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by

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Part II: The ANDES Element

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MEMBRANE TRIANGLES WITH CORNER DRILLING FREEDOMS: II. THE ANDES ELEMENT

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SUMMARY

This is the second article in a three-Part series on the construction of 3-node, 9-dof membrane elements with normal-to-its-plane rotational freedoms (the so-called drilling freedoms) using parametrized variational principles. In this Part, one such element is derived within the context of the Assumed Natural Deviatoric Strain (ANDES) formulation. The higher order strains are obtained by constructing three parallel-to-sides pure-bending modes from which natural strains are obtained at the corner points and interpolated over the element. To attain rank sufficiency, an additional higher order "torsional" mode, corresponding to equal hierarchical rotations at each corner with all other motions precluded, is incorporated. The resulting formulation has five free parameters. When these parameters are optimized against pure bending by energy balance methods, the resulting element is found to coalesce with the optimal EFF element derived in Part I. Numerical integration as a strain filtering device is found to play a key role in this achievement.

1. INTRODUCTION

In the first Part of this article series [1], a 9-dof triangular membrane element with three corner drilling freedoms was constructed within the framework of the extended free formulation (EFF). In the present work, we undertake the derivation of an element with the same freedom configuration, using the Assumed Natural Deviatoric Strain (ANDES) formulation.

ANDES represents a recent variant of the Assumed Natural Strain (ANS) formulation. The latter is in turn a relatively new development. A restricted form of the assumed strain method, not involving natural strains, was introduced in 1969 by Willam [2]. He constructed a 4-node plane stress element by assuming a constant shear strain independent of the direct strains, and using a strain-displacement mixed variational principle; the resulting element is identical to that derivable by selective one-point integration. A different approach advocated by Ashwell and coworkers [3] viewed "strain elements" as a way to obtain appropriate displacement fields by integration of assumed compatible strain fields. (In fact, this was the same technique used by Turner *et. al.* [4] for deriving the constant strain membrane triangle in their celebrated 1956 paper.)

These and other forms of assumed strain techniques were overshadowed in the 1970s by developments in reduced and selective integration methods for displacement models. The assumed strain approach in natural coordinates, inaugurated in a pioneer paper by MacNeal [5], has attracted increased attention since 1980. Among the main contributors we may cite Bathe and Dvorkin [6], Park and Stanley [8,9], Crisfield [7], Simo and Hughes [10], Huang and Hinton [11], and Jang and Pinsky [12]. The name "assumed natural strain" and the acronym ANS are due to Park and Stanley [9].

ANS applications have been focused on plates and shell elements because of the effectiveness of this formulation in producing elements with low distortion sensitivity, balanced stress/displacement accuracy, and which are easily extendible to geometrically nonlinear analysis. These advantages are somewhat counterbalanced by the fact that *a priori* satisfaction of the patch test is not guaranteed, even for flat elements, and *a posteriori* verifications to that effect are required.

The basic steps of the ANS formulation are summarized in Box 1. The narrative assumes that the element to be constructed has nodal displacement degrees of freedom collected in vector $\mathbf{v}$ (these are those nodal variables common with other elements, also called the *visible* degrees of freedom, or *connectors*), elastic modulus matrix $\mathbf{E}$, and volume V. A generally incompatible strain field (that is, one not necessarily derivable from displacements), is built in natural coordinates, transformed into Cartesian coordinates where it is expressed as $\mathbf{e} = \mathbf{B}\mathbf{v}$, and used to compute the stiffness matrix $\mathbf{K}$ by the standard formula $\int_V \mathbf{B}^T \mathbf{E}\mathbf{B}\, dV$. From the standpoint of connected elements, an ANS element looks exactly like a displacement model and can be easily implemented into a standard finite element code. Extensions to geometrically and materially nonlinear analysis are equally straightforward.

ANDES is a variant of ANS that exploits the fundamental decomposition of the stiffness equations described in Box 1 of Part I [1]:

$$\mathbf{K}\mathbf{v} = (\mathbf{K}_b + \alpha\mathbf{K}_h)\mathbf{v} = \mathbf{p}, \tag{1}$$

where $\alpha > 0$ is a scaling coefficient. Assumptions are made only on the "deviatoric" portion $\mathbf{e}_d$ of the element strains, namely the portion that integrates to zero over the element volume: $\int_V \mathbf{e}_d\, dV = \mathbf{0}$.

Thus instead of $\mathbf{e} = \mathbf{B}\mathbf{v}$ we eventually get, by the procedure outlined in Box 2, $\mathbf{e}_d = \mathbf{B}_d\,\mathbf{v}$, and

$$\mathbf{K}_h = \alpha \int_V \mathbf{B}_d^T \mathbf{E} \mathbf{B}_d \, dV. \tag{2}$$

The basic stiffness matrix $\mathbf{K}_b$ is constructed by the same procedure described in Box 2 of Part I. The mean portion of the strains, namely $\bar{\mathbf{e}}$, is left to be determined variationally from the constant stress assumptions used to develop $\mathbf{K}_b$, and has no effect on the stiffness equations.

The main advantage of ANDES over ANS is that elements constructed with the former technique are guaranteed to pass the individual element test of Bergan and Hanssen [13] (a strong form of the patch test that demands pairwise cancellation of surface tractions among adjacent elements in a constant stress state). There are cases when an ANS element and the corresponding ANDES element with $\alpha = 1$ coalesce. The ANDES formulation retains an edge, however, in that the scaling coefficient remains available to improve the element performance. Furthermore, the availability of $\mathbf{K}_h$ helps in the construction of element level error estimators [14] for r and h mesh adaptation.

The variational justification of the ANDES formulation was developed by Felippa and Militello [15,16], to which the reader is referred for details. This justification built on previous work [17,18] on the variational foundations of the ANS formulation. The first ANDES elements constructed using this theory were 9-dof Kirchoff plate bending triangles presented in [19]. The technique has also been used to formulate C^0 plate bending elements [14].

