II. Efficiency, Equity and the Optimal Allocation

Let's begin by defining what economists mean by the optimal allocation of resources. An allocation of resources is (socially) optimal, if it satisfies the following two criteria:

i) It must be an efficient allocation. Efficiency is synonymous with Pareto optimality: a resource allocation is efficient (Pareto optimal) if it is impossible to make one member of society better off without making some other member or members worse off.1

Efficiency is necessary, but not sufficient, for social optimality. There can be an infinite number of Pareto efficient resource allocations. To distinguish among them, we use a second condition:

ii) The final distribution of goods and services must be equitable (i.e., fair). If the form of economic organization is competitive markets, this also requires the initial distribution of resources be equitable.

Equity is a normative concept, with no universal definition; what is equitable to you is equitable to you, but not necessarily equitable to me, and what is equitable to me is equitable to me, but not necessarily equitable to you. Note that an equitable distribution is not necessarily an equal distribution.

These two conditions (efficiency and equity) together define social optimality — the maximization of society's welfare.

The optimal allocation ⇒ i) and ii). But

i) alone ⇒ optimality

ii) alone ⇒ optimality

Note that society consists of its members, but who are those members? It is not up to us as economists to decide who is, and who is not, a member of society. In addition, the definition of society can vary from context to context. For example, one might define society as all sentient beings who currently live in Boulder; or

---

1Note that one cannot determine whether an allocation is efficient until one decides who is, and who is not, an member of society, and this is a normative decision, with ultimate implications for equity; that is, if one decides an individual is not a member of society, their preference don’t count.
alternatively, as current and future Citizens of the U.S. Who's in and who's out is a critical issue. Whether an allocation is either efficient or equitable will often depend on how society is defined.

Note that economists typically assume non-humans are not members of society, but this assumption is not part of neoclassical economics.

THOUGHT: Consider the distinction between a situation where all sentient beings are member of society, and a situation where only humans are member of society, but humans care about animals.

QUESTION: What does it mean to say allocation II is more efficient that allocation I? I am not sure.
Thoughts: If at II everyone is at least as well off as at I and some people are strictly better off, then allocation II is more efficient than I (allocation II is Pareto preferred to allocation I.
What if at II, relative to I, some people are better off and some are worse off.
Can these two states be ranked in terms of their degree of efficiency?
I think not.
If one has the SWF, one can also rank I and II in terms of social welfare.
If two allocations, I and II, are both efficient, then they have the same degree of efficiency—complete efficiency.

The goal of welfare economics is to seek conditions for welfare improvement. That is, we want to determine when a policy will increase social welfare, and when it will decrease social welfare.
Ideally, we want maximization of society's welfare given the state of knowledge and the endowment of scarce resources.
The optimal (or best) allocation of resources is that allocation that results in the greatest social welfare.
(Note: there may not be a unique optimum.)
Now that the optimal allocation of resources has been defined, let's add some more detail by describing a number of conditions that are **necessary and sufficient** for efficiency. Overall efficiency for the economy requires, of course, efficiency in all aspects of the economy:

i) efficiency in production

ii) efficiency in exchange

iii) efficiency in the interface between production and exchange.

We will first examine i) and ii) separately, then we will examine overall efficiency.
II. A. Efficiency in Production

Given the resource base and the state of technical knowledge, efficiency in production is achieved when it is impossible to increase the output of some good without decreasing the output of some other good.

This implies full employment of resources (if the supply of resources is fixed), and it implies that all goods and services are produced in a way that minimizes the opportunity cost of their production (i.e., that there is no waste).

Alternatively, and more generally, one might assume the supply of one or more resources (e.g. labor) is endogenous.

Let's make the definition more explicit by assuming two resources (both in fixed supply) \( \bar{L} \) and \( \bar{K} \) and a two good economy X and Y. Assume the production functions for X and Y are

\[
X = f(L_1, K_1) \quad \text{and} \quad Y = g(L_2, K_2)
\]

(Note: these two production functions restrictively assume no production externalities. How would you incorporate a production externality)

The resource constraints are

\[
\bar{L} \geq L_1 + L_2 \quad \text{and} \quad \bar{K} \geq K_1 + K_2 \quad \text{(also, neither input is a public input)}
\]
For any given level of $Y$ output, efficiency in production implies the

$$\max_{K_1, K_2, L_1, L_2} X = f(L_1, K_1)$$

subject to

$$g(L_2, K_2) \geq \bar{Y},$$

$$K_1 + K_2 \leq \bar{K},$$

$$L_1 + L_2 \leq \bar{L}$$

nonnegativity of $L_1, K_1, L_2$ and $K_2$.

