Calculating the exact compensating variation in logit and nested-logit models with income effects: theory, intuition, implementation, and application ⋆

Anders Karlstrom [a] and Edward R. Morey [b]

[a] *Systems Analysis and Economics, Royal Institute of Technology, SE– 100 44 Stockholm, Sweden. email: AndersK@infra.kth.se*

[b] *Department of Economics, Campus Box 256, University of Colorado Boulder CO 80309, USA. email: Edward.Morey@colorado.edu web: http://www.colorado.edu/Economics/morey/index.html*

Abstract

An exact formula for the expected compensating variation is derived for logit and nested-logit models with income effects. Intuition, examples, and an application are provided. The appendix contains a formal proof. The formula is applied to estimate the E[cv]s salmon anglers in Maine would associate with changes in catch rates at Maine and Canadian Rivers.

Key words: expected compensating variation, income effects, expected expenditure formula

⋆ Presented at the meeting of the American Economic Association, New Orleans, January 2001. It is a substantially revised version of Karlstrom (1998). Daniel McFadden provided encouragement and discussion. The comments of Michael Hanemann, Cathy Kling, Kivanc Kirgiz, Roger von Haefen, the editor and three referees have significantly improved the paper.The first author would also like to thank professor Folke Snickars for his encouragement, and thank Föreningssparbanken and the Royal Institute of Technology, Sweden, for providing financial support for this research. A special thank goes to Tony E. Smith for pointing out non-rigorous arguments, see the Appendix.

1 Introduction

Ever since the derivation of the random utility framework in the early 1970s, it has been explicitly or implicitly recognized that there is no exact formula for the expected compensating variation, $E[cv]$, in such models if they include income effects (a non-constant marginal utility of money). Here, a formula is derived that one can use to calculate the exact compensating variation in discrete-choice logit and nested-logit models. We refer to this formula as the *expected expenditure formula*. The formula involves solving only a finite one-dimensional integral with an analytical integrand. We explain how to apply the formula to estimate $E[cv]$ for both price and quality changes in a number of examples, provide a formal proof, and an intuitive explanation of why it works. The formula is then used to estimate the $E[cv]$s salmon anglers in Maine would associate with a reduction in the catch rates in the Penobscot River, an increase in the Penobscot catch rates, and this increase combined with decreased catch rates at the other Maine salmon rivers. This is a first application of the formula.

An earlier draft of this paper (Karlstrom (1998)) stated and proved the theorem, but did not include an application or provide much in the way of intuition. In a related paper, Dagsvik and Karlstrom (2004) derive a more general result than that presented here. [1]

Section 2 outlines the discrete-choice random-utility model. Section 3 presents the formula and provides an intuitive proof in the context of a simple three alternative logit model with income effects. The policy initially considered is an increase in the price of the first alternative (a deterioration). Section 4 is a numerical example. Section 5 considers a multiple price change: one price decreases and one increases. Section 6 presents the general form of the formula: J alternatives and any change in the quality and price vectors. This is Theorem 1. In Section 7, the formula is first applied to estimate the $E[cv]$s salmon anglers in Maine would associate with a reduction in the catch rates in the Penobscot River, a quality change, then applied to a more complicated scenario. Section 8 summarizes.The appendix includes a formal proof of Theorem 1. Code and data to estimate the $E[cv]$s using the expected expenditure formula can be obtained at http://www.colorado.edu/Economics/morey/expectedcv.html.

[1] Their context is all discrete-choice random-utility models with additive error terms. They derive the distribution of the minimum expenditures required to achieve the utility level associated with some original income, prices and qualities, but at some new vector of prices and qualities. The expected value of that distribution is what is used to calculate the $E[cv]$. Dagsvik and Karlstrom provide no examples and little in the way of intuition.

2 The discrete-choice random-utility model, the cv and the $E[cv]$

Assume a standard discrete-choice random utility model. That is, there are J alternatives and the individual is constrained to choose one of those J alternatives. Income not spent on the alternative is spent the numeraire. The utility associated with alternative j is

$$u_j = v_j + \varepsilon_j \quad j = 1, ..., J. \tag{1}$$

The choice problem is completely deterministic from the individual perspective and the individual chooses the alternative that provides the most utility. From the researcher's perspective, the realized $\boldsymbol{\epsilon} = (\epsilon_1, ..., \epsilon_n)$ are unobserved but a random draw from some known joint density function $f^{\epsilon}(\boldsymbol{\epsilon})$. Assuming budget exhaustion,

$$v_j = v(y - p_j, \mathbf{X}_j, \mathbf{C}) \tag{2}$$

where y is income, p_j is price/cost of alternative j, $\mathbf{X}_j$ is a vector of the characteristics of alternative j and $\mathbf{C}$ is vector of the characteristics of the individual. That is, utility from alternative j is a function of expenditures on the numeraire and the characteristics of alternative j (possibly interacted with characteristics of the individual).

Let

$$\begin{aligned} U \equiv & \max\{u_1, ..., u_J\} \\ = & \max\{[v((y - p_1), \mathbf{X}_1, \mathbf{C}) + \varepsilon_1], ..., [v(y - p_j, \mathbf{X}_J, \mathbf{C}) + \varepsilon_J]\} \end{aligned} \tag{3}$$

U is observed by the individual but not by the researcher; it is a random variable from the researcher's perspective. Denote its expectation $E[U]$.

Consider a change from some initial state $\{y^0, \mathbf{p}^0, \mathbf{X}^0\}$ to some proposed state $\{y^0, \mathbf{p}^1, \mathbf{X}^1\}$. The individual's compensating variation, cv, is that amount of money, l, such that

$$U^0 = \max\{[v((y^0 - l - p_1^1), \mathbf{X}_1^1, \mathbf{C}) + \varepsilon_1], ..., [v((y^0 - l - p_J^1), \mathbf{X}_J^1, \mathbf{C}) + \varepsilon_J]\} \tag{4}$$

where [2]

$$U^0 = \max\{[v((y^0 - p_1^0), \mathbf{X}_1^0, \mathbf{C}) + \varepsilon_1], ..., [v((y^0 - p_J^0), \mathbf{X}_J^0, \mathbf{C}) + \varepsilon_J]\}$$

cv is the amount money that when subtracted from income in the proposed state equates maximum utility in the proposed state with maximum utility in

[2] Note the typical assumption that the magnitudes of the random terms are not affected by the policy; that is $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^0 = \boldsymbol{\varepsilon}^1$.

the initial state.[3] It is deterministic from the individual's perspective but a random variables from the researcher's perspective. Denote its density function $f_{cv}(cv)$ and its expectation $E[cv]$. Calculation of $E[cv]$ is the goal of much welfare analysis.

If one restrictively assumes the utility from alternative j is a linear function of expenditures on the numeraire ($v_j = \beta_0(y - p_j) + g(\mathbf{X}_j, \mathbf{C})$), one has the classic *no income-effects* model. Put simply, in this case the marginal utility from expenditures on the numeraire is a constant, β_0 and $E[cv]$ often has a closed-form solution. For example, if one assumes $f_\varepsilon(\varepsilon)$ has an Gumbel extreme value distribution, $E[cv]$ has the well-known log-sum formula

$$\begin{aligned} E[cv] &= \frac{1}{\beta_0}\left[E[U(y^0, \mathbf{p}^0, \mathbf{X}^0)] - E[U(y^0, \mathbf{p}^1, \mathbf{X}^1)]\right] \\ &= \frac{1}{\beta_0}\left[\ln \sum_{j=1}^{J} e^{V(y^0, p_j^0, \mathbf{X}_j^0)} - \ln \sum_{j=1}^{J} e^{V(y^0, p_j^1, \mathbf{X}_j^1)}\right] \end{aligned} \tag{5}$$

In the case of no income effects, cv is not a function of y; it drops out of the equation.

Income effects exist if the marginal utility of expenditures on the numeraire varies as a function of the level of expenditures on the numeraire and/or varies as a function of the alternative chosen. In such cases, cv varies as a function of y, what one would typically expect. This makes the income-effects discrete-choice RUM of most interest. However, in the case of income effects, there has been no exact formula for the $E[cv]$, only approximations. When the marginal utility of the numeraire varies with the level of income and/or the alternative chosen, things are complicated because the rate at which utility is converted into money typically varies with the alternative chosen in each of the states and their prices. The next section calculates the $E[cv]$ for a price increase in the presence of income effects using the new expected expenditure formula.

In the case of income effects, the practice has been to either assume away income effects, or assume income effects and approximate the $E[cv]$ using the representative consumer approximation, cv^r, where cv^r is the amount of money that when subtracted from income in the new proposed state equates *expected* maximum utility in the new state with *expected* maximum utility in the initial state.[4] McFadden (1999) demonstrates that the representative

[3] We have adopted the convention of defining cv so that it is positive for improvements and negative for deteriorations.