The present paper describes the first application of ANDES to membrane elements with drilling degrees of freedom. The main objective is to illustrate another application of this relatively new technique and assess its advantages and shortcomings when compared to FF and EFF.

2. THE TRIANGULAR ELEMENT

The geometry and degree-of-freedom configuration of triangular element is identical to that developed in Part I, to which the reader is referred for notation, geometric and behavioral relationships.

2.1. Extracting the Higher Order Behavior.

From the EFF development in the Appendix of Part I we learned that the most effective way to exhibit the higher order element behavior is to extract the hierarchical corner rotations $\tilde{\theta}_i$ from the total corner rotations θ_i:

$$\tilde{\theta}_i = \theta_i - \theta_0, \tag{16}$$

where $i = 1, 2, 3$ is the corner index and θ_0 is the rotation of the Constant Strain Triangle (CST):

$$\theta_0 = \frac{1}{4A} \left(x_{23} v_{x1} + x_{31} v_{x2} + x_{12} v_{x3} + y_{23} v_{y1} + y_{31} v_{y2} + y_{12} v_{y3} \right). \tag{17}$$

Box 1 Construction of K by the ANS Formulation

Step S.1. Select locations in the element where "natural straingage" locations are to be chosen. For many ANS elements these gages are placed on *reference lines* (in 2D elements) or *reference planes* (in 3D elements), but this is not a general rule. By appropriate interpolation express the element natural strains $\boldsymbol{\epsilon}$ in terms of the "straingage readings" $\mathbf{g}$ at those locations:

$$\boldsymbol{\epsilon} = \mathbf{A}_\epsilon\, \mathbf{g}, \tag{3}$$

where $\boldsymbol{\epsilon}$ is a strain field in natural coordinates that must include all constant strain states. (For structural elements the term "strain" is to be interpreted in a generalized sense.)

Step S.2. Relate the Cartesian strains $\mathbf{e}$ to the natural strains:

$$\mathbf{e} = \mathbf{T}\boldsymbol{\epsilon} = \mathbf{T}\mathbf{A}_\epsilon \mathbf{g} = \mathbf{A}\mathbf{g} \tag{4}$$

at each point in the element. (If $\mathbf{e} \equiv \boldsymbol{\epsilon}$, or if it is possible to work throughout in natural coordinates, this step is skipped.) The resulting Cartesian strain interpolation is

$$\mathbf{e} = \mathbf{T}\mathbf{A}_\epsilon \mathbf{g} = \mathbf{A}\mathbf{g}. \tag{5}$$

If $\mathbf{T}$ is constant over the element, as in the case of the triangle studied here, the step during which interpolation is effected is irrelevant.

Step S.3. Relate the natural straingage readings $\mathbf{g}$ to the visible degrees of freedom

$$\mathbf{g} = \mathbf{Q}\mathbf{v}, \tag{6}$$

where $\mathbf{Q}$ is a straingage-to-node displacement transformation matrix. Techniques for doing this vary from element to element and it is difficult to state rules that apply to every situation. Often the problem is amenable to breakdown into subproblems; for example

$$\mathbf{g} = \mathbf{Q}_1\mathbf{v}_1 + \mathbf{Q}_2\mathbf{v}_2 + \dots \tag{7}$$

where $\mathbf{v}_1, \mathbf{v}_2, \dots$ are conveniently selected subsets of $\mathbf{v}$. Some of these components may be derivable from displacements while others are not.

Box 1 Construction of K by the ANS Formulation (Continued)

Step S.4. For a three-dimensional element of volume V and elastic modulus matrix $\mathbf{E}$, the element stiffness matrix is given by

$$\mathbf{K} = \mathbf{Q}^T \mathbf{K}_a \mathbf{Q}, \qquad \text{with} \qquad \mathbf{K}_a = \int_V \mathbf{A}^T \mathbf{E} \mathbf{A}\, dV. \tag{8}$$

Should $\mathbf{B} = \mathbf{A}\mathbf{Q}$ be readily available one may use the standard formula

$$\mathbf{K} = \int_V \mathbf{B}^T \mathbf{E} \mathbf{B}\, dV. \tag{9}$$

In general this stiffness matrix does not necessarily pass the individual element test of Bergan and Hanssen [13] (a strong form of the patch test that demands pairwise cancellation of node forces between adjacent elements in constant stress states). For this to happen, $\mathbf{K}$ must admit the decomposition

$$\mathbf{K} = \mathbf{K}_b + \mathbf{K}_h = \upsilon^{-1} \mathbf{L}\mathbf{E}\mathbf{L}^T + \mathbf{K}_h, \tag{10}$$

where $\upsilon = \int_V dV$ is the element volume measure, $\mathbf{L}$ is a force-lumping matrix derivable as discussed in Box 1 of Part I and $\mathbf{K}_h$ is orthogonal to the rigid body and constant strain test motions. In other words, the ANS element must coalesce with the ANDES formulation with $\alpha = 1$. The equivalence may be checked by requiring that

$$\overline{\mathbf{B}} = \overline{\mathbf{A}}\mathbf{Q} = \upsilon^{-1}\, \mathbf{L}^T, \tag{11}$$

where $\overline{\mathbf{A}}$ denotes the mean part of $\mathbf{A}$ (cf. Box 2). As of this writing, no general techniques for explicit construction of ANS fields that satisfy these conditions *a priori* are known.

If the patch test is not satisfied, one should switch to the ANDES formulation by replacing the basic stiffness constructed from constant strain, namely $\upsilon \overline{\mathbf{B}}^T \mathbf{E} \overline{\mathbf{B}}$, with one constructed from constant stress assumptions.