Assuming an interior solution, this constrained maximization problem can be solved by forming the Lagrangian:

$$\mathcal{L} = f(L_1, K_1) + \lambda_1(L - L_1 - L_2) + \lambda_2(K - K_1 - K_2) + \lambda_3(g(L_2, K_2) - Y).$$

Interior solutions are obtained by taking the partial derivative of $\mathcal{L}$ with respect to each of the choice variables ($L_1, K_1, K_2, K_2$) and each of the Lagrangian multipliers ($\lambda_1, \lambda_2, \lambda_3$), and then setting each of these partials equal to zero:

$$\frac{\partial \mathcal{L}}{\partial L_1} = \frac{\partial f}{\partial L_1} - \lambda_1 = 0 \Rightarrow \frac{\partial f}{\partial L_1} = MP^X_L = \lambda_1$$

$$\frac{\partial \mathcal{L}}{\partial K_1} = \frac{\partial f}{\partial K_1} - \lambda_2 = 0 \Rightarrow \frac{\partial f}{\partial K_1} = MP^X_K = \lambda_2$$

$$\frac{\partial \mathcal{L}}{\partial L_2} = \lambda_3 \frac{\partial g}{\partial L_2} - \lambda_1 = 0 \Rightarrow \frac{\partial g}{\partial L_2} = MP^Y_L = \frac{\lambda_1}{\lambda_3}$$

$$\frac{\partial \mathcal{L}}{\partial K_2} = \lambda_3 \frac{\partial g}{\partial K_2} - \lambda_2 = 0 \Rightarrow \frac{\partial g}{\partial K_2} = MP^Y_K = \frac{\lambda_2}{\lambda_3}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_1} = L - L_1 - L_2 = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_2} = K - K_1 - K_2 = 0$$

2A.5 and 2A.6 say that efficiency in production requires the full employment of resources that are in fixed supply.

$^2$Note that the following conditions are not sufficient for a global max, only necessary for a global interior max if the world is differentiable.
\[
\frac{\partial g}{\partial \lambda_3} = g(L_2, K_2) - \bar{Y} = 0
\]

2A.7 just says that the specified amount of \( Y \) must be produced.

Now let's see if 2A.1-2A.4 can be rearranged into a more familiar form.

If we divide 2A.1 by 2A.2, we get

\[
\frac{\text{MP}_L^X}{\text{MP}_K^X} = \frac{\lambda_1}{\lambda_2} ;
\]

2A.8

If we divide 2A.3 by 2A.4, we get

\[
\frac{\text{MP}_L^Y}{\text{MP}_K^Y} = \frac{\lambda_1}{\lambda_2} ,
\]

2A.9

Where \( \text{MRTS}_{LK}^j \) stands for marginal rate of technical substitution between labor and capital in the production of good \( j \); that is \( \text{MRTS}_{LK}^j = -\frac{\partial K}{\partial L} \mid_{j=0} \). Noting that 2A.8 = 2A.9, one obtains the familiar condition that efficiency of production requires, absent production externalities and public inputs, \( \text{MRTS}_{LK}^X = \text{MRTS}_{LK}^Y = \frac{\lambda_1}{\lambda_2} \).

Assuming no externalities or public inputs, efficiency in production requires that the marginal rate of technical substitution between any two resources (inputs) must be equal in the production of all goods.
If $\text{MRTS}_{LK}^X \neq \text{MRTS}_{LK}^Y$, then the output of both industries could be increased if they trade inputs. This result generalizes to the case of $N$ resources and $M$ goods in which case efficiency in production requires that there is full employ of resources and that

$$\text{MRTS}_{jk}^f = \text{MRTS}_{jk}^g, \quad j, k = 1, \ldots, N$$

$$f \neq k$$

$$f, g = 1, \ldots, M$$

$$f \neq g$$

**QUESTION:** Why does efficiency require that we use all of our resources "this period"?

**REMARK:** Nothing has been said about the form of market organization — we have found the conditions for efficiency in production that apply to both a market economy and in a centrally planned economy.

The 2 resource, 2 good case can be represented graphically using an Edgeworth Box.

![Edgeworth Box Diagram](image)

The dimensions of the box are determined by the magnitudes of $\overline{L}$ and $\overline{K}$.

Every feasible allocation of $L$ and $K$ between $X$ and $Y$ is represented by a point in the box.