[4] cv^r is the amount of money l such that $E[\max\{[v((y^0 - l - p_1^1), \mathbf{x}_1^1, \mathbf{c}) + \varepsilon_1], ..., [v((y^0 - l - p_J^1), \mathbf{x}_J^1, \mathbf{c}) + \varepsilon_J]\}] = E[\max\{[v((y^0 - p_1^0), \mathbf{x}_1^0, \mathbf{c}) + \varepsilon_1], ..., [v((y^0 - p_J^0), \mathbf{x}_J^0, \mathbf{c}) + \varepsilon_J]\}]$. See e.g. Viton (1985), Hau (1985), Jara-Diaz and Videla (1989, 1990), Morey et al. (1993), Oppenheim (1995), Verboven (1996) and Morey (1999).

individual approximation can yield a poor approximation if the welfare impact of the policy being valued is large. The representative approximation can be a good approximation; the problem is that one does not know whether it is or not, in a particular context, until one calculates $E[cv]$ exactly.

In response, McFadden (1999) developed an approximation based on simulation. Put simply, for each individual in the sample, one randomly draws an ε vector and calculates the individual's exact cv conditional on that draw. One repeats this hundreds or thousand of times and then approximates the individual's $E[cv]$ with the average of the conditional cv's. This approximation converges to the exact $E[cv]$ as the number of draws increases. McFadden developed a technique to take random draws from a GEV density function. The simulation approximation is relatively easy to implement if each element of the ε vector is independent (for example, in the case of the logit model) but complicated if the ε are not independent. In that case, one must take random draws from a density function of dimension J. This becomes increasingly less tractable as the number of alternatives increases and as the nature of the covariances between the ε becomes more complex. McFadden (1999) and Herriges and Kling (1999) both demonstrate the simulation technique with a small number of alternatives.

As noted by McFadden (2001), the expected expenditure formula for calculating the $E[cv]$ dominates his simulation approximation, and other approximations. It is exact.

3 Calculating the $E[cv]$ with income effects: a simple logit example and a little intuition

This section introduces the expected expenditure formula. Consider a simple three-alternative logit model with simple income effects:

$$u_1 = \beta(y - p_1)^2 + \varepsilon_1 \tag{6}$$

$$u_2 = \beta(y - p_2)^2 + \varepsilon_2 \tag{7}$$

$$u_3 = \beta(y - p_3)^2 + \varepsilon_3 \tag{8}$$

For the calculation of $E[cv]$, denote the initial state $\{y^0, p_1^0, p_2^0, p_3^0\}$ and the proposed state $\{y^0, p_1^1, p_2^0, p_3^0\}$,where $p_1^1 > p_1^0$. That is, things get worse because the price of alternative 1 increases.

First consider the cv rather than its expectation. Depending on the ε draw, an individual will fall into one of three groups:

- Group A: Individuals who do not choose alternative 1 either before or after its price increases.

- Group B: Individuals that choose alternative 1 both before and after its price increases.
- Group C: Individuals that choose alternative 1 before its price increases but not after it increases.

Define m as the expenditure level required to keep an individual at his original utility level after p_1 has increased. It is a random variable with density $f_m(m)$. For individuals in group A, $cv = 0$, or said another way, the expenditure level, m, they need to remain at their original utility level is y^0; that is, $cv = y^0 - y^0$. For individuals in group B, $cv < 0$ and equal to $p_1^0 - p_1^1$. If in group B, the expenditure level required to keep the individual at his original utility level is $m = y^0 + (p_1^1 - p_1^0)$, so $cv = y^0 - (y^0 + (p_1^1 - p_1^0)) = p_1^0 - p_1^1 < 0$. If an individual is in group C, his cv is between 0 and $(p_1^0 - p_1^1)$. In terms of the required levels of expenditures, an individual in group C will require expenditures greater than y^0 and less than $y^0 + (p_1^1 - p_1^0)$. To simplify the notation, let $y^0 + (p_1^1 - p_1^0) \equiv \mu$.

3.1 The $E[cv]$ in terms of $E[m]$, expected compensating expenditures

Turn now to $E[cv]$. Writing the $E[cv]$ as a function the expectation of the expenditure level required to compensate for the price increase, $E[m]$,

$$E[cv] = y^0 - E[m] \tag{9}$$

One proceeds by deriving $E[m]$. The expectation of this level of expenditures can be decomposed into three terms

$$E[m] = c_A + c_B + c_C \tag{10}$$

where c_υ is the contribution to $E[m]$ from those individuals whose epsilon draw causes them to fall in group υ, $\upsilon = A, B, C$. Each of the c_υ can be expressed as the level of expenditures required to make the individual whole, conditional on the individual belonging to group υ, multiplied by the probability that with that level of expenditures the individual will belong to groupt υ.

All individuals in group A require the same expenditure level, $m = y^0$, to make them whole in the new state, so

$$c_A = \Pr(\text{in } A\text{: } y^0) y^0 \tag{11}$$

where $\Pr(\text{in } A\text{: } y^0)$ is the probability that the individual chooses alternative 2 or 3 (not 1) both before and after the price increase with expenditures level y^0 in the original state and with expenditure level y^0 in the new state. The form of $\Pr(\text{in } A\text{: } y^0)$, assuming ε is Gumbel extreme value distributed, is derived below.

Likewise, all individuals in group B require the same expenditure level, μ, to make them whole in the new state, so

$$c_B = \Pr(\text{in } B\text{: } \mu)\mu \tag{12}$$

where $\Pr(\text{in } B\text{: } \mu)$ is the probability that the individual chooses alternative 1 with its higher price and expenditure level μ. $\Pr(\text{in } B\text{: } \mu)$ is formally derived below. Note that $\Pr(\text{in } A\text{: } y^0)$ and $\Pr(\text{in } B\text{: } \mu)$ are both calculated at the total level of expenditures required to make the individual whole: y^0 if the individual is in group A, and μ if the individual is in group B.

Substituting Equations 11 and 12 into Equation 10

$$E[m] = \Pr(\text{in } A\text{: } y^0)y^0 + \Pr(\text{in } B\text{: } \mu)\mu + c_C \tag{13}$$

The derivation of c_C is complicated by the fact that the expenditure level required to make individuals in group C whole is not a constant but rather varies between y^0 and μ as a function of the individuals specific epsilon draw. Derivation of $E[m]$ is derived by deriving c_A, c_B and c_C.

3.2 The density function of compensating expenditures, $f(m)$

$E[m]$ and its three components are derived in the context of the density function of m, $f_m(m)$. $f_m(m)$, denoted for simplicity $f(m)$, is the distribution of the level of expenditures required to make whole the individuals in the population after the price of alternative 1 has increased. It is a mixed discrete-continuous distribution with mass in the range $y^0 \leq m \leq \mu$ with spikes at $m = y^0$ and $m = \mu$, associated with groups A and B, respectively. Specifically,

$$f(m) = \begin{cases} 0 \text{ if } m < y^0 \text{ or } m > \mu \\ \Pr(\text{in } A : y^0) \text{ if } m = y^0 \\ \Pr(\text{in } B : \mu) \text{ if } m = \mu \\ > 0 \text{ and varies with } m \text{ if } y^0 < m < \mu \end{cases} \tag{14}$$

Figure 1A is an example $f(m)$. It was drawn to conform with the numerical example in the next section. In explanation, if $m < y^0$ no one is made whole with expenditures of m. If $m = y^0$, everyone in group A is made whole - they need no extra money because they are not affected by the price increase. As m increases between y^0 and μ, more and more of group C is made whole. If $m = \mu$, everyone, including those in group B are made whole. Note that $[1 - \Pr(\text{in } A : y^0) - \Pr(\text{in } B : \mu)]$ is the probability that, with compensation, an individual is in group C.

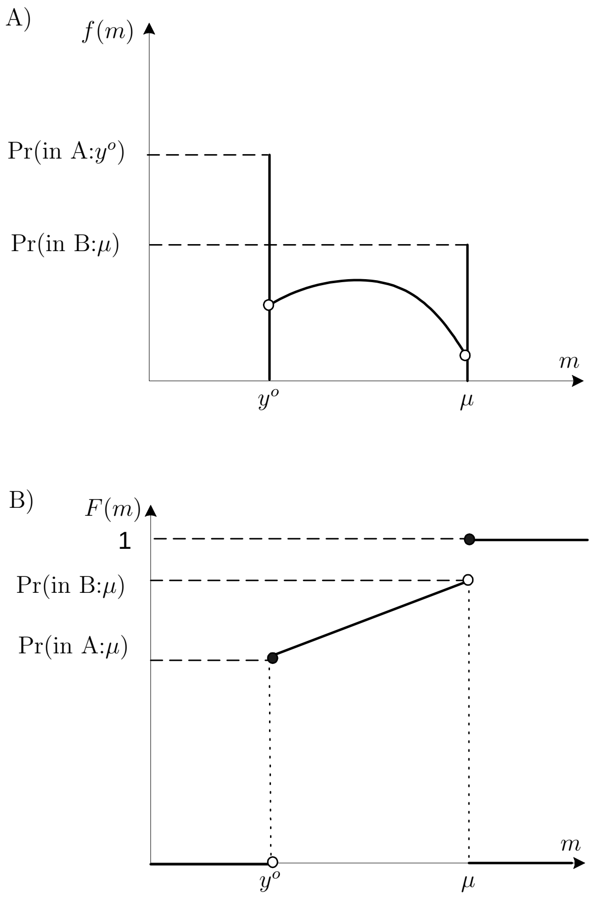

Fig. 1. An example of $f(m)$ and $F(m)$.