Box 2 Construction of $\mathbf{K}_h$ by the ANDES Formulation

Steps H.1 to *H.3*. Identical to the first three steps *S.1* through *S.3* in Box 1.

Step H.4. Split the Cartesian strain field into mean (volume-averaged) and deviatoric strains:

$$\mathbf{e} = \bar{\mathbf{e}} + \mathbf{e}_d = (\overline{\mathbf{A}} + \mathbf{A}_d)\,\mathbf{g}, \tag{12}$$

where $\overline{\mathbf{A}} = \upsilon^{-1} \int_V \mathbf{T}\mathbf{A}_\epsilon \, dV$, and $\mathbf{e}_d = \mathbf{A}_d\,\mathbf{g}$ has mean zero value over V. For elements of simple geometry this decomposition can often be done in advance, and $\mathbf{e}_d$ constructed directly. Furthermore, this step may also be carried out on the natural strains if $\mathbf{T}$ is constant, as is the case for the elements here.

Step H.5. The higher-order stiffness matrix is given by

$$\mathbf{K}_h = \alpha \mathbf{Q}^T \mathbf{K}_d \mathbf{Q}, \qquad \text{with} \qquad \mathbf{K}_d = \int_V \mathbf{A}_d^T \mathbf{E} \mathbf{A}_d \, dV, \tag{13}$$

where $\alpha = j_{22} > 0$ is a scaling coefficient (see Box 1).

It is often convenient to combine the product of $\mathbf{A}$ and $\mathbf{Q}$ into a single strain-displacement matrix called (as usual) $\mathbf{B}$, which splits into $\overline{\mathbf{B}}$ and $\mathbf{B}_d$:

$$\mathbf{e} = \mathbf{A}\mathbf{Q}\mathbf{v} = (\overline{\mathbf{A}} + \mathbf{A}_d)\mathbf{Q}\mathbf{v} = (\overline{\mathbf{B}} + \mathbf{B}_d)\mathbf{v} = \mathbf{B}\,\mathbf{v}, \tag{14}$$

in which case

$$\mathbf{K}_h = \int_V \mathbf{B}_d^T \mathbf{E} \mathbf{B}_d \, dV. \tag{15}$$

From (16) and (17) we readily perceive the fundamental transformation

$$\begin{bmatrix} \tilde{\theta}_1 \\ \tilde{\theta}_2 \\ \tilde{\theta}_3 \end{bmatrix} = \frac{1}{4A} \begin{bmatrix} x_{32} & y_{32} & 4A & x_{13} & y_{13} & 0 & x_{21} & y_{21} & 0 \\ x_{32} & y_{32} & 0 & x_{13} & y_{13} & 4A & x_{21} & y_{21} & 0 \\ x_{32} & y_{32} & 0 & x_{13} & y_{13} & 0 & x_{21} & y_{21} & 4A \end{bmatrix} \begin{bmatrix} v_{x1} \\ v_{y1} \\ \theta_1 \\ v_{x2} \\ v_{y2} \\ \theta_2 \\ v_{x3} \\ v_{y3} \\ \theta_3 \end{bmatrix}, \tag{18}$$

or

$$\tilde{\boldsymbol{\theta}} = \mathbf{H}_{\theta v}\mathbf{v}. \tag{19}$$

The unscaled higher order stiffness of this element fits the *generic template* introduced in Section 6.6 of Part I:

$$\mathbf{K}_h = \mathbf{H}_{\theta v}^T \mathbf{K}_{\theta h} \mathbf{H}_{\theta v}. \tag{20}$$

The main objective of all formulations investigated here, as well as those in Part I, is to construct the 3×3 matrix $\mathbf{K}_{\theta h}$, which represents the higher order stiffness in terms of the hierarchical rotations $\tilde{\boldsymbol{\theta}}$.

Guided by these considerations, we begin by decomposing the visible degree of freedom vector into basic (CST) and higher order, as follows:

$$\mathbf{v} = \mathbf{v}_b + \mathbf{v}_h = \mathbf{v}_b + \mathbf{P}\tilde{\boldsymbol{\theta}}, \tag{21}$$

where

$$\mathbf{v}_b = \begin{bmatrix} v_{x1} \\ v_{y2} \\ \theta_0 \\ v_{x2} \\ v_{y2} \\ \theta_0 \\ v_{x3} \\ v_{y3} \\ \theta_0 \end{bmatrix}, \qquad \mathbf{v}_h = \begin{bmatrix} 0 \\ 0 \\ \tilde{\theta}_1 \\ 0 \\ 0 \\ \tilde{\theta}_2 \\ 0 \\ 0 \\ \tilde{\theta}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{\theta}_1 \\ \tilde{\theta}_2 \\ \tilde{\theta}_3 \end{bmatrix}. \tag{22}$$

To simplify the problem of building higher order strain fields, we further split the hierarchical rotations into mean and deviatoric:

$$\begin{bmatrix} \tilde{\theta}_1 \\ \tilde{\theta}_2 \\ \tilde{\theta}_3 \end{bmatrix} = \begin{bmatrix} \bar{\theta} \\ \bar{\theta} \\ \bar{\theta} \end{bmatrix} + \begin{bmatrix} \theta'_1 \\ \theta'_2 \\ \theta'_3 \end{bmatrix} \tag{23}$$

where $\bar{\theta} = \frac{1}{3}(\tilde{\theta}_1 + \tilde{\theta}_2 + \tilde{\theta}_3)$ and $\theta'_i = \tilde{\theta}_i - \bar{\theta}$. Consequently $\theta'_1 + \theta'_2 + \theta'_3 = 0.$ In matrix form

$$\tilde{\boldsymbol{\theta}} = \bar{\boldsymbol{\theta}} + \boldsymbol{\theta}', \tag{24}$$