The contract curve is made up of all the allocations of $L$ and $K$ that are efficient. At a point off the contract curve, we can increase production of $Y$ without decreasing production of $X$ by moving toward the contract curve (e.g., point A to point B). Assuming no production externalities, at each point on the curve $\text{MRTS}_{LK}^X = \text{MRTS}_{LK}^Y$.

For this graph, notice that the maximum amount of $X$ that can be produced given $\overline{L}$, $\overline{K}$, and given $\overline{Y} = 20$ is $X = 30$; the maximum amount of $X$ that can be produced given $\overline{L}$, $\overline{K}$, and given $\overline{Y} = 40$ is $X = 10$. 
If one plots all these points, one obtains the production possibilities frontier.

Efficiency in production requires that society operate on this frontier. Pt. A is inefficient, Pt. B is impossible, and Pts. C & D are efficient. Notice that efficiency in production can be identified independent of any equity considerations. This curve would shift out if there was technical progress and/or an increase in $L$ or $K$. The slope of the production possibilities frontier is called the marginal rate of transformation in the production of $Y$ and $X$:

$$MRT_{YX} = \frac{dX}{dY}$$

The $MRT_{YX}$ expresses the opportunity cost of $Y$ in terms of units of $X$ foregone when society is on the frontier. As $Y$ increases, we see that society has to give up more and more $X$ to get another unit of $Y$. This case is known as the decreasing marginal rate of transformation.

**QUESTION:** Consider necessary and sufficient conditions for the production possibilities set to be convex.
II. B. Efficiency in Exchange

Given a fixed output of goods and services, efficiency in exchange (consumption) within a society requires that the goods and services be allocated among the society's members such that it is impossible to increase the utility of any one member without decreasing the utility of another member. Let's make efficiency in exchange more concrete by assuming a society with fixed quantities of 2 goods, $\bar{X}$ and $\bar{Y}$ (the manna from heaven model), and only 2 members, A and B. Denote the utility functions for A and B by $U^A(X^A, Y^A)$ and $U^B(X^B, Y^B)$, respectively. Note that we are restricted assuming utility is not a function of leisure. Also note that we have no externalities in consumption and assume neither good is a public good. For any given utility level for B, $\bar{U}^B$, efficiency in exchange requires that $(X^A, Y^A), (X^B, Y^B)$ solve the following problem:

\[
\begin{align*}
\text{max} & \quad U^A(X^A, Y^A) \\
\text{subject to} & \quad U^B(X^B, Y^B) \geq \bar{U}^B, \\
& \quad Y^A + Y^B \leq \bar{Y}, \\
& \quad X^A + X^B \leq \bar{X}, \\
& \quad \text{nonnegativity of } X^A, Y^A, X^B, Y^B
\end{align*}
\]

**Question:** Would the solution to this problem change if one monotonically transformed either utility function?

Assuming an interior solution, this constrained optimization problem can be solved by forming the Lagrangean:

\[ L = U^A(X^A, Y^A) + \gamma_1(\bar{X} - X^A - X^B) + \gamma_2(\bar{Y} - Y^A - Y^B) + \gamma_3(U^B(X^B, Y^B) - \bar{U}^B). \]

An interior solution satisfies the first order conditions:

\[
\begin{align*}
\frac{\partial L}{\partial X^A} &= \frac{\partial U^A}{\partial X^A} - \gamma_1 = 0 \quad \Rightarrow \quad \frac{\partial U^A}{\partial X^A} = \gamma_1 \quad 2B.1 \\
\frac{\partial L}{\partial Y^A} &= \frac{\partial U^A}{\partial Y^A} - \gamma_2 = 0 \quad \Rightarrow \quad \frac{\partial U^A}{\partial Y^A} = \gamma_2 \quad 2B.2 \\
\frac{\partial L}{\partial X^B} &= \gamma_3 \frac{\partial U^B}{\partial X^B} - \gamma_1 = 0 \quad \Rightarrow \quad \gamma_3 \frac{\partial U^B}{\partial X^B} = \gamma_1 \quad 2B.3
\end{align*}
\]
\[ \frac{\partial q}{\partial y^b} = y_3 \frac{\partial u^b}{\partial u^b} - y_2 = 0 \implies y_3 \frac{\partial u^b}{\partial u^b} = y_2 \]  

2B.4

\[ \frac{\partial q}{\partial y_1} = \bar{x} - x^a - x^b = 0 \]  

2B.5

\[ \frac{\partial q}{\partial y_2} = \bar{y} - y^a - y^b = 0 \]  

2B.6

2B.5 and 2B.6 say that efficiency in exchange requires that all goods are consumed.

\[ \frac{\partial q}{\partial y_3} = u^b(x^b, y^b) - \bar{u}^b = 0 \]  

2B.7

2B.7 just says that individual B must achieve the specified utility level \( \bar{u}^b \)

Now let's see if 2B.1-2B.4 can be rearranged to a more familiar form.