The cdf of m, $F(m)$ has the following properties[5]

$$F(m) = \begin{cases} 0 \text{ if } m < y^0. \\ \Pr(\text{in } A : y^0) \text{ if } m = y^0. \\ \text{increases from } \Pr(\text{in } A : y^0), \\ \text{approaching } [1 - \Pr(\text{in } B : \mu)] \text{ as } m \text{ increases, if } y^0 < m < \mu. \\ 1 \text{ if } m \geq \mu. \end{cases} \tag{15}$$

Figure 1B is the cdf, $F(m)$ corresponding to $f(m)$, where $dF(m) = f(m)dm$. $F(m)$ is the proportion of the population that is compensated for the increase in the price of alternative 1 with an expenditure level of m.[6]

[5] Note that $[1 - \Pr(\text{in } B : \mu)]$ is the probability that the individual is in A or C.
[6] Note that the section between y^0 and μ will not typically be linear.

$E[m]$ in terms of $f(m)$ is

$$\begin{aligned}E[m] &= \Pr(\text{in } A\text{: } y^0)y^0 + \Pr(\text{in group } B\text{: } \mu)\mu + c_C \\ &= \Pr(\text{in } A\text{: } y^0)y^0 + \Pr(\text{in group } B\text{: } \mu)\mu \\ &+ \int_{>y^0}^{<\mu} mf(m)dm\end{aligned} \tag{16}$$

where $c_C = \int_{>y^0}^{<\mu} mf(m)dm = E[m : \text{in } C] \cdot [1 - \Pr(\text{in } A : y^0) - \Pr(\text{in } B : \mu)]$; that is, c_C is our expectation of the expenditure level required to compensate those in group C, multiplied by the probability that the individual is, with compensation, in group C.

Concentrating on the last term in Equation 16 and using the fact that $f(m) = \frac{dF(m)}{m}$

$$c_C = \int_{>y^0}^{<\mu} mf(m)dm = \int_{>y^0}^{<\mu} m\frac{dF(m)}{dm}dm \tag{17}$$

Writing this in terms of $S(m) \equiv 1 - F(m)$, where $S(m)$ is the proportion of the population that is not compensated with an expenditure level of m.

$$c_C = \int_{>y^0}^{<\mu} m\frac{dF(m)}{dm}dm = \int_{>y^0}^{<\mu} m\frac{d[1 - S(m)]}{dm}dm = -\int_{>y^0}^{<\mu} m\frac{dS(m)}{dm}dm \tag{18}$$

As we soon show, it is easy to determine $\frac{dS(m)}{dm}$ and then integrate Equation 18 to obtain c_C. $S(m)$ has the following properties

$$S(m) = \begin{cases} 1 \text{ if } m < y^0 \text{ (everyone uncompensated)} \\ 1 - \Pr(\text{in } A : y^0) \text{ if } m = y^0 \text{ (those in } B \text{ and } C \text{ are uncompensated)} \\ \text{declines from } [1 - \Pr(\text{in } A : y^0)] \text{ approaching } \Pr(\text{in } B : \mu) \text{ , if } y^0 < m < \mu \\ 0 \text{ if } m \geq \mu \text{ (everyone compensated)} \end{cases} \tag{19}$$

Figure 2A is obtained by flipping Figure 1B vertically and horizontally. Figure 2A is the $S(m)$ corresponding to Figures 1A and B. $S(m)$ can be viewed as a survival function - survivors (those who remain uncompensated) decline as m increases; none remain when $m = \mu$.

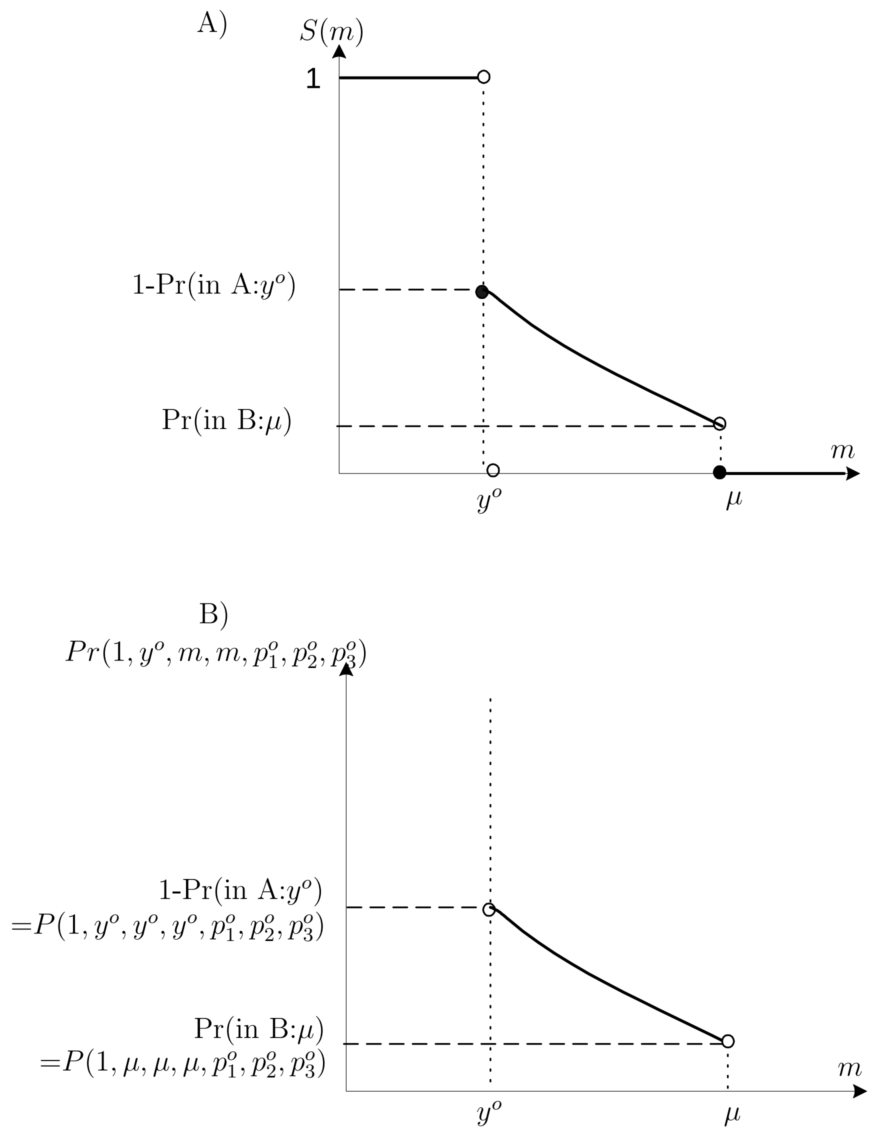

Fig. 2. Figure A) $S(m)$ - the survivor function. Figure B) $Pr(1 : y^o, m, m, p_1^o, p_2^o, p_3^o)$.

3.3 The derivation of $\Pr($*in A:* $y^0)$, $\Pr($*in group C:* $\mu)$ *and* c_C *assuming* $f_\varepsilon(\varepsilon)$ *is Gumbel Extreme Value*

The derivation of Pr(in A: y^0), Pr(in group C: μ) and c_C requires that one specify $f_\varepsilon(\varepsilon)$. Consider the probability of choosing alternative k as a function of different expenditure levels depending on the alternative chosen. Denote this probablity $\Pr(k : m_1, m_2, m_3, p_1, p_2, p_3)$ where m_i is the level of expenditures if the individual chooses alternative i. Assuming the ε are independent draws from the Gumbel Extreme Value distribution

$$\Pr(k : m_1, m_2, m_3, p_1, p_2, p_3) \tag{20}$$
$$= \frac{\exp(\beta(m_k - p_k)^2)}{\exp(\beta(m_1 - p_1)^2) + \exp(\beta(m_2 - p_2)^2) + \exp(\beta(m_3 - p_3)^2)}$$

This is the standard logit probability that an individual will choose alternative k given expenditures m_i and price p_i associated with alternative i.[7] In explanation, imagine that one's budget, in addition to the price, varies as a function of the alternative consumed.

To calculate $\Pr(\text{in } A : y^0)$ and $\Pr(\text{in } B : \mu)$, use Equation 20. Specifically,

$$\Pr(\text{in } A\text{: } y^0) = 1 - P(1 : y^0, y^0, y^0, p_1^0, p_2^0, p_3^0) \tag{21}$$
$$= 1 - \frac{\exp(\beta(y^0 - p_1^0)^2)}{\exp(\beta(y^0 - p_1^0)^2) + \exp(\beta(y^0 - p_2^0)^2) + \exp(\beta(y^0 - p_3^0)^2)}$$

Note that $P(1 : y^0, y^0, y^0, p_1^0, p_2^0, p_3^0)$ is the the simple logit probability that the individual will choose alternative 1 in the initial state, so $1 - P(1 : y^0, y^0, y^0, p_1^0, p_2^0, p_3^0)$ is the probability that an individual will initially choose one of the other alternatives; that is, he is in group A. If the individual does not choose alternative 1 in the initial state, he will not choose it after its price has increased.