which in terms of the nodal displacement vector becomes

$$\mathbf{v} = \mathbf{v}_b + \mathbf{P}(\bar{\boldsymbol{\theta}} + \boldsymbol{\theta}'), \tag{25}$$

where **P** is the 9×3 matrix shown above. The deviatoric corner rotations define the linear deviatoric-rotation field:

$$\theta' = \theta'_1 \zeta_1 + \theta'_2 \zeta_2 + \theta'_3 \zeta_3, \tag{26}$$

which integrates to zero over the element. For future use we note the matrix relation

$$\begin{bmatrix} \theta'_1 \\ \theta'_2 \\ \theta'_3 \\ \bar{\theta} \end{bmatrix} = \tfrac{1}{3} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \tilde{\theta}_1 \\ \tilde{\theta}_2 \\ \tilde{\theta}_3 \end{bmatrix} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} - \tfrac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\} \begin{bmatrix} \tilde{\theta}_1 \\ \tilde{\theta}_2 \\ \tilde{\theta}_3 \end{bmatrix} \tag{27}$$

or

$$\begin{bmatrix} \boldsymbol{\theta}' \\ \bar{\theta} \end{bmatrix} = (\mathbf{J}' - \bar{\mathbf{J}})\, \tilde{\boldsymbol{\theta}}. \tag{28}$$

The hierarchical rotation decomposition is associated with a similar decomposition of the higher order strains:

$$\mathbf{e}_d = \mathbf{e}_b + \mathbf{e}_t, \tag{29}$$

where subscripts b and t identify "pure bending" and "torsional" strain fields, respectively. The former is generated by the deviatoric rotations $\boldsymbol{\theta}'$ whereas the latter is generated by the mean hierarchical rotation $\bar{\theta}$. We now proceed to examine these two components in turn.

2.2. The Pure-Bending Field.

This field is produced by pure inplane-bending modes associated with the deviatoric corner rotations θ'_i, $i = 1, 2, 3$. One way to visualize the nature of these modes is to think of a tiny triangle superposed on a thin plane beam bent to constant curvature in its plane. Place the triangle centroid at neutral axis height. Then rotate the triangle so that its 3 sides align in turn with the bending direction.

From this visualization it follows that the *reference lines* mentioned in Box 1 are the triangle sides. The *straingage locations* are chosen at the triangle corners. The *natural strains* are the three direct strains parallel to the triangle sides, traversed in the counterclockwise sense. These strains are collected in the vector

$$\boldsymbol{\epsilon}_b = [\, \epsilon_{b21} \quad \epsilon_{b32} \quad \epsilon_{b13} \,]^T \,. \tag{30}$$

The natural strain ϵ_{jk} at corner i will be written $\epsilon_{jk|i}$, the bar being used for reading convenience. Vector $\boldsymbol{\epsilon}_b$ at corner i is denoted by $\boldsymbol{\epsilon}_{bi}$. Our objective is to construct the 3×3 matrices $\mathbf{Q}_{bi}$ that relate natural straingage readings to the deviatoric rotations:

$$\boldsymbol{\epsilon}_{b1} = \mathbf{Q}_{b1} \boldsymbol{\theta}', \qquad \boldsymbol{\epsilon}_{b2} = \mathbf{Q}_{b2} \boldsymbol{\theta}', \qquad \boldsymbol{\epsilon}_{b3} = \mathbf{Q}_{b3} \boldsymbol{\theta}', \tag{31}$$

Once these are known the natural bending strains can be easily obtained by linear interpolation over the triangle: $\boldsymbol{\epsilon}_b = (\mathbf{Q}_{b1} \zeta 1 + \mathbf{Q}_{b2} \zeta_2 + \mathbf{Q}_{b3} \zeta_3) \boldsymbol{\theta}' = \mathbf{Q}_b \, \boldsymbol{\theta}'$.

Consider the natural strain $\epsilon_{b21}(P)$ at an arbitrary point P of the triangle. Denote by $d_{21|P}$ the signed distance from the centroid to P measured along the internal normal to side 21. In particular, for the corners we have

$$d_{21|3} = \frac{4A}{3\ell_{21}}, \qquad d_{21|1} = d_{21|2} = -\tfrac{1}{2} d_{21|3} = -\frac{2A}{3\ell_{12}}. \tag{32}$$

We shall assume that $\epsilon_{b21|P}$ depends only on $d_{21|P}$ divided by the side length ℓ_{21}, which introduces a distance scaling. These dimensionless ratios will be called $\chi_{21|P} = d_{21|P}/\ell_{21}$, which specialized to the corners become

$$\chi_{21|3} = \frac{4A}{3\ell_{21}^2}, \qquad \chi_{21|1} = \chi_{21|2} = -\frac{2A}{3\ell_{21}^2}. \tag{33}$$

Formulas for corners 2 and 3 are obtained by cyclic permutation. According to the assumption just stated, the natural straingage readings $\epsilon_{b21|i}$ at corner i depend only on $\chi_{21|i}$, multiplied by as yet unknown weighting factors. This can be written in matrix form as follows:

$$\begin{aligned}
\boldsymbol{\epsilon}_{b1} &= \begin{bmatrix} \epsilon_{b21|1} \\ \epsilon_{b32|1} \\ \epsilon_{b13|1} \end{bmatrix} = \begin{bmatrix} \rho_1\chi_{21|1} & -\rho_2\chi_{21|1} & \rho_4\chi_{21|1} \\ \rho_5\chi_{32|1} & \rho_3\chi_{32|1} & -\rho_3\chi_{32|1} \\ -\rho_1\chi_{13|1} & \rho_4\chi_{13|1} & \rho_2\chi_{13|1} \end{bmatrix} \begin{bmatrix} \theta'_1 \\ \theta'_2 \\ \theta'_3 \end{bmatrix} = \mathbf{Q}_{b1}\boldsymbol{\theta}', \\
\boldsymbol{\epsilon}_{b2} &= \begin{bmatrix} \epsilon_{b21|2} \\ \epsilon_{b32|2} \\ \epsilon_{b13|2} \end{bmatrix} = \begin{bmatrix} \rho_2\chi_{21|2} & -\rho_1\chi_{21|2} & \rho_4\chi_{21|2} \\ \rho_4\chi_{32|2} & \rho_1\chi_{32|2} & -\rho_2\chi_{32|2} \\ -\rho_3\chi_{13|2} & \rho_5\chi_{13|2} & \rho_3\chi_{13|2} \end{bmatrix} \begin{bmatrix} \theta'_1 \\ \theta'_2 \\ \theta'_3 \end{bmatrix} = \mathbf{Q}_{b2}\boldsymbol{\theta}', \\
\boldsymbol{\epsilon}_{b3} &= \begin{bmatrix} \epsilon_{b21|3} \\ \epsilon_{b32|3} \\ \epsilon_{b13|3} \end{bmatrix} = \begin{bmatrix} \rho_3\chi_{21|3} & -\rho_3\chi_{21|3} & \rho_5\chi_{21|3} \\ \rho_4\chi_{32|3} & \rho_2\chi_{32|3} & -\rho_1\chi_{32|3} \\ -\rho_2\chi_{13|3} & \rho_4\chi_{13|3} & \rho_1\chi_{13|3} \end{bmatrix} \begin{bmatrix} \theta'_1 \\ \theta'_2 \\ \theta'_3 \end{bmatrix} = \mathbf{Q}_{b3}\boldsymbol{\theta}',
\end{aligned} \tag{34}$$

where ρ_1 through ρ_5 are dimensionless weight factors to be determined on the basis of energy balancing for rectangular mesh units, as discussed in Section 3. The distribution and sign of these factors is made on the basis of triangular symmetries.

The strain field is energy orthogonal if

$$\rho_1 + \rho_2 = 2\rho_3, \qquad \rho_4 + \rho_5 = 0, \tag{35}$$

but these conditions will not be assumed *a priori*. The optimal element described in Section 3.2 will be found, however, to satisfy (35).

The natural strains can be related to Cartesian strains by the transformation

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{12} \\ \epsilon_{23} \\ \epsilon_{31} \end{bmatrix} = \begin{bmatrix} c_{21}^2 & s_{21}^2 & s_{21}c_{21} \\ c_{32}^2 & s_{32}^2 & s_{32}c_{32} \\ c_{13}^2 & s_{13}^2 & s_{13}c_{13} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix} = \mathbf{T}^{-1}\mathbf{e}. \tag{36}$$

where $c_{21} = x_{21}/\ell_{21}$, $s_{21} = y_{21}/\ell_{21}$, $c_{32} = x_{32}/\ell_{32}$, $s_{32} = y_{32}/\ell_{32}$, $c_{13} = x_{13}/\ell_{13}$ and $s_{13} = y_{13}/\ell_{13}$. The inverse of this relation is

$$\begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix} = \frac{1}{4A^2} \begin{bmatrix} y_{23}y_{13}\ell_{21}^2 & y_{31}y_{21}\ell_{32}^2 & y_{12}y_{32}\ell_{13}^2 \\ x_{23}x_{13}\ell_{21}^2 & x_{31}x_{21}\ell_{32}^2 & x_{12}x_{32}\ell_{13}^2 \\ (y_{23}x_{31} + x_{32}y_{13})\ell_{21}^2 & (y_{31}x_{12} + x_{13}y_{21})\ell_{32}^2 & (y_{12}x_{23} + x_{21}y_{32})\ell_{13}^2 \end{bmatrix} \begin{bmatrix} \epsilon_{12} \\ \epsilon_{23} \\ \epsilon_{31} \end{bmatrix} \tag{37}$$

or, in compact matrix notation, $\mathbf{e} = \mathbf{T}\boldsymbol{\epsilon}$. Note that $\mathbf{T}$ is constant over the triangle. Combining with (30) we get the Cartesian corner strains as $\mathbf{e}_{bi} = \mathbf{B}_{bi}\boldsymbol{\theta}' = \mathbf{T}\mathbf{Q}_{bi}\boldsymbol{\theta}'$, $i = 1, 2, 3$. The Cartesian strains are obtained by linearly interpolating over the element:

$$\mathbf{e}_b = (\mathbf{B}_{b1}\zeta_1 + \mathbf{B}_{b2}\zeta_2 + \mathbf{B}_{b3}\zeta_3)\boldsymbol{\theta}' = \mathbf{B}_b\boldsymbol{\theta}'. \tag{38}$$

2.3. The Torsional Field.

The higher order stiffness produced by the pure bending fields alone is rank deficient (2 instead of 3) because of the deviatoric constraint $\sum \theta'_i = 0$. To complete the construction of a rank-sufficient higher order stiffness we need to build a strain field associated with the degree of freedom setting $\theta_i = \bar{\theta}$, others zero. This may be viewed as forcing each corner of the triangle to rotate by the same amount while the corner displacements are precluded. A displacement-based solution to this problem is provided by the cubic field of the QST triangle constructed by Felippa [21] and developed by Carr [22] as membrane component for refined analysis of thin shells. The QST expansion is