If we divide 2B.1 by 2B.2, one obtains

\[ \frac{\left( \frac{\partial u^a}{\partial x^a} \right)}{\left( \frac{\partial u^a}{\partial y^a} \right)} = \frac{y_1}{y_2} \]  

2B.8

If we divide 2B.3 by 2B.4, one obtains

\[ \frac{y_3 \left( \frac{\partial u^b}{\partial x^b} \right)}{y_3 \left( \frac{\partial u^b}{\partial y^b} \right)} = \frac{y_1}{y_2} \]  

2B.9

where \( MRS_{xy}^k = -\frac{dy^k}{dx^k} |_{u^k = 0} \) Noting that 2B.8 = 2B.9, one obtains the familiar condition that efficiency in exchange requires that (if there are no externalities in consumption, or public goods)

\[ MRS_{xy}^a = MRS_{xy}^b = \frac{\lambda_1}{\lambda_2} \]

Assuming no consumption externalities or public goods, efficiency in exchange requires that all goods are consumed and that the marginal rate of substitution between any two goods be the same for all consumers.

This result generalizes to M goods and K consumers. In this case, efficiency in exchange requires that all goods are consumed and
\[ \text{MRS}_{jk}^f = \text{MRS}_{jk}^g \quad j, k = 1, 2, \ldots M \]
\[ f, g = 1, 2, \ldots K \]

QUESTION: Why does efficiency in exchange require that there is no savings for "the future"?

REMARK: Efficiency in exchange is identified independent of any equity considerations.

REMARK: Nothing has been said about the form of economic organization. We have found the conditions for efficiency in exchange that apply to both a market economy and a centrally planned economy.

NOTE.: Don't confuse efficiency with pure competition.

QUESTION: Consider how the mathematical conditions for efficiency would change if there were externalities in consumption and/or one or both of the commodities were public goods.

The 2 consumer, 2 good case can be represented graphically using an Edgeworth Box.

The dimensions of the box are determined by the magnitudes of \( \bar{X} \) and \( \bar{Y} \).

Every feasible allocation of X and Y between the 2 consumers is represented by a point in the box.

The negative of the slopes of the indifference curves are the \( \text{MRS}_{XY} \). The contract curve is made up of all the allocations of X and Y that are efficient. At every point on this contract curve \( \text{MRS}_{XY}^A = \text{MRS}_{XY}^B \). For this graph the max amount of \( U^A \) given \( \bar{X}, \bar{Y} \), and \( \bar{U}^B = 10 \) is \( U^A = 25 \). The max amount of \( U^A \) given \( \bar{X}, \bar{Y} \), and \( \bar{U}^B = 20 \) is \( U^A = 20 \).
II. C. Overall Efficiency

Above we said that efficiency in production is necessary but not sufficient for overall efficiency and that efficiency in exchange is also necessary but not sufficient: Overall efficiency further requires efficiency in the interface between production and exchange.

To make this clearer, let's again consider a society with 2 individuals, 2 resources, and 2 goods. Overall efficiency requires that it is impossible to increase the welfare of some member of society without decreasing the welfare of some other member.

For any given utility level for consumer B, \( U^B \), overall efficiency requires that \((X^A, Y^A), (X^B, Y^B), (L_1, K_1), (L_2, K_2)\). Solve the following problem:

\[
\begin{align*}
\text{max} & \quad U^A(X^A, Y^A) \\
\text{subject to} & \quad f(L_1, K_1) \geq X, \\
& \quad g(L_2, K_2) \geq Y, \\
& \quad U^B(X^B, Y^B) \geq U^B, \\
& \quad X^A + X^B \leq X, \\
& \quad Y^A + Y^B \leq Y, \\
& \quad L_1 + L_2 \leq \bar{L}, \\
& \quad K_1 + K_2 \leq \bar{K}, \\
& \quad \text{nonnegativity}.
\end{align*}
\]

Assuming an interior solution, this constrained optimization problem can be solved by forming the Lagrangean:

\[
\mathcal{L} = U^A(X^A, Y^A) + \gamma_1(f(L_1, K_1) - X^A - X^B) + \gamma_2(g(L_2, K_2) - Y^A - Y^B) \\
+ \gamma_3(U^B(X^B, Y^B) - U^B) + \lambda_1(L_1 - L_2) + \lambda_2(\bar{K} - K_1 - K_2).
\]

If we derived all the 1st order conditions, we would obtain the following conditions for overall efficiency.