Equation 20, evaluated at $(1, \mu, \mu, \mu, p_1^1, p_2^0, p_3^0)$, is the probability that the individual will stick with alternative 1 given a higher price for alternative 1, p_1^1, and expenditure level μ; that is

$$\Pr(\text{in } B : \mu) = P(1 : \mu, \mu, \mu, p_1^1, p_2^0, p_3^0)$$
$$= \frac{\exp(\beta(\mu - p_1^1)^2)}{\exp(\beta(\mu - p_1^1)^2) + \exp(\beta(\mu - p_2^0)^2) + \exp(\beta(\mu - p_3^0)^2)} \tag{22}$$

The first two terms in Equation 13 are therefore easily calculated.[8]

[7] For example, if $m_i = y^0 \; \forall i$

$$\Pr(1 : y^0, y^0, y^0, p_1, p_2, p_3)$$
$$= \frac{\exp(\beta(y^0 - p_1)^2)}{\exp(\beta(y^0 - p_1)^2) + \exp(\beta(y^0 - p_2)^2) + \exp(\beta(y^0 - p_3)^2)}$$

and if $m_1 = y^0$ and $m_1 = m_2 = m$

$$\Pr(1 : y^0, m, m, p_1, p_2, p_3)$$
$$= \frac{\exp(\beta(y^0 - p_1)^2)}{\exp(\beta(y - p_1)^2) + \exp(\beta(m - p_2)^2) + \exp(\beta(m - p_3)^2)}$$

[8] As an aside, note that it is always the case that $\Pr(1 : y^0, \mu, \mu, p_1^0, p_2^0, p_3^0)$ equals $\Pr(1 : \mu, \mu, \mu, p_1^1, p_2^0, p_3^0)$; that is, the probability that an individual will choose alter-

The derivative $\frac{dS(m)}{dm}$, what is needed to determine c_C , is also derived from Equation 20. For a price increase in p_1, as is explained below,

$$\frac{\partial S(m)}{\partial m} = \frac{\partial \Pr(1 : y^0, m, m, p_1^0, p_2^0, p_3^0)}{\partial m} \text{ for } y^0 < m < \mu \tag{23}$$

Simply put, the decrease in the proportion of population that remains uncompensated as m increases is equal to the decrease in the probability that an individual will choose alternative 1 as the expenditures on the other alternatives increases, holding constant expenditures on alternative 1 at y^0.

Given Equation (18),

$$c_C = -\int_{>y^0}^{<\mu} m \frac{\partial \Pr(1 : y^0, m, m, p_1^0, p_2^0, p_3^0)}{\partial m} dm \tag{24}$$

Note that $\frac{\partial \Pr(1:y^0,m,m,p_1^0,p_2^0,p_3^0)}{\partial m}$ is varying the level of expenditures associated with alternatives 2 and 3 (the alternatives the individual switches to), holding constant at y^0 the expenditures if alternative 1 is chosen, and prices are evaluated at their original levels.[9] Note that $\frac{\partial \Pr(1:y^0,m,m,p_1^0,p_2^0,p_3^0)}{\partial m}$ is the derivative of the probability associated with the alternative individuals will switch away from (in this case alternative 1) and m is the the expenditure level associated with the alternative they would switch to (in this case, alternatives 2 or 3). Looking ahead, this is a general result.

To understand why the right and left-hand sides of Equation 23 are equal, consider the properties of

$$\begin{aligned} &\Pr(1 : y^0, m, m, p_1^0, p_2^0, p_3^0) \\ &= \frac{\exp(\beta(y^0 - p_1^0)^2)}{\exp(\beta(y^0 - p_1^0)^2) + \exp(\beta(m - p_2^0)^2) + \exp(\beta(m - p_3^0)^2)} \end{aligned} \tag{25}$$

This is the probability that an individual will choose alternative 1 given the initial price vector, given expenditures of y^0 if he consumes alternative 1 and given expenditures of m if he consumes one of the other alternatives. It, like $S(m)$, is a survivor function where survivors are those that stick with alternative 1 as a function of m. $P(1 : y^0, m, m, p_1, p_2, p_3)$ is decreasing in m; that is,

---

native 1 at the new prices and expenditure level μ for each alternative equals the probability that alternative 1 will be chosen at the old prices, expenditure level y_0 if alternative 1 is chosen and expenditure level μ if one of the other two alternatives is chosen. There are two ways to calculate Pr(in $B : \mu$). Equivalences of this sort will sometimes prove useful when calculating an $E[cv]$.

[9] That is, for an increase in the price of only alternative 1, those in group c choose alternative 1 before its price increases, but not afterwards, so the relevant prices are p_1^0, p_2^0 and p_3^0. This will not be the case for an improvement.

the probability that an individual will choose alternative 1 declines the more he is compensated, $m - y^0 > 0$, if he chooses some other alternative [10] .

Figure 2B plots $P(1 : y^0, m, m, p_1^0, p_2^0, p_3^0)$ as a function of m for $y^0 < m < \mu$. Examining the Figure, $P(1 : y^0, y^0, y^0, p_1^0, p_2^0, p_3^0)$ is the probability that an individual initially chooses alternative 1 when $m = y^0$ and $P(1 : \mu, \mu, \mu, p_1^0, p_2^0, p_3^0)$ equals the probability associated with choosing alternative 1 at the initial prices and at the expenditure level μ.

Figure 2B is identical to the middle section of Figure 2A. $S(m)$ and $P(1 : y^0, m, m, p_1^0, p_2^0, p_3^0)$ are equal and equal to $1 - \Pr(\text{in group } A\text{: } y^0)$ at $m = y^0$. Like, $S(m)$, $P(1 : y^0, m, m, p_1^0, p_2^0, p_3^0)$ approaches $\Pr(\text{in } B : \mu)$ as m approaches μ. For $y^0 < m < \mu$, $P(1 : y^0, m, m, p_1^0, p_2^0, p_3^0)$ is the proportion of the population that sticks with alternative 1 if they are offered expenditure level m for switching to another alternative. And $S(m)$ is the proportion of the population that is compensated with expenditure level m for the higher p_1.They are equal.

$$\Pr(1 : y^0, m, m, p_1^0, p_2^0, p_3^0) = S(m), \quad y^0 < m < \mu \tag{26}$$

So

$$\frac{\partial \Pr(1 : y^0, m, m, p_1^0, p_2^0, p_3^0)}{\partial m} = \frac{\partial S(m)}{\partial m}, \quad y^0 < m < \mu \tag{27}$$

This completes the derivation of Equation 24. Plugging in the specific numerical amounts, Equation 24 can be calculated with no further simplification almost instantaneously using *Maple* or *Mathematica*.

The above example is an application of Theorem 1, presented in Section 6. The appendix provides a proof of Theorem 1.

4 A simple numerical example

Consider the following simple numerical example of the above result. For equations (6), (7) and (8), assume $\beta = .1$, $y^0 = 100$, $p_1^0 = 94.5$, $p_2^0 = 95$, $p_3^0 = 96$ and $p_1^1 = 95$. Equation 20 is used to calculate the probabilities associated with

[10] Said the other way, $1-$ $P(\cdot)$ is the probability that an individual will not choose alternative 1 as a function of the expenditure level, m, if he choses one of the other alternatives, holding constant at y^0 the expenditure level if he chooses alternative 1. He chooses another alternative if the additional expenditures he gets for choosing another alternative, $m - y^0$, are sufficient to achieve a utility level greater than or equal to the utility he would have gotten from sticking with alternative 1 with an expenditure level of y^0.

each of the alternatives in the initial state.

x	$P(x; 100, 100, 100, 94.5, 95, 96)$
1	$.54583 = 1 - \Pr(\text{in } A : 100)$
2	.32289
3	.13128

In the initial state, the probability that an individual will choose alternative 1 is .54583, so the probability that they don't is $\Pr(\text{in } A : 100) = .454\,17$. Therefore, given the numerical assumptions, the first term in equation 13 , equation 21, is

$$\begin{aligned}
\Pr(\text{in } A\text{: } y^0) &= 1 - P(1 : y^0, y^0, y^0, p_1^0, p_2^0, p_3^0) \\
&= 1 - \frac{\exp(\beta(y^0 - p_1^0)^2)}{\exp(\beta(y^0 - p_1^0)^2) + \exp(\beta(y^0 - p_2^0)^2) + \exp(\beta(y^0 - p_3^0)^2)} \\
&= 1 - \frac{\exp(.1(100 - 94.5)^2)}{\exp(.1(100 - 94.5)^2) + \exp(.1(100 - 95)^2) + \exp(.1(100 - 96)^2)} \\
&= 0.454\,17 \qquad (28)
\end{aligned}$$

so $c_A = \Pr(\text{in } A : y^0)y^0 = (0.454\,17)100 = \$45.417$

For those individuals who choose alternative 1 both before and after the change (group B), $cv = p_1^0 - p_1^1 = -.50$, and

$$\mu \equiv y^0 + (p_1^1 - p_1^0) = 100 + .50 = 100.5 \qquad (29)$$

¿From equation (22) we have

$$\begin{aligned}
\Pr(inB : 100.5) &= P(1 : 100.5, 100.5, 100.5, 95, 95, 96) \\
&= \frac{\exp(.1(100.5 - 95)^2)}{\exp(.1(100.5 - 95)^2) + \exp(.1(100.5 - 95)^2) + \exp(.1(100.5 - 96)^2)} \\
&= .42232
\end{aligned}$$

so that the second term in equation 13 is $\Pr(\text{in } B : 100.5)100.5 = (.42232)100.5 = \$42.443$.