$$u_x = \begin{bmatrix} v_{x1} \\ v_{x,x|1} \\ v_{x,y|1} \\ v_{x2} \\ v_{x,x|2} \\ v_{x,y|2} \\ v_{x3} \\ v_{x,x|3} \\ v_{x,y|3} \\ v_{x0} \end{bmatrix}^T \begin{bmatrix} \zeta_1^2(3-2\zeta_1) - 7\zeta_1\zeta_2\zeta_3 \\ \zeta_1^2(x_{21}\zeta_2 - x_{13}\zeta_3) + (x_{13}-x_{21})\zeta_1\zeta_2\zeta_3 \\ \zeta_1^2(y_{21}\zeta_2 - y_{13}\zeta_3) + (y_{13}-y_{21})\zeta_1\zeta_2\zeta_3 \\ \zeta_2^2(3-2\zeta_2) - 7\zeta_1\zeta_2\zeta_3 \\ \zeta_2^2(x_{32}\zeta_3 - x_{21}\zeta_1) + (x_{21}-x_{32})\zeta_1\zeta_2\zeta_3 \\ \zeta_2^2(y_{32}\zeta_3 - y_{21}\zeta_1) + (y_{21}-y_{32})\zeta_1\zeta_2\zeta_3 \\ \zeta_3^2(3-2\zeta_3) - 7\zeta_1\zeta_2\zeta_3 \\ \zeta_3^2(x_{13}\zeta_1 - x_{32}\zeta_2) + (x_{32}-x_{13})\zeta_1\zeta_2\zeta_3 \\ \zeta_3^2(y_{13}\zeta_1 - y_{32}\zeta_2) + (y_{32}-y_{13})\zeta_1\zeta_2\zeta_3 \\ 27\zeta_1\zeta_2\zeta_3 \end{bmatrix} \tag{39}$$

where $v_{x,x|i}$ and $v_{x,y|i}$ denote $\partial u_x/\partial x$ and $\partial u_x/\partial y$, respectively, evaluated at corner i. A similar interpolation holds for the y displacement component u_y. The torsional mode with unit rotations $\theta_i = \bar{\theta} = 1$ is imposed by setting the QST nodal displacements to

$$v_{xi} = v_{yi} = v_{x,x|j} = v_{y,y|j} = 0, \quad v_{x,y|j} = -\bar{\theta}, \quad v_{y,x|j} = \bar{\theta}, \quad i = 0, 1, 2, 3, \;\; j = 1, 2, 3. \tag{40}$$

Differentiating (39) with respect to x and y and setting the freedoms to (40), we obtain the torsional strain field

$$\begin{aligned} e_{txx} &= -\frac{1}{A}\Big[\zeta_1 y_{23}(y_{31}\zeta_3 - y_{12}\zeta_2) + \zeta_2 y_{31}(y_{12}\zeta_1 - y_{23}\zeta_3) + \zeta_3 y_{12}(y_{23}\zeta_2 - y_{31}\zeta_1)\Big]\bar{\theta}, \\ e_{tyy} &= \frac{1}{A}\Big[\zeta_1 x_{32}(x_{21}\zeta_2 - x_{13}\zeta_3) + \zeta_2 x_{13}(x_{32}\zeta_3 - x_{21}\zeta_1) + \zeta_3 x_{21}(x_{13}\zeta_1 - x_{32}\zeta_2)\Big]\bar{\theta}, \\ 2e_{txy} &= -\frac{1}{A}\Big[\zeta_1 x_{32}(y_{31}\zeta_3 - y_{12}\zeta_2) - \zeta_1 y_{23}(x_{21}\zeta_2 - x_{13}\zeta_3) \\ &\qquad + \zeta_2 x_{13}(y_{12}\zeta_1 - y_{23}\zeta_3) - \zeta_2 y_{31}(x_{32}\zeta_3 - x_{21}\zeta_1) \\ &\qquad + \zeta_3 x_{21}(y_{23}\zeta_2 - y_{31}\zeta_1) - \zeta_3 y_{12}(x_{13}\zeta_1 - x_{32}\zeta_2)\Big]\bar{\theta}, \end{aligned} \tag{41}$$

where A is the triangle area. In matrix form

$$\mathbf{e}_t = \begin{bmatrix} e_{txx} \\ e_{tyy} \\ 2e_{txy} \end{bmatrix} = \mathbf{B}_t\,\bar{\theta}. \tag{42}$$

This strain field is compatible, varies quadratically, and vanishes at the corners and centroid. Integrating over the triangle and using the fact that $x_{12} + x_{23} + x_{31} = 0$ and $y_{12} + y_{23} + y_{31} = 0$ it may be verified that all strain components are energy orthogonal.

The field (41) appears unduly complicated. Conversion to natural strains through the transformation (36) reveals, however, its intrinsic simplicity:

$$\boldsymbol{\epsilon}_t = \begin{bmatrix} \epsilon_{t21} \\ \epsilon_{t32} \\ \epsilon_{t13} \end{bmatrix} = \mathbf{T}^{-1}\mathbf{e}_t = \tfrac{3}{2} \begin{bmatrix} \chi_{21|1}\,\zeta_{21}\zeta_3 \\ \chi_{32|2}\,\zeta_{32}\zeta_1 \\ \chi_{13|3}\,\zeta_{13}\zeta_2 \end{bmatrix} \bar{\theta}, \tag{43}$$

where $\zeta_{21} = \zeta_2 - \zeta_1$, $\zeta_{32} = \zeta_3 - \zeta_2$ and $\zeta_{13} = \zeta_1 - \zeta_3$. For future use, it is of interest to consider a *midpoint-filtered* version of (43), obtained by evaluating it at the three triangle midpoints 4, 5, 6 and then interpolating linearly over the triangle:

$$\boldsymbol{\epsilon}_t^m = \begin{bmatrix} \epsilon_{t21}^m \\ \epsilon_{t32}^m \\ \epsilon_{t13}^m \end{bmatrix} = 3 \begin{bmatrix} \chi_{21|1}\,\zeta_{21} \\ \chi_{32|2}\,\zeta_{32} \\ \chi_{13|3}\,\zeta_{13} \end{bmatrix} \bar{\theta}. \tag{44}$$