Overall efficiency, in the absence of externalities and public goods, requires that there is

a) Efficiency in production

This requires full employment of resources

\[
\begin{align*}
\bar{L} &= L_1 + L_2 \\
\bar{K} &= K_1 + K_2
\end{align*}
\]
and

$$\text{MRTS}^X_{LK} = \text{MRTS}^Y_{LK} = \frac{\lambda_1}{\lambda_2}$$

b) Efficiency in exchange

This requires that all goods are consumed

$$f(L_1, K_1) = X^A + X^B \quad g(L_2, K_2) = Y^A + Y^B$$

and

$$\text{MRS}^A_{XY} = \text{MRS}^B_{XY} = \frac{\gamma_1}{\gamma_2}$$

c) and efficiency in the interface between production and exchange

$$\text{MRS}^A_{XY} = \text{MRS}^B_{XY} = \frac{\gamma_1}{\gamma_2} = \text{MRT}^L_{XY} = \text{MRT}^K_{XY}$$

where

$$\text{MRT}^L_{XY} = \frac{\partial g}{\partial L_2} = \frac{\text{MP}^Y_L}{\text{MP}^X_L} = \text{the rate at which society substitutes X for Y as society reallocates L to X production from Y production.}$$

$$\text{MRT}^K_{XY} = \frac{\partial g}{\partial K_2} = \frac{\text{MP}^Y_K}{\text{MP}^X_K} = \text{the rate at which one can substitute X for Y as one reallocates K from X production to Y production.}$$

As noted above, overall efficiency requires that $$\text{MRT}^L_{XY} = \text{MRT}^K_{XY} = \text{MRT} = \frac{\gamma_1}{\gamma_2}$$

The marginal rate of transformation, as noted above, is the slope of the production possibilities frontier. The efficiency condition for the interface between production and exchange says overall efficiency requires that each consumer's MRS$_{XY}$ must equal MRT$_{XY}$; that is, the rate at which consumers are willing to substitute one good for another, holding utility
constant, must equal the technical rate at which manufacturing can substitute output of the one good for output of
the other.

So to review overall efficiency requires:

a) efficiency in production,
b) efficiency in exchange, and
c) \( \text{MRS}_{XY}^i = \text{MRT}_{XY}^i \), \( \forall i \)
QUESTION: Why did we spend so much time deriving the conditions for overall efficiency in the economy?

ANSWER:

1. Overall efficiency is a necessary, but not sufficient, condition for an optimal allocation of society's scarce resources;

2. We want to now see under what conditions a market system will and will not achieve overall efficiency in the allocation of its resources;

3. There are situations in which it is impossible for a market economy to achieve overall efficiency. These situations are called Market Failures. Common Property resources are one type of market failure that is common with natural and environmental resources. Market failures are rampant with natural and environmental resources.

and

4. To determine how much information one needs to know to figure out whether an allocation is efficient. (Resource constraints, state of technical knowledge, everyone's preferences.)
II. D. Equity and Social Welfare

Overall efficiency is a necessary, but not sufficient condition for the max of social welfare.\(^3\)
Optimality also requires that the allocation of resources is "equitable," "fair." across members of society.

How do we figure out which point is "most fair?" To answer this question we need a Social Welfare Function (SWF). A SWF ranks all the alternative states of society.

A state of society is defined in terms of the allocation of goods and services, \((X_A^1, X_A^2, X_B^1, X_B^2)\).

So \(SW = w(X_A^1, X_A^2, X_B^1, X_B^2)\).\(^4\)

Does the social welfare function \(SW = w(X_A^1, X_A^2, X_B^1, X_B^2)\) imply interpersonal comparisons? Yes. Are the cardinal properties of \(w(.)\) important? No.

Let \(X^k \equiv (X_{A1}^k, X_{A2}^k, X_{B1}^k, X_{B2}^k)\) denote allocation \(k\)

We usually assume that all member of society have "worth" in the following sense
\[
X^m \geq A \text{ and } B \text{ and either } X^m > A X^n \text{ or } X^m > B X^n \Rightarrow w(X^m) > w(X^n)
\]
where \(\geq\) indicates "weakly preferred" (notation not quite right). In words, social welfare is higher in allocation \(m\) than in \(n\) if some members strictly prefer \(m\) over \(n\), and no member is worse off in \(m\).

