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Overview

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This book covers advanced techniques for the analysis of linear elastic structures by the Finite Element Method (FEM). It has been constructed from Notes prepared for the course **Advanced to Finite Element Methods** or AFEM. This course has been taught at the Department of Aerospace Engineering Sciences, University of Colorado at Boulder since 1990. It is offered every 2 or 3 years. AFEM is a continuation of **Introduction to Finite Element Methods**, or IFEM.

§1.1. CONTENTS

The course embodies five Parts:

I **Review of Advanced Variational Methods**. The formulation of problems of engineering and physics in Strong, Weak and Variational Form.

II. **Three-Dimensional Finite Elements**: Axisymmetric iso-P elements. Solid elements: bricks, wedges, tetrahedra, pyramids. Infinite elements.

III. **High Performance Element Formulations**. The free formulation. The assumed natural strain (ANS) formulation and its variants. The patch test. Variational crimes. Drilling freedoms.

IV. **Beams, Plates and Shells**. C^1 and C° beams. Kirchhoff plate bending elements. Reissner-Mindlin (C°) plate bending elements. Facet and quadrilateral shell elements. Treatment of junctures. Transition elements.

V. **Miscellaneous and Special Project Topics**. Discussion of term projects by students.

Understanding Advanced Finite Element Methods require deeper knowledge of Variational Calculus than the "recipe" level of IFEM. Accordingly, Part I of this course deals with that topic. Some of the material has been extracted from the course **Variational Methods in Mechanics** (ASEN 5637), which is no longer offered.

§1.2. WHERE THE MATERIAL FITS

The field of Mechanics can be subdivided into four major areas:

$$\text{Mechanics} \begin{cases} Theoretical \\ Applied \\ Computational \\ Experimental \end{cases}$$

Theoretical Mechanics deals with fundamental laws and principles of mechanics studied for their intrinsic value. *Applied Mechanics* transfers this theoretical knowledge to scientific and engineering applications, especially as regards the construction of mathematical models of physical phenomena. *Computational Mechanics* solves specific problems by combining mathematical models with numerical methods implemented on digital computers, a process called *simulation*. *Experimental Mechanics* puts physical laws, mathematical models and numerical simulations to the ultimate test of observation.

Computational Mechanics is strongly interdisciplinary. The major contributing disciplines are pictured in Figure 1.1. This course will focus on Finite Element Methods. Aspects of the other

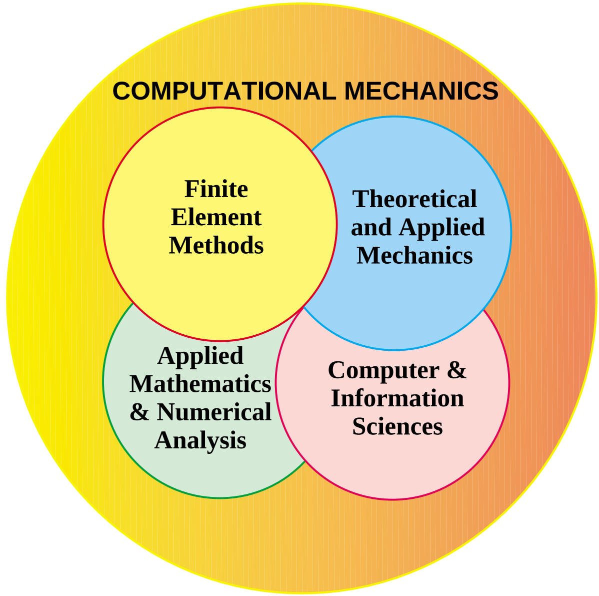

Figure 1.1. The "pizza slide:" Computational Mechanics integrates aspects of four disciplines.

contributing disciplines: Applied Mathematics and Numerical Analysis, Computer Sciences, and Applied Mechanics, will be covered as necessary, but they will not represent the major focus.