To facilitate combination with the bending field, it is convenient to define the "spread" matrix form of (44) in which each column receives one third of the strain:

$$\boldsymbol{\epsilon}_t^m = \begin{bmatrix} \epsilon_{t21}^m \\ \epsilon_{t32}^m \\ \epsilon_{t13}^m \end{bmatrix} = \begin{bmatrix} \chi_{21|1}\zeta_{21} & \chi_{21|1}\zeta_{21} & \chi_{21|1}\zeta_{21} \\ \chi_{32|2}\zeta_{32} & \chi_{32|2}\zeta_{32} & \chi_{32|2}\zeta_{32} \\ \chi_{13|3}\zeta_{13} & \chi_{13|3}\zeta_{13} & \chi_{13|3}\zeta_{13} \end{bmatrix} \begin{bmatrix} \bar{\theta} \\ \bar{\theta} \\ \bar{\theta} \end{bmatrix} = \mathbf{Q}_t^m \begin{bmatrix} \bar{\theta} \\ \bar{\theta} \\ \bar{\theta} \end{bmatrix} \tag{45}$$

3. THE STIFFNESS MATRIX

Having constructed the higher order strain fields, the computation of the higher-order stiffness can proceed according to the general rules laid out in Box 2. The bending and torsional strain fields are combined as

$$\mathbf{e}_d = \mathbf{B}_b\boldsymbol{\theta}' + \mathbf{B}_t\bar{\theta} = (\mathbf{B}_b\mathbf{J}' - \mathbf{B}_t\bar{\mathbf{J}})\,\tilde{\boldsymbol{\theta}} = \mathbf{B}_d\tilde{\boldsymbol{\theta}}, \tag{46}$$

where $\mathbf{J}'$ and $\bar{\mathbf{J}}$ are the numerical matrices in (28). We shall evaluate the higher order stiffness in terms of $\tilde{\boldsymbol{\theta}}$, namely

$$\mathbf{K}_{\theta h} = \int_A \mathbf{B}_d^T (h\mathbf{E})\mathbf{B}_d \, dA \tag{47}$$

where h is the plate thickness, by numerical quadrature. The 9×9 higher order stiffness $\mathbf{K}_h$ then follows from the congruential transformation (19). At this point, however, we still have the undetermined ρ_i coefficients present in $\mathbf{B}_d$.

3.1. The Optimal Element.

For reasons that will be immediately apparent, we are particularly interested in *three point quadrature rules* defined parametrically by

$$\int_A F(\zeta_1, \zeta_2, \zeta_3)\, dA \approx \frac{A}{3}\Big[F\,(\xi, \eta, \eta) + F\,(\eta, \xi, \eta) + F\,(\eta, \eta, \xi)\Big] \tag{48}$$

where $0 \le \xi \le 1$ and $\eta = \frac{1}{2}(1 - \xi)$. In practice the two most interesting rules of this type are $\xi = 2/3$ (the interior-three-point rule) and $\xi = 0$ (the midpoint rule), both of which exhibit quadratic accuracy. But in the present context it is instructive to leave ξ free, excluding only the cases $\xi = 1$ (corners) and $\xi = 1/3$ (centroid). A symbolic analysis with Macsyma, described fully in Section 2 of Part III [20], shows that the choice

$$\rho_1 = 0, \quad \rho_2 = 1 - \xi, \quad \rho_3 = \tfrac{1}{2}(1 - \xi), \quad \rho_4 = \rho_5 = 0, \tag{49}$$

has the following properties:

1. It achieves pure-bending energy balance for rectangular mesh units of *arbitrary* aspect ratio, a test discussed in detail in Section 2 of Part III.

2. Let $\mathbf{K}_{\theta h}(\xi)$ be the stiffness obtained with the integration rule (48) and the choice (49) for the ρ coefficients. Then the scaled stiffness

$$\mathbf{K}_{\theta h} = \frac{2}{8(\xi - 1)^2(\xi - \frac{1}{3})^2}\mathbf{K}_{\theta h}(\xi), \tag{50}$$

   is independent of ξ, and coincides with that of the optimal EFF element derived in Part I [1].

For practical calculations, it is convenient to use the midpoint rule $\xi = 0$ in which case $\mathbf{K}_{\theta h} = \frac{9}{4}\mathbf{K}_{\theta h}(0)$ for $\rho_2 = 1$, $\rho_3 = \frac{1}{2}$, others zero. If these are replaced in (34), the matrices $\mathbf{Q}_{bi}$ reduce to the simple form

$$\mathbf{Q}_{b1} = \begin{bmatrix} 0 & -\chi_{21|1} & 0 \\ 0 & \frac{1}{2}\chi_{32|1} & -\frac{1}{2}\chi_{32|1} \\ 0 & 0 & \chi_{13|1} \end{bmatrix}, \quad \mathbf{Q}_{b2} = \begin{bmatrix} \chi_{21|2} & 0 & 0 \\ 0 & 0 & -\chi_{32|2} \\ -\frac{1}{2}\chi_{13|2} & 0 & \frac{1}{2}\chi_{13|2} \end{bmatrix}, \quad \mathbf{Q}_{b3} = \begin{bmatrix} \frac{1}{2}\chi_{21|3} & -\frac{1}{2}\chi_{21|3} & 0 \\ 0 & \chi_{32|3} & 0 \\ -\chi_{13|3} & 0 & 0 \end{bmatrix}. \tag{51}$$

The seven-interior-point quadrature rule was also tried, but then it was found impossible to construct an energy-balanced element. Because this rule accounts for quadratic strain variations in the torsional mode, the foregoing negative result suggests that linear strain variations are required to attain an optimal element.