Note that at $m = 100.5$, individuals who initially chose alternative 1 are indifferent between alternatives 1 and either 2 or 3.

Using Equation 24

$$c_C = -\int_{100}^{100.5} m\frac{\partial}{\partial m}\left(\frac{\exp(.1(100-94.5)^2)}{\exp(.1(100-94.5)^2)+\exp(.1(m-95)^2)+\exp(.1(m-96)^2)}\right)dm$$
$$= 12.383 \tag{30}$$

This integral was calculated almost instantaneously using *Maple* in the context of Scientific Word.[11]

Plugging the three terms into Equation 13

$$E[m] = \Pr(\text{in } A: \ y^0)y^0 + \Pr(\text{in } B: \ \mu_{11})\mu_{11} + c_C$$
$$= \$45.417 + \$42.443 + \$12.383 = \$100.24 \tag{31}$$

so

$$E[cv] = y^0 - E[m] = 100 - 100.24 = -\$0.24 \tag{32}$$

That is, the $E[cv]$ associated with increasing p_1 from \$94.50 to \$95.00 is $-\$0.24$.

Figures 1 and 2 were drawn with this numerical example in mind. This example has been simple but should be sufficient to demonstrate how one proceeds with a number of more general models. For the moment, continue to assume a change that involves only a deterioration in alternative 1.

(1) Adding in characteristics of the alternatives or individual-specific characteristics only adds additional terms to the conditional indirect utility functions. For example, $u_j = \beta(y - p_j)^2 + \varepsilon_j$ is replaced by $u_j = \beta_0(y^0 - p_j)^2 + \boldsymbol{\beta}_1'\mathbf{X}_j + \varepsilon_j$. In which case, $\beta(y^0 - p_j)^2$ will everywhere be replaced by $\beta_0(y^0 - p_j)^2 + \boldsymbol{\beta}_1'\mathbf{X}_j$, and $\beta(m - p_j)^2$ with $\beta_0(m - p_j)^2 + \boldsymbol{\beta}_1'\mathbf{X}_j$. Generalizing in this way allows one to consider deteriorations in alternative 1 that involve either a price increase, a quality decrease, or both. Note that generalizing in this way, does not affect the form of Equation 24 and allows one to calculate $E[cv]$ for a deterioration in quality.

(2) One can easily change the nonlinear way in which $(y - p_j)$ enters into the conditional indirect utility functions, Equations 6 through 8. For example, if in the previous numerical example, one replaces $.1(y - p_j)^2$ with $.1(y - p_j)^{.5}$, the $E[cv]$ for increasing p_1^0 frin 94.5 to 95 is $-\$0.17$, rather than $-\$0.24$. If one assumes $.05(y - p_j) + .05(y - p_j)^{.5}$, $E[cv]$ drops to $-\$0.07$.

[11] Note that $c_C = 12.383 = E[m : \text{in } C] \cdot [1 - .32289 - .13128 - .42232] = E[m : \text{in group } C] \cdot .12351$. Which implies that $E[m : \text{in group } C] = 100.26$. That is, the expected value of the expenditure level required to make inidividuals in group C whole is \$100.26, which, as required, is greater than the level of expenditures required to make whole those who never choose to consume alternative 1, and less than the amount required to compensate those that chose alternative 1 even after its price increased.

(3) Increasing the number of other alternatives adds additional terms to the denominator of equation 20, one for each additional alternative. This results in additional terms in $\frac{\partial P(1{:}y^0, m, p_1^0, p_2^0, \ldots, p_N^o)}{\partial m}$, but does not fundamentally change the form of the solution, or the fact that one has to calculate only a single integral, Equation 24. It just has more terms.

(4) Assuming a nested-logit model, rather than a simple logit model increases the complexity of the equation 20 probabilities, but the nested-logit form of these probabilities is well known.

5 An improvement for some and a deterioration for others

Consider now a policy that affects two alternatives: an increase in the price of alternative 1, $p_1^1 > p_1^0$ and a decrease in the price of alternative 2, $p_2^1 < p_2^0$.[12] Whether it is an improvement for an individual will depend on the magnitude of the two price changes and the individual's ε draw. It can be an improvement for some and a deterioration for others. Continue to assume the ε are independent draws from an Gumbel Extreme Value distribution. Depending on the ε draw, an individual will fall into one of five groups:

- Group A: Individuals that choose alternative 3 both before and after the changes.
- Group B: Individuals that choose alternative 1 both before and after its price increases.
- Group C: Individuals that choose alternative 2 both before and after its price decreases.
- Group D: Individuals that switch from alternative 3 to alternative 2 (no one will switch from 3 to 1)[13]
- Group E: Individuals that switch away from alternative 1 (to either 2 or 3).[14]

For individuals in group A, $cv = 0$ and $m = y^0$. For individuals in group B, $cv = p_1^0 - p_1^1 < 0$ and the expenditure level required to keep the individual at his original utility level is $y^0 + (p_1^1 - p_1^0) \equiv \mu_{\max} > y^0$. For individuals in

[12] Note that an $p_2^1 < p_2^0$ and $p_1^1 = p_1^0$ (an improvement for all) is a special case. A section on such an improvement was deleted from this paper. A copy of the section can be found at the previously mentioned web link.

[13] More generally, group D includes individuals that initially choose one of the alternatives that is not impacted by the policy and then switches to one of the impacted alternatives, one that has improved.

[14] In the special case $p_2^1 < p_2^0$ and $p_1^1 = p_1^0$, these five groups would still all exist but Group E would include only those who switch from alternative 1 to alternative 2. No one would switch from 1 to 3 due to a decrease in the price of alternative 2.

group C, $cv = p_2^0 - p_2^1 > 0$. If in group C, the expenditure level required to keep the individual at his original utility level, u^o, is $y^0 + (p_2^1 - p_2^0) \equiv \mu_{\min} < y^0$. Note that the density function, $f(m)$, for this scenario has three spikes.

For individuals in group D, $(p_2^0 - p_2^1) > cv > 0$. To be made whole, an individual in group D will require expenditures greater than $\mu_{\min}$ and less than y^0.

Individuals in group E can be decomposed into those switching to alternative 2, and those that switch to alternative 3. Those switching to the unaffected alternative, 3, are made worse off by the change, $cv < 0$. Some that switch to alternative 2 find the change an improvement, some a deterioration. To be made whole, individuals in group E that consider the change an improvement require an expenditure greater than $\mu_{\min}$ and less y^0. Those that find it a deterioration require an expenditure level greater than y^0 and less than $\mu_{\max}$.

Expanding on Equation 10,

$$E[m] = c_A + c_B + c_C + c_D + c_E$$

Given the above definitions of the 5 groups, all individuals in group A require the same expenditure level, y^0, to make them whole in the new state, so,

$$c_A = \Pr(\text{in } A : y^0) y^0$$

where [15]

$$\Pr(\text{in } A : y^0) = P(3 : y^0, y^0, y^0, p_1^0, p_2^1, p_3^0) \tag{33}$$

Likewise, all individuals in group B require the same expenditure level, $\mu_{\max}$, to make them whole in the new state, so

$$c_B = \Pr(\text{in } B\text{: } \mu_{\max}) \mu_{\max}$$

where [16]

$$\Pr(\text{in } B : \mu_{\max}) = P(1 : y^0, \mu_{\max}, \mu_{\max}, p_1^0, p_2^1, p_3^0) \tag{34}$$

All individuals in group C require the same expenditure level, $\mu_{\min}$, to make them whole in the new state, so

$$c_C = \Pr(inC : \mu_{\min}) \mu_{\min}$$

[15] Note that this probability is evaluated at the lowest price for both alternatives 1 and 2, even though they do not occur simaltaneously.

[16] Note that $P(1 : y^0, \mu_{\max}, \mu_{\max}, p_1^0, p_2^1, p_3^0) = P(1 : \mu_{\max}, \mu_{\max}, \mu_{\max}, p_1^1, p_2^1, p_3^o)$; that is, the probability that the individual will choose alternative 1 at the new prices with expenditure level y^0 if alternative 1 is chosen, and expenditure level $\mu_{\max}$ if one of the other alternatives is chosen, is equivelant to the probability that alternative 1 is chosen at the higher price for alternative 1, the lower price for alternative 2, and an expenditure level of $\mu_{\max}$ on each alternative.

where[17]

$$\Pr(inC : \mu_{\min}) = P(2 : y^0, \mu_{\min}, y^0, p_1^0, p_2^1, p_3^0) \tag{35}$$

For group D, those who switch away from alternative 3,

$$c_D = -\int_{>\mu_{\min}}^{<y^0} m \frac{\partial P(3 : y^0, m, y^0, p_1^0, p_2^1, p_3^0)}{\partial m} dm \tag{36}$$

where alternative 2 is the alternative they switch to. For group E, those who switch away from alternative 1,

$$-c_E = \int_{>\mu_{\min}}^{<y_o} m \frac{\partial P(1 : y^0, m, y^0, p_1^0, p_2^1, p_3^0)}{\partial m} dm + \int_{>y^0}^{<\mu_{\max}} m \frac{\partial P(1 : y^0, m, m, p_1^0, p_2^1, p_3^0)}{\partial m} dm \tag{37}$$

The first term is the contribution to c_E from those in group E that find the change an improvement (a subset of those that switch from alternative 1 to 2). The second term is associated with individuals in group E that need an increased expenditure to be made whole.[18] This subset of group E can switch either to alternative 2 or 3, so the partial is taken with respect to expenditure level associated with alternative 2 and 3.