The initial Chapters deal in fact with a branch of Applied Mathematics called Variational Methods. More precisely, the application of those methods to the construction of mathematical models of mechanical systems. Various formulations that differ on the selection of primary variables are presented. Subsequent Chapters will use such formulations for building numerical approximation schemes in the form of discrete models.

The theoretical basis of Variational Methods is *Variational Calculus* or VC. Two VC "flavors", called *standard variational calculus* or SVC and *nonstandard variational calculus* or NSVC, are mentioned below. Explanation of the technical differences between the two, however, is left for specialized courses.

§1.3. THE ANALYSIS PROCESS

Recall from IFEM that the analysis process by computer methods can be characterized by the stages diagrammed in Figure 1.2. This is an expansion of a similar figure in IFEM. The stages are *idealization*, *discretization* and *solution*.

Idealization, also called *mathematical modeling*, leads to a *mathematical model* of the physical system. In Figure 1.1 this model has been subdivided into three broad classes: Strong Form (SF), Weak Form (WF) and Variational Form (VF). These are discussed further in the rest of this Chapter.

§1.4. THE BIG PICTURE

Figure 1.3 depicts three alternative forms of a mathematical model. The yellow circles zoom into the three smaller circles of Figure 1.2.

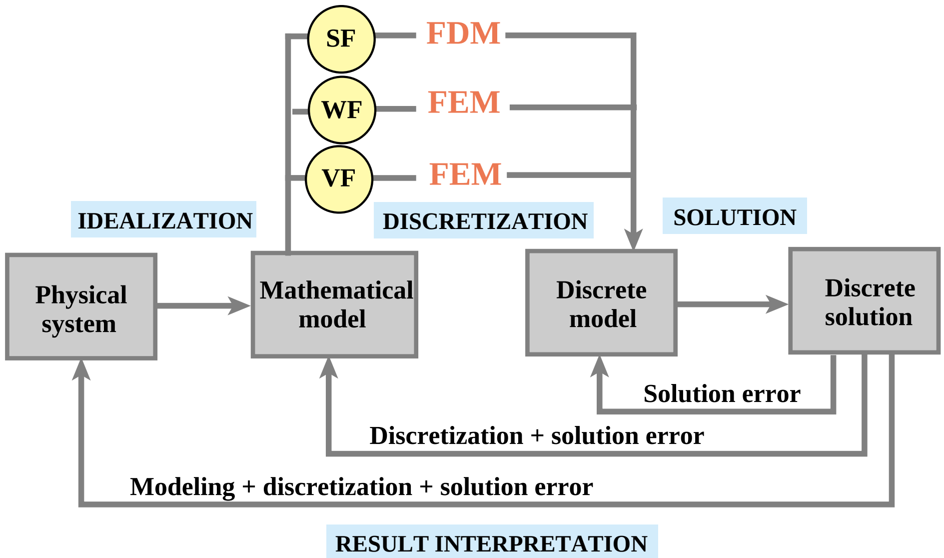

Figure 1.2. The main stages of computer-based simulation: idealization, discretization and solution. This is a slightly expanded version of a similar picture shown in Chapter 1 of IFEM.

SF **Strong Form**. Presented as a system of *ordinary or partial differential equations* in space and/or time, complemented by appropriate boundary conditions. Ocassionally this form may be presented in integraodifferential form, or reduce to algebraic equations

WF **Weak Form**. Presented as a *weighted integral equation* that "relaxes" the strong form into a domain-averaging statement.

VF **Variational Form**. Presented as a *functional* whose stationary conditions generate the weak and strong forms.

Variational Calculus or VC is a set of rules and techniques by which one can pass from one of these forms to another.

§1.5. FORM TRANSFORMATIONS

Much of variational theory and practice is concerned with the *transformation* of one form into another. As diagrammed in Figure 1.4, the following transformation paths are always possible:

From SF to WF and vice-versa.

From VF to WF, or from VF to SF.

The last two transformation constitute an important part of standard variational calculus (SVC). The rules to pass from VF to SF essentially represent a generalization of the differentiation rules of ordinary calculus.

