Homework Exercises for Chapter 3
Solutions

EXERCISE 3.1

$$\Pi_{\text{TPE}}[T] = \tfrac{1}{2}\int_V k\,(\nabla T)^2\,dV + \int_V h\,T\,dV + \int_{S_T} \hat{q}T\,dS$$
$$= \tfrac{1}{2}\int_V k\left[\left(\frac{\partial T}{\partial x_1}\right)^2 + \left(\frac{\partial T}{\partial x_2}\right)^2 + \left(\frac{\partial T}{\partial x_3}\right)^2\right] dV + \int_V k\,T\,dV + \int_{S_T} \hat{q}T\,dS. \quad \text{(E3.3)}$$

EXERCISE 3.2

(a) Let Π^{st}_{TPE} denote the standard functional for heat conduction with prescribed temperature and flux BC on S_u and S_q, derived previously. Take the convective flux residual $q^u_n - \chi(u - u_0)$ of the RBC, multiply by the variation of the conjugate quantity δu and integrate over the convection surface S_r. Upon integration by parts one gets

$$\delta\Pi_{\text{TPE}} = \delta\Pi^{st}_{\text{TPE}} - \int_{S_r} \chi(u - u_0)\,\delta u\,dS. \quad \text{(E3.4)}$$

The last term is the first variation of $-\int_{S_r} \frac{1}{2}\chi(u - u_0)^2\,dS$ with respect to u. Consequently

$$\Pi_{\text{TPE}} = \tfrac{1}{2}\int_V \rho\Delta u\cdot\Delta u\,dV + \int_V s\,u\,dV - \int_{S_q} \hat{q}\,u\,dS - \tfrac{1}{2}\int_{S_r} \chi(u - u_0)^2\,dS. \quad \text{(E3.5)}$$

(b) Figure E3.1 shows a possible graphical representation out of many possibilities. In the figure u, ρ and σ have been replaced by T, $-k$ and $-h$, respectively, for better fit with the thermal problem.

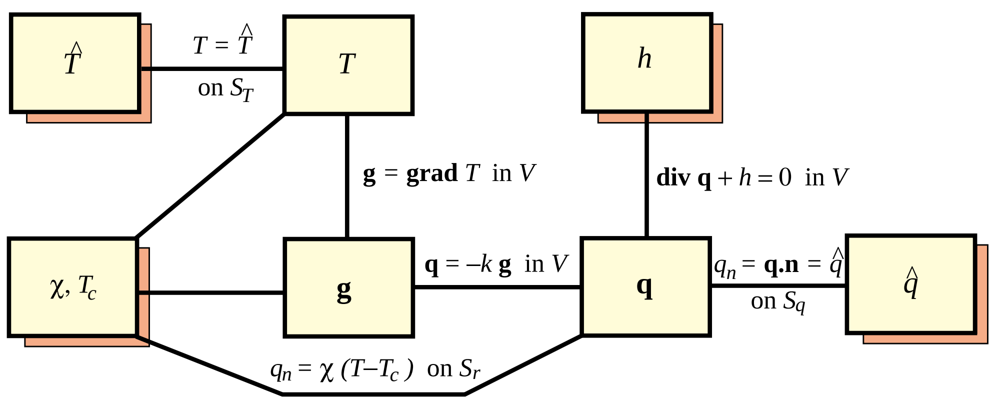

Figure E3.1. A Strong Form Tonti diagram for the thermal conduction problem with convection BC (E3.1).

EXERCISE 3.3 An exact linearization is possible:

$$\sigma(T^4 - T_r^4) = \sigma(T - T_r)(T^3 + T^2T_r + TT_r^2 + T_r^3) = \sigma(T - T_r)h_r. \quad \text{(E3.6)}$$

where $h_r = T^3 + T^2T_r + TT_r^2 + T_r^3$. If h_r is "frozen" during variation, the previous exercise shows that the appropriate surface term to add to the functional is

$$-\tfrac{1}{2}\int_{S_r} \sigma h_r(T - T_r)^2. \quad \text{(E3.7)}$$

EXERCISE 3.4

The PBC residual is $u - \hat{u}$ over S_u. Multiply by the variation of the conjugate quantity: δq_n, and integrate over S_u. Upon integration by parts and passing to the functional one obtains the given term with the $-$ sign.