s$ substituting $\phi \to -\delta u$, $\psi \to u$ and $\alpha \to \rho$ to get

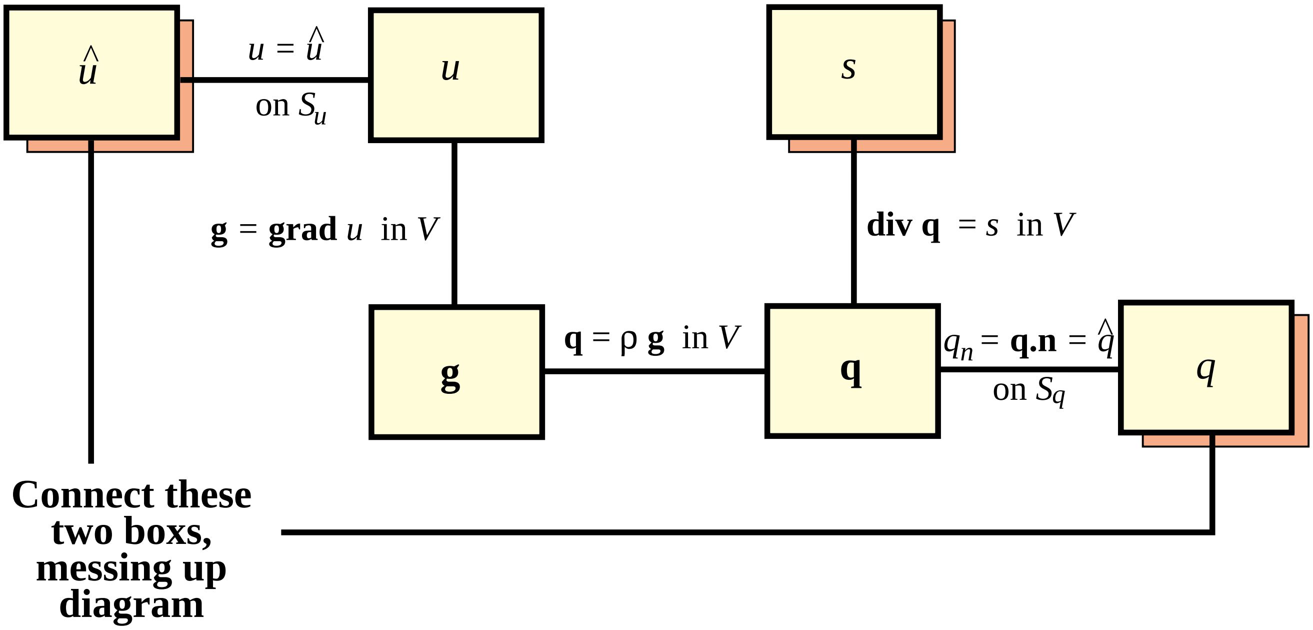
$$-\int_V \left(\delta u \ \nabla \cdot (\rho \nabla u) + \nabla \delta u \cdot (\rho \nabla u)\right) dV = -\int_S \delta u \ \rho \ \nabla u \cdot \mathbf{n} \, dS.$$

Rearrange terms, separate the surface integral in two portions and note that $\delta u = 0$ on S_u because $u = \hat{u}$ there:

$$\begin{aligned}
\int_V \nabla \cdot (\rho \nabla u) \ \delta u \, dV &= \int_V \rho \nabla u \cdot \nabla \delta u \ dV - \int_S \rho \ \nabla u \cdot \mathbf{n} \, \delta u \ dS \\
&= \int_V \rho \nabla u \cdot \nabla \delta u \ dV - \int_{S_u} \rho \ \nabla u \cdot \mathbf{n} \, \cancelto{0}{\delta u} \ dS - \int_{S_q} \rho \ \nabla u \cdot \mathbf{n} \, \delta u \ dS \\
&= \int_V \rho \nabla u \cdot \delta \, \nabla u \ dV - \int_{S_q} \rho \ \nabla u \cdot \mathbf{n} \, \delta u \ dS.
\end{aligned}$$

This relation is used in the development of the primal functional

Robin Boundary Conditions (Ch. 3 Exercises)



Examples in heat conduction:
convection (Exercise 3.2)
radiation (Exercise 3.3)