2

Decomposition of Poisson Problems

Tonti Decomposition

What it is:

Graphical representation of Strong Form (SF) using two auxiliary variables

intermediate variable

flux variable

in addition to the *primary variable*

Advantages:

Visualization of WF and VF derivations
Clarification of BCs

The Poisson Equation

General form for an isotropic medium

$$\nabla \cdot (k \nabla u) = s$$

If k is constant (homogeneous medium)

$$k \nabla^2 u = s$$

If source $s = 0$ vanishes, it reduces to Laplace's equation

$$\nabla^2 u = 0$$

The Poisson Equation (Cont'd)

Written in full (in 1D, 2D, 3D):

$$\frac{\partial}{\partial x_1}\left(k\frac{\partial u}{\partial x_1}\right) = s$$

$$\frac{\partial}{\partial x_1}\left(k\frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(k\frac{\partial u}{\partial x_2}\right) = s$$

$$\frac{\partial}{\partial x_1}\left(k\frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(k\frac{\partial u}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(k\frac{\partial u}{\partial x_3}\right) = s$$

For uniform *k*:

$$k\frac{\partial^2 u}{\partial x_1^2} = s, \qquad k\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right) = s, \qquad k\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}\right) = s$$

The Poisson's Equation Models Many Important Steady-State Problems in Engineering & Physics

Linear (Fourier Law) Heat Conduction (Notes, S2.3)

Potential Flow (Notes, S2.4)

Electrostatics (Notes, S2.5)

Magnetostatics (Notes, S2.6, vector form of eq)

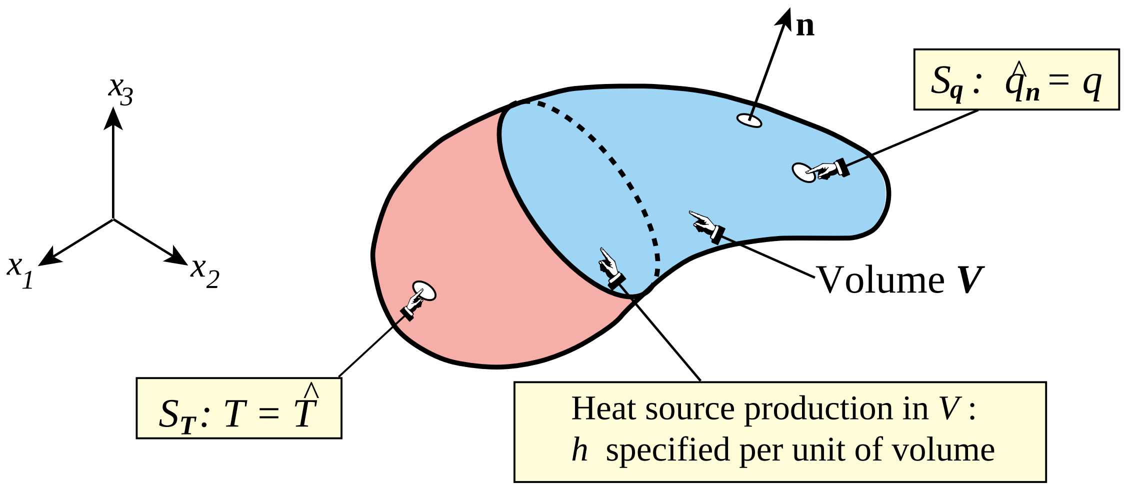
Bar & Cables (Exercise 2.2, 1D form)

St. Venant's Torsion (Exercise 2.4, 2D form)

Laterally loaded membranes (2D form)

Flow in Porous Media

The Heat Conduction Problem



Heat Conduction: Field Equations
(in full component notation)

Kinematic equation (KE): defines temperature gradient

$$\begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = \begin{bmatrix} \partial T/\partial x_1 \\ \partial T/\partial x_2 \\ \partial T/\partial x_3 \end{bmatrix}$$

Constitutive equation (CE): Fourier's law of heat conduction (isotropic body)

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = -k \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}$$

Balance equation (BE): heat flux equilibrium

$$\frac{\partial q_1}{\partial x_1} + \frac{\partial q_2}{\partial x_2} + \frac{\partial q_3}{\partial x_3} + h = 0$$

Heat Conduction: Boundary Conditions

Primary Boundary Condition (PBC): specified boundary temperature

$$T = \hat{T} \qquad \text{on } S_T$$

Flux Boundary Condition (FBC): specified boundary heat flux

$$q_1 n_1 + q_2 n_2 + q_3 n_3 = \hat{q}_n \quad \text{on } S_q$$

These are the classical BCs for Heat Conduction. Radiation and Convection BCs (which are generally nonlinear) are not considered here