The following transformation paths are generally impossible under the framework of *standard variational calculus* (SVC):

From SF to VF.

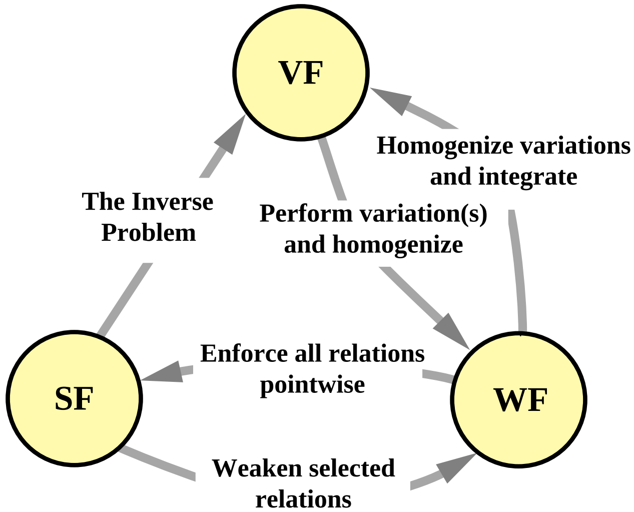

Figure 1.3. Diagram sketching Strong, Weak and Variational Forms, and relationships between form pairs. Weak Forms are also called weighted-residual equations, Galerkin equations, variational equations, variational statements, and integral statements in the literature.

From WF to VF.

Passing from a given SF to a VF is called the Inverse Problem of Variational Calculus, and may be viewed as a generalization of the problem of integrating arbitrary functions. It is therefore understandable that no general solution to this problem exists. Under *extended variational calculus* (EVC), however, such paths become possible.

§1.6. WHY VARIATIONAL METHODS?

The Strong Form (SF) states problems in ordinary or partial differential equation format. This is an old and well studied branch of calculus and mathematical physics. For example the famous Newton's Second Law: $F = ma$, is a Strong Form.

Why then the interest in Weak and Variational Forms? The following reasons may be offered.

1. Unification: the functional of the VF embodies all properties of the modeled system, including field equations, natural boundary conditions and conservation laws. Since functionals are scalars, and scalars are invariant with respect to coordinate transformations, the VF provides automatically for transformations between different coordinate frames.

2. VFs and WFs are the basis for technically important computer-based discrete methods of approximation, notably the Finite Element Method (FEM).

3. VFs, and to less extent WFs, directly characterizes "overall" quantities of interest to scientists and engineers. Examples: mass, momentum, energy. Mathematically these forms are said to lead naturally into *conservation laws*.

4. VFs clarify and systematize the treatment of boundary and interface conditions, particularly in connection with discretization schemes. [WFs are also useful in the handling of BCs, but no so powerful.]

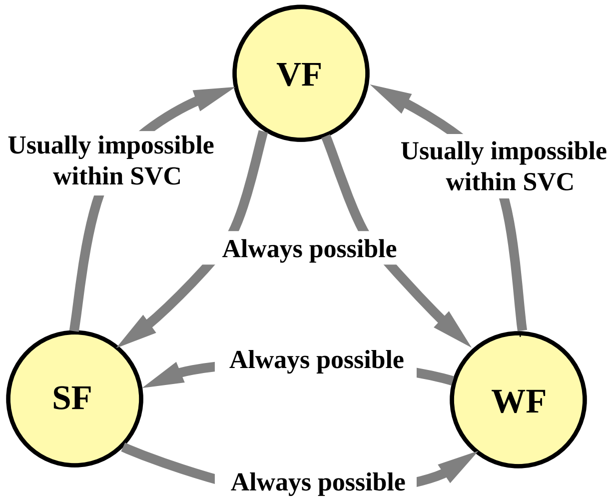

Figure 1.4. Feasibility of transformations between SF, WF and VF.

5. VFs permit a deeper and more powerful mathematical treatment of questions of existence, stability, error bounds, convergence of numerical solutions, etc. More importantly, they provide general guidelines on how to achieve desirable behavior of the related discrete schemes. [WFs are better than SFs in this regard but not as satisfactory as VFs.]