For equations (6)-(8) consider an example where, $\beta = .1$, $y^0 = 100$, $p_1^0 = 90$, $p_2^0 = 92$, $p_3^0 = 91$, $p_1^1 = 91$ and $p_2^1 = 91.5$, so $\mu_{\min} = 99.5$ and $\mu_{\max} = 101$. In which case,

$$\begin{aligned} \Pr(\text{in } A : 100) &= P(3 : 100, 100, 100, 90, 91.5, 91) \\ &= \frac{\exp(.1(100-91)^2)}{\exp(.1(100-90)^2) + \exp(.1(100-91.5)^2) + \exp(.1(100-91)^2)} \\ &= 0.123\,41 \end{aligned} \tag{38}$$

$$\begin{aligned} \Pr(\text{in } B : 101) &= P(1 : 100, 101, 101, 90, 91.5, 91) \\ &= \frac{\exp(.1(100-90)^2)}{\exp(.1(100-90)^2) + \exp(.1(101-91.5)^2) + \exp(.1(101-91)^2)} \\ &= 0.420\,66 \end{aligned} \tag{39}$$

So $c_A = (0.123\,41)100 = \$12.341$ and $c_B = (0.420\,66)101 = \$42.487$.

[17] Note that $P(2 : y^0, \mu_{\min}, y^0, p_1^0, p_2^1, p_3^0) = P(2 : y^0, y^0, y^0, p_1^0, p_2^0, p_3^o)$

[18] Note that this second terms disapears if $p_1^1 = p_1^0$ because no one is made worse off if only the price of alternative 2 decreases.

$$\Pr(\text{in } C : 99.5) = \frac{\exp(.1(99.5-91.5)^2)}{\exp(.1(100-90)^2)+\exp(.1(99.5-91.5)^2)+\exp(.1(100-91)^2)}$$
$$= 2.3217 \times 10^{-2} \quad (40)$$

and $c_C = (2.3217 \times 10^{-2})99.5 = \$2.3101$. In total, 72.597% $(.123 + .421 + .02)$ do not switch alternatives because of the price changes.

Now consider the 27.403% that do switch. For those who switch from alternative 3 to alternative 2 (Group D),

$$c_D = -\int_{>\mu_{\min}}^{<y^0} m \frac{\partial P(3 : y^0, m, y^0, p_1^0, p_2^1, p_3^0)}{\partial m} dm$$
$$= \int_{>99.5}^{<100} m \frac{\partial P(3 : y^0, m, y^0, p_1^0, p_2^1, p_3^0)}{\partial m} dm$$
$$= -\int_{99.5}^{100} m \frac{\partial \left[\frac{\exp(.1(100-91)^2)}{\exp(.1(100-90)^2)+\exp(.1(m-91.5)^2)+\exp(.1(100-91)^2)}\right]}{\partial m} dm$$
$$= \$0.366\,5 \quad (41)$$

which is their contribution to the expected level of expenditures.

For group E, those who switch away from alternative 1, we have

$$\int_{>\mu_{\min}}^{<y^0} m \frac{\partial P(1 : y^0, m, y^0, p_1^0, p_2^1, p_3^0)}{\partial m} dm$$
$$= \int_{99.5}^{100} m \frac{\partial P(1 : 100, m, 100, 90, 91.5, 91)}{\partial m} dm$$
$$= \int_{99.5}^{100} m \frac{\partial}{\partial m} \left(\frac{\exp(.1(100-90)^2)}{\exp(.1(100-90)^2)+\exp(.1(m-91.5)^2)+\exp(.1(100-91)^2)} \right) dm$$
$$= -2.4504 \quad (42)$$

and

$$\int_{>y^0}^{<\mu_{\max}} m\frac{\partial P(1:y^0,m,m,p_1^0,p_2^1,p_3^0)}{\partial m}dm$$
$$=\int_{100}^{101} m\frac{\partial P(1:100,m,m,90,91.5,91)}{\partial m}dm$$
$$=\int_{100}^{101} m\frac{\partial}{\partial m}\left(\frac{\exp(.1(100-90)^2)}{\exp(.1(100-90)^2)+\exp(.1(m-91.5)^2)+\exp(.1(m-91)^2)}\right)dm$$
$$=-40.67 \tag{43}$$

So, $c_E = -(-2.4504 - 40.67) = \$43.12$. Therefore, $E[m] = 12.341 + 42.487 + 2.3101 + 0.3665 + 43.12 = \$100.62$ and $E[cv] = 100 - 100.62 = -\$0.62 < 0$.

6 The general theorem

The preceeding and following examples are applications of Theorem 1.

Theorem 1: Assume a change in qualities and prices. Assume the individual experiences the same ε vector both before and after the change. Then $E[m]$, our expectation of the amount of money needed to keep the individual at the original utility level after the change, is

$$E[m] = -\sum_i \int_{\underline{\mu}}^{\mu_{ii}} y\,dP_i(y) = \sum_i \left\{ \mu_{ii}P_i(\mu_{ii}) - \int_{\underline{\mu}}^{\mu_{ii}} y\,dP_i(y) \right\} \tag{44}$$

where i indexes the n alternatives and $P_i(y)$ is the choice probability

$$P_i(y) = P_i(g_1(y), ..., \nu_i^0(y^0), ..., g_J(y))$$

where

μ_{ii} is the amount of money the individual would need in the new state to achieve the original utility level if he chooses alternative i both before and after the change

the lower integral limit $\underline{\mu} = \min_k \mu_{kk}$

$\nu_j^s(\cdot)$ is the deterministic component of the conditional indirect utility function function for alternative j evaluated at $(\mathbf{X}_j^s, p_j^s)$, $s = 0, 1$. Using (2) we define $\nu_i(y) = v_i(y - p_i^s, \mathbf{X}_j^s, \mathbf{C})$.

$g_j() = \max\left[v_j^0(y^0), v_j^1(y)\right]$

$P_i(y)$ is the probability of choosing alternative i given the vector of deterministic components of utility, $(g_1(y), ..., \nu_i^0(y^0), ..., g_n(y))$.

Theorem 1 applies to all additive RUM, not just GEV models. In the GEV case, the choice probabilities are analytical and the distribution of compensating variations can be derived analytically. In this case, $E[cv]$ is a one-dimensional integral.[19] The theorem holds for any changes in prices and/or qualities, and with any indirect utility functions, not only the linear form assumed in our examples. The appendix contains a proof.

Relating Theorem 1 to the earlier examples, in the first numerical example (a price increase of \$0.50 in alternative 1)

$$
\begin{aligned}
&\sum_i \left\{ \mu_{ii} P_i(\mu_{ii}) - \int_{\underline{\mu}}^{\mu_{ii}} y \frac{\partial P_i(y)}{\partial y} dy \right\} \\
&= 100.5(P_1(100.5)) - \int_{100}^{100.5} y \frac{\partial P_1(y)}{\partial y} dy \\
+100(P_2(100)) \quad &- \int_{100}^{100} y \frac{\partial P_2(y)}{\partial y} dy \\
&+100(P_3(100)) - \int_{100}^{100} y \frac{\partial P_3(y)}{\partial y} dy \\
&= 100.5(P_1(100.5)) - \int_{100}^{100.5} y \frac{\partial P_1(y)}{\partial y} dy + 100(P_2(100)) + 100(P_3(100)) \\
&= 100.5(.42232) + 100.0(.32289) + 100(.13128) \\
&- \int_{100}^{100.5} y \frac{\partial}{\partial y} \left(\frac{\exp(.1(100-94.5)^2)}{\exp(.1(100-94.5)^2) + \exp(.1(y-95)^2) + \exp(.1(y-96)^2)} \right) dy \\
&= 87.86 - (-12.383) = \$100.24
\end{aligned}
$$

In the second numerical example (a price increase of \$1.00 in alternative 1 and a decrease of \$0.50 in alternative 2)

[19] For other densities, the choice probabilities will typically not be analytical. In these cases, $E[m]$ involves a $J-1$ dimensional integral, where J is the number of alternatives.

$$
\begin{aligned}
&\sum_i \left\{ \mu_{ii} P_i(\mu_{ii}) - \int\limits_{\underline{\mu}}^{\mu_{ii}} y \frac{\partial P_i(y)}{\partial y} dy \right\} \\
&= 101(P_1(101)) - \int\limits_{99.5}^{101} y \frac{\partial P_1(y)}{\partial y} dy \\
&\quad +99.50(P_2(99.5)) - \int\limits_{99.5}^{99.5} y \frac{\partial P_2(y)}{\partial y} dy \\
&\quad +100(P_3(100)) - \int\limits_{99.5}^{100} y \frac{\partial P_3(y)}{\partial y} dy \\
&= 101(.42066) - \int\limits_{99.5}^{101} y \frac{\partial P_1(y)}{\partial y} dy + 99.50(.023217)) - 0 + 100(.12341) - \int\limits_{99.5}^{100} y \frac{\partial P_3(y)}{\partial y} dy
\end{aligned}
$$

$$
\begin{aligned}
&= 57.138 - \int\limits_{99.5}^{100} y \frac{\partial P_1(y)}{\partial y} dy - \int\limits_{100}^{101} y \frac{\partial P_1(y)}{\partial y} dy - \int\limits_{99.5}^{100} y \frac{\partial P_3(y)}{\partial y} dy \\
&= 57.138 - (-2.4504) - (-40.67) - (-.3665) = \$100.62
\end{aligned}
$$

As is well known, there is an exact formula - the logsum formula - for the $E[cv]$ for GEV models with a constant marginal utility of money. For GEV models with no income effects, the logsum formula can be derived as a special case of the expected expenditure formula.