Note, that this is a restrictive assumption.

\(^3\)Note that strictly speaking, one could imagine SWF where efficiency was not a necessary condition for optimality, such SWF would, in general, not be increasing in the welfare of its members.

\(^4\)Consider, in contrast, the more restrictive social welfare function \(SW = s(U^A(X_A^1, X_A^2), U^B(X_B^1, X_B^2))\). The latter says that social welfare is a function of the utility levels of its members. Is this a restrictive assumption? Yes, is it one you are willing to make? Does this social welfare function imply that the individual's have cardinal preferences? Does this social welfare function imply interpersonal comparisons? The answer to the first question is no. Individual might only have ordinal preferences, which implies that any monotonic transformation of each individual's utility function would also represent the individual's preferences. However, \(SW = s(U^A, U^B)\) is not invariant to monotonic transformation of either or both individual's utility function. \(SW = s(U^A(X_A^1, X_A^2), U^B(X_B^1, X_B^2))\) requires that the cardinal properties of \(U^A(X_A^1, X_A^2)\) and \(U^B(X_B^1, X_B^2)\) are preserved, where the cardinal properties of \(U^A(X_A^1, X_A^2)\) and \(U^B(X_B^1, X_B^2)\) reflect part of society's interpersonal weighting of individuals A and B. Therefore, the answer to the second question is yes if \(SW = s(U^A, U^B)\); to rank states of society, interpersonal comparisons must be made.

Notes on Welfare Economics - II. Efficiency, Equity and the Optimal Allocation Edward Morey - September 9, 2002
This is just one of many properties one might assume social welfare functions should possess.\footnote{Again, remember that individuals are deemed to not be members of society have no worth.}

Society wants to utilize its resources so as to maximize social welfare.

\[
\max_{\text{wrt allocation of resources}} \quad \text{SW}
\]

(subject to)

- initial endowment of resources
- technical knowledge and
- individual preferences.
**Digression: The Arrow Impossibility Theorem**

Did Arrow win a Nobel prize by proving that a SWF cannot exist? - **NO.** Arrow's Impossibility Theorem states that no SWF exists that satisfies the following properties:

1) Individual preference must count: If the utility of one individual increases while the utility of everyone else doesn't decrease, then social welfare cannot decrease.

2) The ranking of states aren't imported. That is, the states aren't ranked by a dictator or social custom; states cannot be ranked by someone or something outside of society.

3) Society's ranking between any two states of society cannot be influenced by the addition or removal of some other state from the choice set. [Property (3) rules out social welfare functions that exhibit intensity of society's preferences over states of society.

4) The ranking must be consistent (i.e., transitive): $A \geq B \Rightarrow B \geq C \Rightarrow A \geq C$.

Arrow proved that there is no SWF that fulfills these four properties. In retrospect, this is not too surprising. For example, one might consider majority voting as a way to rank states, but majority voting will often violate condition 4.

If one views the SWF as a constitution (the rules a society uses to rank states), and if one imposes Arrow's four conditions on this constitution, the constitution will be unable to rank all possible states.

The most stringent of the four conditions is the third, which says society's preferences cannot have intensity.

If we drop condition 3), or any of the other conditions, a SWF can exist. SWF can exist, there just cannot be one that fulfills Arrow's four conditions.
In our simple world of two resources, two goods, two individuals, and no externalities or public goods, social welfare is maximized by choosing a resource allocation and consumption bundles to solve the following problem:

\[
\begin{align*}
\text{Max } SW &= w(X^1_1, X^1_2, X^1_3, X^2_1) \\
\text{s.t.} \\
\bar{L} &= L_1 + L_2 \\
\bar{K} &= K_1 + K_2 \\
X &= X^A + X^B \\
Y &= Y^A + Y^B \\
X &= f(L_1, K_1) \\
Y &= g(L_2, K_2)
\end{align*}
\]
II. E. Informational Requirements for Social Optimality

An important thing to note is the amount of information that is required to determine the optimal allocation of resources. One needs to know:

- all the production functions (that is, the complete state of technical knowledge)
- the resource constraints
- everyone's preferences

and

- the SWF.
II. F. Intertemporal Allocations: A Review of Time Preference and Technical Progress