3.2. The Combined Natural Strain Field.

Having chosen the optimal ρ coefficients and the midpoint integration rule, it is possible to obtain the complete higher-order natural strain field. This is done by combining the bending matrices (51) with the filtered torsional strain expression (45):

$$\epsilon_d = (\mathbf{Q}_b\mathbf{J}' - \mathbf{Q}_t^m\bar{\mathbf{J}})\,\tilde{\boldsymbol{\theta}} = \mathbf{Q}_d\tilde{\boldsymbol{\theta}} = (\mathbf{Q}_{d1}\zeta_1 + \mathbf{Q}_{d2}\zeta_2 + \mathbf{Q}_{d3}\zeta_3)\tilde{\boldsymbol{\theta}}, \tag{52}$$

where

$$\mathbf{Q}_{d1} = \begin{bmatrix} -\chi_{21|1} & -2\chi_{21|1} & -\chi_{21|1} \\ 0 & \frac{1}{2}\chi_{32|1} & -\frac{1}{2}\chi_{32|1} \\ \chi_{13|1} & \chi_{13|1} & 2\chi_{13|1} \end{bmatrix}, \quad \mathbf{Q}_{d2} = \begin{bmatrix} 2\chi_{21|2} & \chi_{21|2} & \chi_{21|2} \\ -\chi_{32|2} & -\chi_{32|2} & -2\chi_{32|2} \\ -\frac{1}{2}\chi_{13|2} & 0 & \frac{1}{2}\chi_{13|2} \end{bmatrix},$$
$$\mathbf{Q}_{d3} = \begin{bmatrix} \frac{1}{2}\chi_{21|3} & -\frac{1}{2}\chi_{21|3} & 0 \\ \chi_{32|3} & 2\chi_{32|3} & \chi_{32|3} \\ -2\chi_{13|3} & -\chi_{13|3} & -\chi_{13|3} \end{bmatrix}. \tag{53}$$

Evaluation at the midpoints gives

$$\mathbf{Q}_{d4} = \begin{bmatrix} \frac{1}{2}\chi_{21|4} & -\frac{1}{2}\chi_{21|4} & 0 \\ \chi_{32|4} & 2\chi_{32|4} & \chi_{32|4} \\ -2\chi_{13|4} & -\chi_{13|4} & -\chi_{13|4} \end{bmatrix}, \quad \mathbf{Q}_{d5} = \begin{bmatrix} -\chi_{21|5} & -2\chi_{21|5} & -\chi_{21|5} \\ 0 & \frac{1}{2}\chi_{32|5} & -\frac{1}{2}\chi_{32|5} \\ \chi_{13|5} & \chi_{13|5} & 2\chi_{13|5} \end{bmatrix},$$
$$\mathbf{Q}_{d6} = \begin{bmatrix} 2\chi_{21|6} & \chi_{21|6} & \chi_{21|6} \\ -\chi_{32|6} & -\chi_{32|6} & -2\chi_{32|6} \\ -\frac{1}{2}\chi_{13|6} & 0 & \frac{1}{2}\chi_{13|6} \end{bmatrix} \tag{54}$$

where $\chi_{ji|4} = \frac{1}{2}(\chi_{ji|1} + \chi_{ji|2})$, etc. Note that the structure of $\mathbf{Q}_{d4}$, $\mathbf{Q}_{d5}$, $\mathbf{Q}_{d6}$ mimics that of $\mathbf{Q}_{d3}$, $\mathbf{Q}_{d1}$, and $\mathbf{Q}_{d2}$, respectively, the only change being the evaluation point.

3.3. Fast Computation of $\mathbf{K}_h$.

With the explicit strain expressions of Section 3.2 we are now in a position to try for the fastest computation of $\mathbf{K}_h$. For this we proceed as follows. First evaluate

$$\mathbf{E}_n = \mathbf{T}^T \mathbf{E} \mathbf{T}, \tag{55}$$

which may be interpreted as a stress-strain matrix in natural coordinates. Then apply the midpoint rule, which for uniform thickness h yields

$$\mathbf{K}_{\theta h} = \frac{9}{4} \frac{Ah}{3} (\mathbf{Q}_{d4}^T \mathbf{E}_n \mathbf{Q}_{d4} + \mathbf{Q}_{d5}^T \mathbf{E}_n \mathbf{Q}_{d5} + \mathbf{Q}_{d6}^T \mathbf{E}_n \mathbf{Q}_{d6}) \tag{56}$$

Finally, transform to physical coordinates via (20), in which advantage should be taken of the special form (18) of $\mathbf{H}_{\theta v}$. These are essentially the same computational steps described in Appendix 2 of [19] for the ‘AQR’ ANDES plate bending triangle.

4. CONCLUDING REMARKS

We have presented the derivation of a plane stress triangle with drilling freedoms using the assumed natural deviatoric strain (ANDES) formulation. It is somewhat surprising that the optimal choice in the energy-balance sense described in Section 2 of Part III [20] coalesces with the optimal EFF element. This result suggest that this may in fact be the best available triangular element with the present freedom configuration.

Numerical integration is seen to play a crucial role in achieving an optimal element. The key effect is the function of the 3-point rule as a *strain filtering device* for the torsional mode. Note that strain filtering was not needed for the EFF derivation in Part I, which dealt throughout with quadratic displacements and linear strains.

Despite the coalescence, the ANDES derivation displays a different flavor than EFF. The formulation offers greater flexibility in that one is not restricted to compatible strain fields, allowing element developers to bypass detailed kinematic analysis. By way of contrast, the present element was formulated in two months whereas the derivation of the final EFF form took over one year. The difference may become more appreciable as one proceeds to shells and solid elements.

On the other hand, FF and EFF do provide explicitly the internal displacement field. This knowledge is useful in the calculation of consistent node force vectors — a topic further treated in Sections 3–4 of Part III — consistent mass matrices, and geometric stiffness matrices. In cases where the same element is available from both assumed-strain and assumed-displacement formulations (the present element as well as DKT being examples), one would prefer the latter for tasks that demand knowledge of internal displacements.

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