Heat Conduction: Summary of Governing Equations

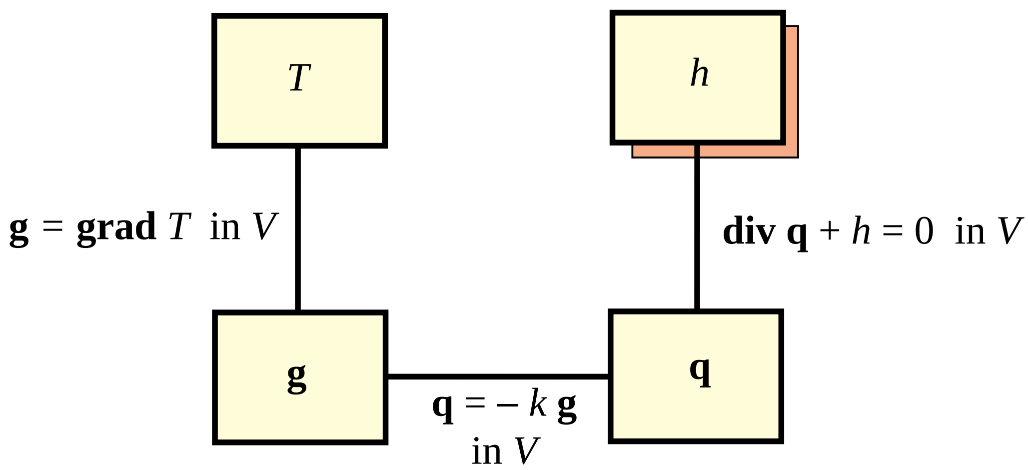
grad/div notation

KE:	$\mathbf{grad}\, T = \mathbf{g}$	in V,
CE:	$-k\mathbf{g} = \mathbf{q}$	in V,
BE:	$\mathbf{div}\, \mathbf{q} + h = 0$	in V,
PBC:	$T = \hat{T}$	on S_T,
FBC:	$\mathbf{q}.\mathbf{n} = q_n = \hat{q}_n,$	on S_q.

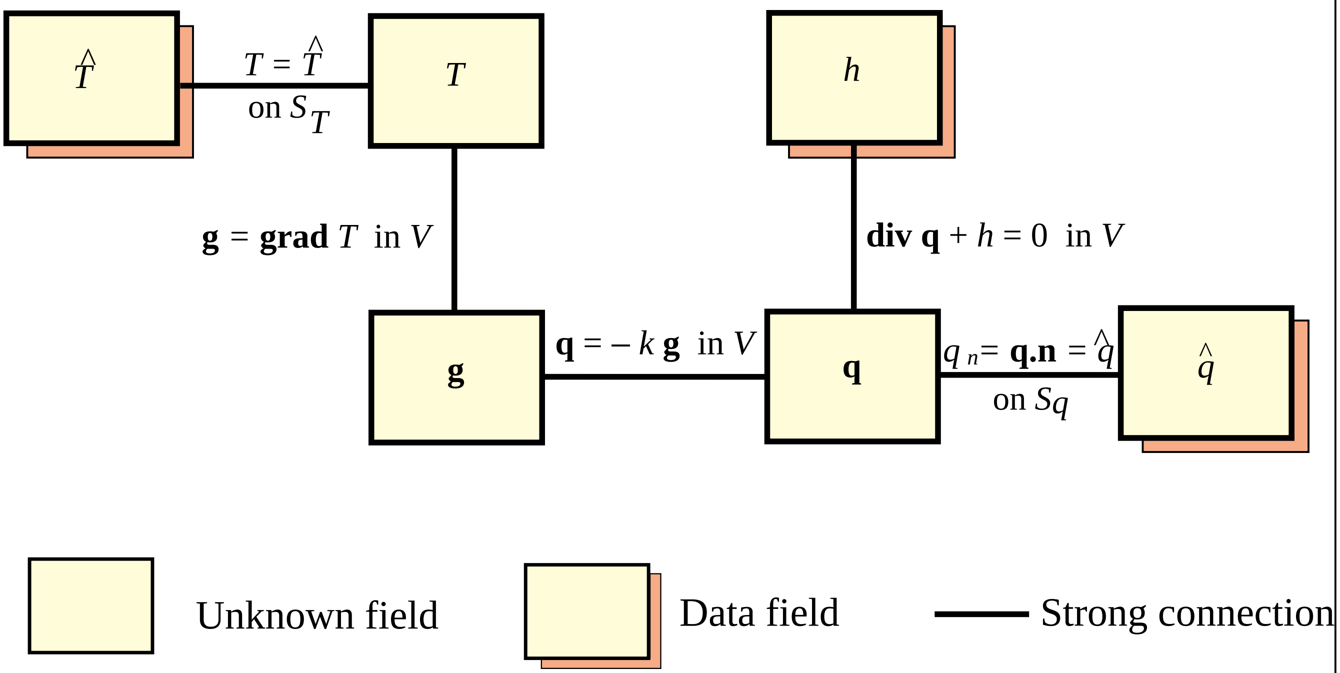
indicial (component) notation

KE:	$T_{,i} = g_i$	in V,
CE:	$-kg_i = q_i$	in V,
BE:	$q_{i,i} + h = 0$	in V,
PBC:	$T = \hat{T}$	on S_T,
FBC:	$q_i n_i = q_n = \hat{q}_n,$	on S_q.

The Tonti Diagram for the Field Equations of Heat Conduction



The Extended Tonti Diagram for the Governing Equations of Heat Conduction



The Components of the Extended Tonti Diagram for a Strong Form