§1.7. METHODS OF APPROXIMATION: DISCRETIZATION

Transforming a SF to WF or VF does not make a problem easy to solve. Complicated problems still have to be treated by *methods of approximation*. These may be hand-based or (since the advent of the digital computer) computer-based.

The essence of all approximation methods is *discretization*. Continuum mathematical models stated in SF, WF or VF have an infinite number of degrees of freedom. Through a discretization method this is reduced to a finite number, yielding *algebraic* equations than can be solved in a reasonable time.

Each form: SF, WF and VF has a *natural* class of discretization methods than can be constructed from it. This attribute is illustrated in Figure 1.5.

§1.7.1. Finite Difference Method

The natural discretization class for SFs is the *finite difference method* (FDM). These are constructed by replacing derivatives by differences. This class is easy to generate and program for regular domains and boundary conditions, but runs into difficulties when geometry or boundary conditions become arbitrary. The other problem with conventional FDM is that the approximate solution is only obtained at the grid points, and extension to other points is not always obvious or even possible. Nevertheless the FDM class is theoretically *general* in that any problem stated in WF or VF can be put into SF.

§1.7.2. Weighted Residual Methods

The natural discretization class for WFs is the *weighted residual method* (WRM). There are well known WRM subclasses: Galerkin, Petrov-Galerkin, collocation, subdomain, finite-volume, least-squares. Sometimes these subclasses, excluding collocation, are collectively called *trial function methods*, an alternative name that accurately reflects the discretization technique. Unlike the FDM, trial-function methods yield approximate solutions defined *everywhere*. Before computers such analytical solutions were obtained by hand, a restriction that limited considerably the scope and accuracy of the approximations. That barrier was overcome with the development of the Finite Element Method (FEM) on high speed computers.

One particularly important subclass of WRM is the Finite Volume Method or FVM, which is used extensively in computational gas dynamics.

§1.7.3. Rayleigh-Ritz Methods

The natural discretization class for VFs is the *Rayleigh-Ritz method* (RRM). Although historically this was the first trial-function method, it is in fact a special subclass of the Galerkin weighted-residual method. The Finite Element Method was originally developed along these lines, and remains the most powerful computer based RRM.

Note that FEM, like FDM, can be viewed as an *universal* approximation method, because any problem can be placed in WF. This statement is no longer true, however, if one restricts FEM to the subclass of Rayleigh-Ritz method, which relies on the VF.

REMARK 1.1

In complex problems treatable within today's computer technology, combinations of these numerical methods, sometimes with a "sprinking" of analytical techniques, are common. Some examples serve to illustrate the richness of possibilities:

1. Fluid-structure interaction: FEM for the structure, FDM or FVM for the fluid.

2. Structural dynamics: FEM in space, FDM in time.

3. Semi-analytical methods: some space directions are treated by FEM (or FDM), while others are treated analytically. The so-called *methods of lines* is a prime example.

4. Finite difference schemes may be constructed from VF and WF in combination with some FEM ideas. The resulting schemes are collectively known as Finite Difference Energy Methods (FDEM). More recently the so-called *mesh free method* has emerged through a blend of FDM and FEM technques.

§1.8. *BOUNDARY ELEMENT METHODS: WHERE ARE YOU?

In addition to FEM and FDM, Boundary Element Methods (BEM) represents a third important class of computer-based discretization methods. The BEM is essentially a dimensionality-reducing technique that combines analytical reduction of one space dimension with the FEM discretization of the remaining space dimension(s). It does not have the generality of FEM or FDM, as it is primarily restricted (in its "pure" form) to linear problems with known fundamental solutions.