We have used the expected expenditure formula in a number of contexts to show that McFadden simulation approximation, with enough draws, gives the same answer.

7 An application: a nested-logit model with income effects and a quality change

This section provides a real data example; it uses the expected expenditure formula to first calculate the estimated $E[cv]$s with income effects for a simple quality change: halving the catch rate at the Penobscot river, an important salmon river in Maine. Then $E[cv]$ are then calculated for a more complicated scenario: increasing the Penoboscot catch rate by 10% while decreasing the catch rates at each of the other Maine site by 25%. This is an improvement for some anglers and a deterioration for others.

The model and data is described in Morey et al. (1993). That paper estimated a repeated nested-logit model of participation and site choice for salmon fishing in Maine and Canada by Maine residents. Income effects were included in the model: $\beta_0(y - p_j) + \beta_1(y - p_j)^{.5}$, along with catch rates and the squareroot of catch rates.[20]

7.1 *Calculating the $E[cv]$ for each angler in the sample*

There are 9 alternatives (8 fishing sites and nonparticipation). The first change being valued is a deterioration in quality (decreased catch rates), and, as noted, the model is nested logit.

Estimation of the $E[cv]$ for each salmon angler follows the same steps as in the deterioration example (Sections 3 and 4). Note that since the Maine model has 50 choice occasions, the calculated $E[cv]$ is per choice occasion, so needs to be multiplied by 50 to get it in annual terms.

Consider a decrease in the catch rate at the Penobscot River, an angler will fall into one of the three groups, A, B, and C, defined in Section 3. As in the price increase example,

$$E[cv] = y^0 - E[m] \tag{45}$$

where

$$E[m] = \Pr(\text{in } A\text{: } y^0)y^0 + \Pr(\text{in } B\text{: } \mu)\mu + c_C \tag{46}$$

$\Pr(\text{in } A\text{: } y^0)$ is the probability that an angler will not choose the Penobscot at the initial catch rates. $\Pr(\text{in } B\text{: } \mu)$ is the probability that the angler chooses the Penobscot with an expenditure level of μ and the lower catch rate, and c_C is the contribution to $E[m]$ from those whose epsilon draw causes them to choose the Penobscot before the catch decreases but not afterwards.

Note that here expenditure levels refer to income per choice occasion, so y^0 is income per choice occasion (see Morey et al. (1993) for details). μ is the level of expenditures required per choice occasion to compensate those in Group B: those who choose the Penobscot both before and after its catch decreases. μ equals y^0 plus the additional amount of expenditure required to keep utility from fishing the Penobscot equal to what it was before catch decreased.

[20] The code and data for this model, and different policy scenarios, can be found at the aforementioned web page. Ox code (Doornik 1998) code is provided. Gauss code is in the works.

7.1.1 The probability of choosing the Penobscot

Calculation of the three terms in Equation 46 requires the derivation of the probability that an angler will choose the Penobscot. Without loss of generality, denote the Penobscot alternative 1, and nonparticipation alternative 0. Sites $1-5$ are the Maine sites and $6-8$ are the Canadian sites. The probability of choosing the Penobscot (Morey 1993, equation 4 with $j=1$) is

$$P_1(V_0, V_1, V_2, .., V_8) = \frac{e^{sV_1} \left[I_p\right]^{(1/t)-1} \left(T_M\right)^{(t/s)-1}}{I} \tag{47}$$

where V_j is the deterministic part of the conditional indirect utility function for alternative j, $T_M = \sum_{j=1}^{5} e^{sV_j}$, $T_C = \sum_{j=6}^{8} e^{sV_j}$, $I_p = (T_M)^{t/s} + (T_C)^{t/s}$ and $I = e^{V_0} + [I_p]^{(1/t)}$. s and t are the standard dissimilarity (logsum) parameters for a three-level nested-logit model. If $t = s$ the model reduces to a two-level nest. The probability of not choosing the Penobscot either before or after it deteriorates is

$$\Pr(\text{in } A : y^0) = 1 - P_1(V_0^0, V_1^0, V_2^0, .., V_8^0) \tag{48}$$

where V_j^0 is V_j evaluated at the initial conditions, including y^0. And

$$\Pr(\text{in } B\text{: } \mu) = P_1(V_0^0(\mu), V_1^1(\mu), V_2^0(\mu), .., V_8^0(\mu)) \tag{49}$$

where $V_j^0(\mu)$ is V_j evaluated at its initial cost and characteristics levels but at expenditure level μ, and $V_1^1(\mu)$ is V_1 (the conditional indirect utility function for the Penobscot) evaluated at its initial cost, lower catch rate, and expenditure level μ. The first two terms in Equation 46 are therefore easily calculated.

7.1.2 Turn now to the calculation of c_C

For a deterioration to alternative 1

$$c_C = -\int_{>y^0}^{<\mu} m \frac{\partial P_1(V_0^0(m), V_1^0(y^0), V_2^0(m), .., V_8^0(m))}{\partial m} dm \tag{50}$$

This is Equation 24 rewritten in terms of the Maine model. Calculation of c_C therefore has two steps: differentiate the probability of choosing the Penobscot with respect to the level of expenditures on all alternatives except the Penobscot, and then integrate it multiplied by m over the finite interval y^0 to μ.

Differentiate $P_1(V_0, V_1, ..., V_8)$ with respect to the level of expenditures on all alternatives except alternative 1

$$\frac{\partial}{\partial m}P_1(m) \equiv \frac{\partial}{\partial m}P_i(V_0(m), V_1(y^0), V_2(m), .., V_8(m)) = \sum_{k\neq 1} \frac{\partial P_1}{\partial V_k}\frac{\partial V_k}{\partial m} \quad (51)$$

Note that total expenditures are held constant at y^0 if alternative 1 is chosen.[21]

Substitute Equation 51 into Equation 50 and integrate it to get c_C. We used the Romberg rule, but any simple numerical integration rule will work. The integration can be done in *Maple* or *Mathematica*. The integrands are very smooth, making it easy to quickly and efficiently calculate the integrals to any desired degree of accuracy.

Assuming a two-level nested-logit model, the $E[cv]$ were calculated to 4 significant digits for the 169 individuals in the sample.[22] . The mean annual $E[cv]$ for halving the Penobscot catch rate is $-\$348.72$, the median $-\$241.94$, and the range in the sample is zero to $-\$1291.91$. The representative consumer approximation, cv^r, is accurate, but we only knows this because the exact $E[cv]$ is calculated. In comparison to the deterioration, the mean $E[cv]$ for increasing the Penobscot catch rate by 10% is \$63. 40.

Consider now a more complicated scenario: increasing the Penoboscot catch rate by 10% while decreasing the catch rates at each of the other Maine site by 25%. This could happen if restoration resources are allocated more to the Penobscot, the most popular Maine Salmon river. The annual $E[cv]$s range from $-\$185.47$ to \$339. 48 with a mean of $-\$1.0881$ and a median of $-\$22.$ 377, more anglers are made worse off than are made better off by this scenario.

[21] Our approach is exact is one calculates the derivatives of the choice probabilities analytically. In practice, one can approximate them numerically. In which case, one is approximating the expected expenditure formula and the derived $E[cv]$ is no longer exact. A referee asked how this approximation would compare to cv^r as an approximation to the exact $E[cv]$. As the magnitude of the policy on welfare increases, the advantage of using the $E[cv]$ with numerical derivatives increases; it also increases as the accuracy of the numerical derivatives increases. Of course, analtyicaly derivatives are always preferred from a theoretical perspective. If analytical derivatives are used in parameter estimation, the code for their calculation is already available. Programs like *Maple* and *Mathematica* very accurately calculate complicated derivatives, often analytically.

[22] The calculation code is on the web page. This model with income effects fufills all of the regularity conditions and explains participation and site choice significantly better than the model with no income effects. We were unable to use the exact three-level model reported in Morey et al (1993), the estimated parameters in that model violate a global regularity condition.

8 Concluding remarks

In summary,

- A formula is derived to calculate the exact expected compensating variation in discrete-choice random-utility models with income effects. We refer to this formula as the *expected expenditure formula*. In any Generalized Extreme Value (GEV) model (for example, logit and nested-logit models), the formula involves solving only a finite one-dimensional integral with an analytical integrand.
- Two simple examples are presented: a price increase and a price increase combined with a price decrease, both in terms of a three-alternative logit model.
- The formula is then used to derive the $E[cv]$ for changes in catch rates from a estimated nested-logit model of salmon fishing.