Wilbur has some intertemporal utility function

\[ U = U(X_1, X_2) \]

where

\[ X_1 \] = amount of X Wilbur consumes in period one

\[ X_2 \] = amount of X Wilbur consumes in period two

\[ U = U(X_1, X_2) \] is perfectly general

Now add the restriction that \( U(X_1, X_2) \) is strictly quasiconcave

\[ \Rightarrow \] indifference curves look like

![Indifference Curves]

and not like

![Indifference Curves](or)

What is Wilbur’s $\text{MRS}_{x_1, x_2}$?

$$-\frac{\partial X_2}{\partial X_1} \bigg|_{\partial U = 0} ^{\text{MRS}_{x_1, x_2}} = \frac{\partial U(X_1, X_2)}{\partial X_1} \frac{\partial X_1}{\partial U(X_1, X_2)} = 1 + \delta (X_1, X_2)$$

Given the utility function, how does one determine the MRS?

If $\delta > 1$, the individual, on the margin, prefers the present to the future. If $\delta = 1$ he is indifferent, and if $\delta < 1$, the individual, on the margin, prefers the future to the present.

Note that if an individual always prefers the present to the future, the slope of his or her indifference curves are always negative and greater than one in absolute terms.
For example, $\delta(X_1^0, X_2^0)$ is Wilbur’s rate of time preference when $X_1 = X_1^0$ and $X_2 = X_2^0$,

where $1 + \delta(X_1^0, X_2^0) = MRS_{X_1, X_2}(X_1^0, X_2^0)$

\[ = - \frac{\partial X_1}{\partial X_2} |_{\mathcal{U}=0} = -MRS_{X_1, X_2} = -(1+\delta) \]

That is,

Wilbur’s person rate of discount $\delta(X_1, X_2)$ is not a constant if $U(X_1, X_2)$ is strictly quasiconcave, but as one would expect, the rate at which Wilbur will trade, on the margin, current for future consumption varies as a function of the allocation between now and next period.

Therefore

$\delta$ is a constant indep of $X_1$ and $X_2$ only if Wilbur’s indifference curves are straight lines

If Wilbur has no time preference

$MRS_{X_1, X_2} = 1 \quad (\delta = 0)$

If Wilbur’s preferences are strictly quasiconcave, Wilbur cannot have a constant rate of time preference
Now consider tech $\Delta$ over time

tech $\Delta$ means that the state of tech knowledge has changed.

What is the best and most general way to represent this in a world of no externalities?

With different production functions for $X_1$ and $X_2$.

That is

$X_1 = f(L_1, K_1)$ and $X_2 = g(L_2, K_2)$

The rate of tech progress is then the rate at which $X_2$ can be substituted for $X_1$

\[
\frac{\partial X_2}{\partial X_1} \bigg|_{K=K, L=L} = \text{MRT}_{X_1X_2}
\]

\[
\frac{\partial g(L_2, K_2)}{\partial L_2} = \frac{\partial g(L_2, K_2)}{\partial K_2} = \frac{\partial f(L_1, K_1)}{\partial L_1} \frac{\partial f(L_1, K_1)}{\partial K_1}
\]

\[
= 1 + t(X_1, X_2)
\]

where $t$ is rate of technical progress ($t>0$ is technical progress)

In general $t$ is not a constant

\[
\text{slope} = \text{MRT}_{X_1X_2} = 1 + t(X_1, X_2)
\]
What would one have to assume for $t$ to be constant?

What if $X_1 = f(L_1, K_1)$ and $X_2 = \alpha f(L_2, K_2)$?
Derive conditions for intertemporal efficiency by an individual (utility max) assuming \( U(X_1, X_2) \) is strictly quasiconcave.

Choose

\[
L_1, K_1, L_2, K_2 \text{ to } \max U(X_1, X_2)
\]

\[
st. \quad X_1 = f(L_1, K_1)
\]

\[
X_2 = g(L_2, K_2)
\]

\[
\overline{L} = L_1 + L_2
\]

\[
\overline{K} = K_1 + K_2
\]

If one assumes an interior solution, efficiency will require

\[
\text{MRS}_{X_1X_2} = \text{MRT}_{X_1X_2} \iff \delta(X_1, X_2) = t(X_1, X_2)
\]

\[
\frac{\partial X_2}{\partial X_1} \bigg|_{\partial U = 0} = -\frac{\partial X_2}{\partial X_1} \bigg|_{\overline{K} = \overline{K}, \overline{L} = \overline{L}}
\]

The ability to reach such an interior solution is enhanced because \( \delta \) and \( t \) change as resources are reallocated between \( X_1 \) and \( X_2 \).
If one restrictively assumes

\[ \text{MRS}_{X_1, X_2} = 1 + \delta \]