Originally BEMs were based on a fourth form not shown in Figures 1.3–1.5: the *integro-differential form* or IDF. Over the past decade substantial attention has been given to "merging" BEMs within the framework of the Finite Element Method. The effort has been motivated by the idea of integrating FEM and BEM in

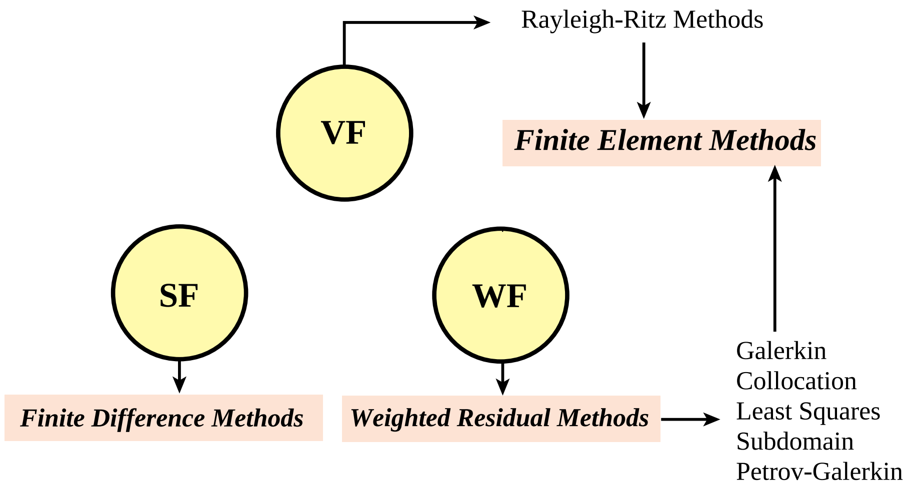

Figure 1.5. Strong, Weak and Variational Forms as source of numerical approximation methods.

the same programming framework. Thus a subclass of BEM called Variational Boundary Element Methods (VBEM) has emerged. These methods can be constructed from VFs and WFs with nonstandard application of trial functions. As of this writing, the future and importance of such methods is not clear.

§1.9. AN EXAMPLE

The following simple example serve to illustrate the three forms introduced in §1.2 and connect them with additional terminology common in applied mathematics.

Consider a function $y = y(x)$, sketched in Figure 1.6, that satisfies the ordinary differential equation

$$\boxed{y'' = y + 2 \quad \text{in} \;\; 0 \leq x \leq 2.} \tag{1.1}$$

Here primes denote derivative with respect to x. This is a Strong Form because (1.1) is to be satisfied *at each point* of the interval $0 \leq x \leq 2$. This interval is called the *problem domain*. By itself (1.1) is not sufficient to determine $y(x)$ and must be complemented with two boundary conditions. Two examples:

$$y(0) = 1, \qquad y(2) = 4, \tag{1.2}$$

$$y(0) = 1, \qquad y'(0) = 0. \tag{1.3}$$

(The first one is that pictured in Figure 1.6.) Equation (1.1) together with (1.2) defines a *boundary value problem* or BVP. Equation (1.1) together with (1.3) defines an *initial value problem* or IVP. BVPs usually model problems in spatial domains whereas IVPs model problems in the time domain.

A *residual function* associated to (1.1) is $r(x) = y'' - y - 2$. The SF (1.1) is equivalent to saying that $r(x) = 0$ at each point in the problem domain $x \in [0, 2]$. The boundary condition residual for (1.2) is $r_0 = x(0) - 1$, $r_2 = x(2) - 4$. Multiply the ODE residual $r(x)$ by a *weight function* $w(x)$ and integrate over $[0, 2]$. Multiply r_0 and r_2 by weights w_0 and w_2 and add the three terms to get