References

[1] Dagsvik, J.K. and Karlstrom, A. 2004, Compensated variations and Hicksian choice probabilities in random utility models that are nonlinear in income, forthcoming in the *Review of Economic Studies*.

[2] Doornik, J.A., 1998, *Object-Oriented Matrix Programming using Ox*, London: Timberlake Consultants Ltd and Oxford.

[3] Hau, T.D.-K., 1985, A hicksian approach to cost-benefit analysis with discrete choice models, *Economica*, 52, pp. 479–490.

[4] Herriges, J.A. and Kling, C.L. 1999, Nonlinear income effects in random utility models, *The Review of Economics and Statistics*, vol. 81, pp. 62–72.

[5] Jara-Diaz, S.R. and Videla, J.I., 1989, Detection of income effect in mode choice: Theory and application, *Transportation Research*, vol. 23B, pp. 393–400.

[6] Jara-Diaz, S.R. and Videla, J.I., 1990, Welfare implications of the omission of income effect in mode choice models, *Journal of Transport Economics and Policy*, pp. 83–93.

[7] Karlstrom, A., 1998, Hicksian welfare measures in a nonlinear random utility framework, working paper presented at the 1st KFB Conference in Transportation Economics, Borlange, Sweden, June 1998, Royal Institute of Technology, Stockholm, Sweden.

[8] Lindberg, P.O., Eriksson, E.A., and Mattsson, L.-G., 1995, Invariance of achieved utility in random utility models, *Environment and Planning A*, vol. 27, pp. 121–142.

[9] McFadden, D., 1999, Computing Willingness-to-Pay in Random Utility Models, in J. Moore, R. Riezman, and J. Melvin (eds.), *Trade, Theory and Econometrics: Essays in honour of John S. Chipman*, Routledge, London.

[10] McFadden, D., 2001, Disaggregate Behavioral Travel Demand's RUM Side: A 20-Year Retroperspective, , in *Transportation Research :The Leading Edge*, Elsevier Science, pp. 17–64.

[11] Morey, E., 1999, TWO RUMS unCLOAKED: Nested-Logit Models of Site Choice and Nested-Logit Models of Participation and Site Choice, chapter 3 in Valuing Recreation and the Environment (Kling and Herriges, eds.), Edward Elgar.

[12] Morey, E., Rowe, R.D. and Watson, M. (1993), A repeated nested-logit model of Atlantic salmon fishing, *American Journal of Agricultural Economics*, 75, pp. 578–592.

[13] Oppenheim, N., 1995, The integrability problem, *Regional Science and Urban Economics*, 25, pp. 85–108.

[14] Verboven, F., 1996, The nested logit model and representative consumer theory, *Economics Letters*, 50, pp. 57–63.

[15] Viton, P., 1985, On the interpretation of income variables in discrete choice models, *Economic Letters*, 17, pp. 203–206.

Appendix

In this appendix we provide a proof of the general Theorem 1. The theorem was first proved in Karlstrom (1998). The first author would like to thank Chuang-Zuong Li and an anonymous referee for helpful suggestions and intuition. In particular, Tony E. Smith has been very helpful in correcting non-rigorous arguments.

Proof: We assume that the cumulative distribution function (CDF) of $\boldsymbol{\epsilon}$ is generated by a continuous, nonvanishing probability density f^{ϵ} with zero probability of ties,

$$\mathrm{F}^{\epsilon}(\mathbf{x}) = \int_{-\infty}^{x_1} \dots \int_{-\infty}^{x_n} f^{\epsilon}(\xi_1, ..., \xi_n) d\xi_1 ... d\xi_n$$

Also, note that given the event that an individual switches from alternative i to k, the money needed to keep utility constant is finite with probability one and bounded from below by μ_{kk} and from above by μ_{ii} [23]

[23] This follows from the assumption that the individual can attain a given utility level with finite income for any alternative. The bounds are derived in McFadden (1999).

By definition of the expected value, it suffices to show that the unconditional distribution of the expenditure to restore utility is given by $1-\sum_i P_i(y)$ indicated in the theorem, that is, one want to show that

$$Pr(m \geq y) = \sum_i P_i(y), \quad y \leq \mu_{ii} \tag{52}$$

As an intermediate result [24], first consider an individual attaining utility u^0 before the change. Denote the indirect utility function $\nu_i(y) = v(y-p_j, \mathbf{X}_j, \mathbf{C}))$. Given the underlying probability space $(\Omega, \mathcal{F}, \mathcal{P})$, for each $\omega \in \Omega$, $u_j(y,\omega) = \nu_j(y) + \epsilon_j(\omega)$ The conditional expenditure function μ_j is defined by

$$\nu_j(\mu_j(u^0,\omega)) + \epsilon_j(\omega) = u^0 \tag{53}$$

By increasing monotonicity of ν_j,

$$\mu_j(u^0,\omega) = \nu_j^{-1}(u^0 - \epsilon_j(\omega)) \tag{54}$$

Hence, for all u^0, ω, y we have

$$\mu_j(u^0,\omega) = \nu_j^{-1}(u^0 - \epsilon_j(\omega)) \geq y \Leftrightarrow \epsilon_j(\omega) \leq u^0 - \nu_j(y,\omega) \tag{55}$$

Then,

$$\begin{aligned} Pr(m \geq y \mid u^0) = Pr(\mu_1(u^0) \geq y, ..., \mu_J(u^0) \geq y \mid u^0) = \\ Pr(\epsilon_j \leq u^0 - \nu_j(y), \forall j) = \mathrm{F}^{\epsilon}(u^0 - \nu_1(y), ..., u^0 - \nu_n(y)) \end{aligned} \tag{56}$$

Without loss of generality, assume event B_i, i.e. the individual choose alternative i before the change, attaining utility $u^0 = \nu_i^0(y^0) + \epsilon_i$, where superscript denotes the state, i.e. $\nu_j^s(\cdot)$, $s = 0, 1$ are the indirect utilities evaluated at attribute levels and prices before and after the change, respectively. Remember that the probability of choosing alternative i before the change can, with an informal notation, be written [25]

$$\begin{aligned} \int \{Pr(\epsilon_1 \leq \nu_i^0(y^0)+x_i-\nu_1(y^0), \epsilon_i \in [x_i, x_i+dx_i], ..., \epsilon_J \leq \nu_i^0(y^0)+x_i-\nu_J^0(y^0))\} \\ = \int_{-\infty}^{\infty} \mathrm{F}_i^{\epsilon}(\nu_i^0(y^0) + x_i - \nu_1^0(y^0), ..., x_i, ..., \nu_i^0(y^0) + x_i - \nu_J^0(y^0))dx_i \end{aligned} \tag{57}$$

[24] This appears as Lemma 1 in Karlstrom (1998).

[25] This notation is short and useful, but informal, see Lindberg et al. (1995). More formally, when necessary, we can write

$$\int_{-\infty}^{\infty} [\int_{-\infty}^{x_i} ... \int_{\infty}^{x_i} f(x_1, ..., x_J)dx_1...dx_{i-1}dx_{i+1}...dx_J]dx_i$$

where F^ϵ_i denotes derivative with respect to the i:th argument.

Hence, since the individual choose alternative i before the change, in addition to the inequalities (55), we also have that $\epsilon_j \leq \nu^0_i(y^0) + \epsilon_i - \nu^0_j(y^0)$. Now, for $y \leq \mu_{ii}$, we have

$$Pr(m \geq y \ \cap \ B_i) =$$
$$\int \{Pr(\epsilon_1 \leq \nu^0_i(y^0)+x_i-g_1(y), ..., \epsilon_i \in [x_i, x_i+dx_i], ..., \epsilon_J \leq \nu^0_i(y^0)+x_i-g_J(y))\} =$$
$$= \int_{-\infty}^{\infty} F^\epsilon_i(\nu^0_i(y^0) + x_i - g_1(y), ..., x_i, ..., \nu^0_i(y^0) + x_i - g_J(y))dx_i \quad (58)$$

where $g_j(y) = \max\{\nu^0_j(y^0), \nu^1_j(y)\}$. Note that this is by definition the choice probability $P_i(y)$ used in the theorem.

The events B_i are mutually disjunct and exhaustive, so summing over these events results in the unconditional distribution. That is,

$$Pr(m \geq y) = \sum_i \int_{-\infty}^{\infty} F^\epsilon_i(\cdot)d\epsilon_i \equiv \sum_i P_i(y) \ , \quad y \leq \mu_{ii} \quad (59)$$

If we are interested in the mean of the expenditure needed to restore utility, we can take the expected value. Noting that the income level μ_{ii} is required for individual that continue to choose the same alternative i, we arrive at the expected expenditure formula

$$\mathbf{E}[m] = \sum_i \{\mu_{ii}P_i(\mu_{ii}) - \int_{-\infty}^{\mu_{ii}} y \, dP_i(y)\} \quad (60)$$

given our assumptions of the underlying distribution f^ϵ. The lower integral limit can be replaced by $\underline{\mu} \equiv \min_k \mu_{kk}$, by noting that $P_i(y)$ is independent of y, for all $y < \underline{\mu}$.