It will be more difficult to achieve an interior solution

For example, if \( \delta > t \forall \text{ feasible } X_1 \text{ and } X_2 \)

The individual will only consume in period 1 \((X_2 = 0)\)

Alternatively if \( \delta < t \forall \text{ feasible } X_1 \text{ and } X_i \)

\((X_i = 0)\) and individual only consumes if period 2

If one adds the additional restriction that \( t \) is a constant,

Then consumption in only one period is, ceteris paribus, more likely.
In this problem

Efficiency in the allocation is any allocation of $\bar{K}$ and $\bar{L}$ where it is impossible to make Wilbur better off.
II. G. Intertemporal Efficiency

I have defined optimality for a one period static world. Since a major problem addressing natural resource economics is how our scarce natural resources should be utilized over time, we should be concerned with intertemporal (dynamic) efficiency and optimality.

Let's therefore define efficiency and optimality in a multiple period world with more than one individual. This is easily done if we continue to maintain our implicit assumption of certainty. For simplicity, assume a two period world with one good and two individuals. At the beginning of period 1, the world is endowed with a fixed amount of L and K.

Let

\[ X_1 \] amount of X produced in period 1.
\[ X^B_1 \] amount of X consumed by individual B in period 1.
\[ L_1 \] amount of labor used to produce X in period 1.
\[ K_2 \] amount of capital used to produce X in period 2.

etc...

Intertemporal optimality is achieved by maximizing

\[
\text{SWF}(U^A, U^B) \quad \text{Note I have defined social welfare in terms of utility levels}
\]

s.t.

\[
U^A = U^A(X^A_1, X^A_2) \quad \text{Each of these utility functions reflects the respective individual's personal rate of time preference.}
\]
\[
U^B = U^B(X^B_1, Y^B_2) \quad \text{There is technical progress from period 1 to period 2, indicated by the production function in period 2 being different from the production function in period 1}
\]
\[
X_1 = f(L_1, K_1)
\]
\[
X_2 = g(L_2, K_2)
\]
\[
X_1 = X^A_1 + X^B_1 \quad X_2 = X^A_2 + X^B_2
\]
\[
L = L_1 + L_2 \quad K = K_1 + K_2
\]
If we solved this problem, we would determine that intertemporal optimality requires intertemporal efficiency.

The Lagrangean is

\[
\mathcal{L} = \text{SWF} \left[ U^A(X^A, Y^A), U^B(X^B, Y^B) \right] + \gamma_1 [f(L_1, K_1) - X^A - X^B] \\
+ \gamma_2 [g(L_2, K_2) - Y^A - Y^B] \\
+ \lambda_1 (\bar{L} - L_1 - L_2) \\
+ \lambda_2 (\bar{K} - K_1 - K_2).
\]

The conditions for intertemporal efficiency are

\[
(1) \quad \text{MRS}^A_{X_1X_2} = 1 + \text{PRD}^A = 1 + \text{PRD}^B = \text{MRS}^B_{X_1X_2}
\]

The personal rate of discount (PRD) depends on the values of \(X_1\) and \(X_2\) in the bundle. Dynamic efficiency requires that the personal rate of time preference is the same for all individuals.

The personal rate of discount (PRD) depends on the values of \(X_1\) and \(X_2\) in the bundle. Dynamic efficiency requires that the personal rate of time preference is the same for all individuals.

\[
\text{MRS}^A_{X_1X_2} = \text{MRS}^B_{X_1X_2}
\]

(2)

(3) \quad \text{MRT}^A_{X_1X_2} = 1 + \text{tech} = \text{MRS}^B_{X_1X_2} = 1 + \delta

rate at which tech = the rate at which society wants to substitute
X_1 and X_2 can be substituted for one another
tech = the rate of tech- nical progress current for future consumption
Where $\delta$ is the social rate of discount.

(3) says $\text{tech} = \delta = \text{PDR}^i \ \forall \ i$

This third efficiency condition requires that the rate people want to substitute current for future consumption equals the rate technical progress allows them to substitute current for future consumption.

So, as long as we maintain our certainty assumption, our static welfare results generalize to a multiple period world. Things are not so simple when we introduce uncertainty.
QUESTION: Consider how one might define intertemporal efficiency in a world of uncertainty.

Now we know that it takes an awful lot of information to figure out what is an optimal allocation of resources (both in a static or dynamic world). It requires more information than we usually have.