$$\boxed{\int_0^2 r(x) w(x)\, dx + r_0 w_0 + r_2 w_2 = 0.} \tag{1.4}$$

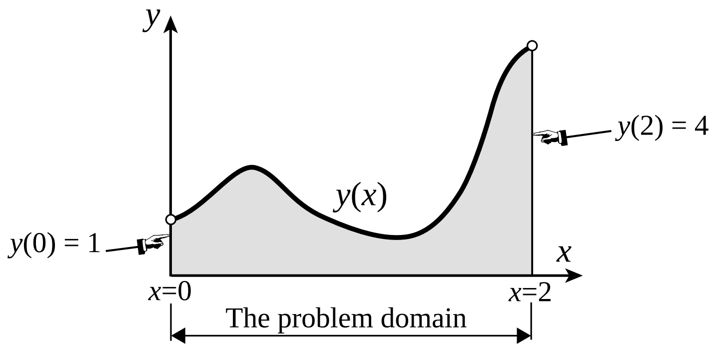

Figure 1.6. Function $y(x)$ for the example in §1.9. This function has to satisfy the boundary conditions $y(0) = 1$ and $y(2) = 4$, which together with the ODE (1.1) represents a *boundary value problem*, or BVP.

This is a *weighted integral form*. It is a Weak Form statement. Obviously a solution of the BVP (1.1)–(1.2) satisfies (1.4). However the possibility is open that other functions not satisfying that BVP may verify (1.4). Thus the qualifier "weak."

If w, w_0 and w_2 are formally written as the variations of functions v, v_0 and v_2, respectively, (we have not defined what a variation is, so what follows has to be accepted on faith) then (1.4) becomes

$$\boxed{\int_0^2 r(x)\,\delta v(x)\,dx + r_0\,\delta v_0 + r_2\,\delta v_2 = 0.} \tag{1.5}$$

Here δ is the variation symbol. The v's are technically called *test functions*. Equation (1.5) is called a *variational statement*, which leads directly to the important Galerkin and Petrov-Galerkin forms.

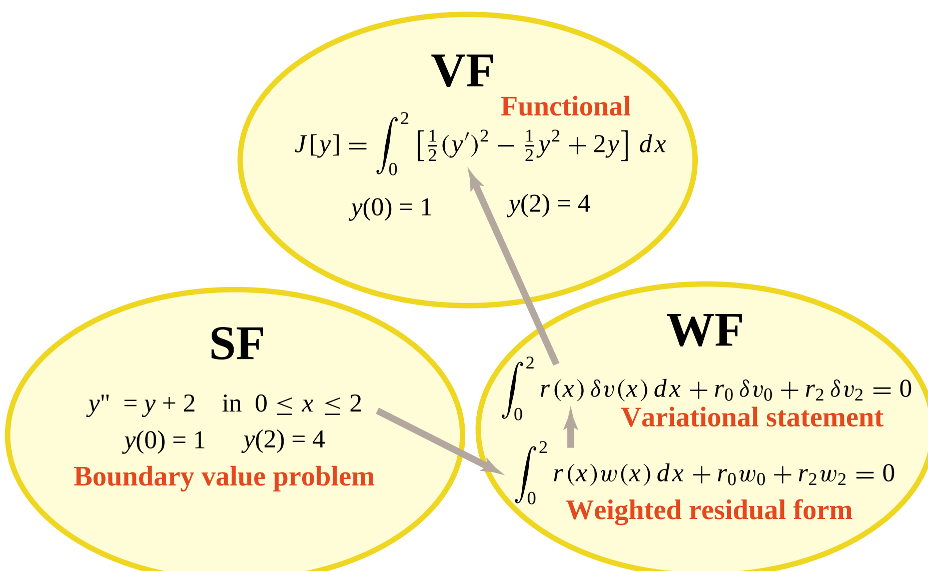

Figure 1.7. Diagramatic representation of the example forms (1.0)–(1.5).

Finally for the BVP (1.1)-(1.2) the Inverse Problem of VC has a solution. The functional

$$\boxed{J[y] = \int_0^2 \left[\tfrac{1}{2}(y'')^2 - \tfrac{1}{2}y^2 + 2y\right] dx} \tag{1.6}$$

when restricted to the class of functions satisfying $y(0) = 1$ and $y(2) = 4$ becomes stationary in the VC sense when $y(x)$ satisfies (1.1), which is called the *Euler-Lagrange equation* of (1.6). This is an example of a Variational Form.