Final Exam – Part I: Closed Book (25%)

Mathematical Methods in Chemical Engineering
CHEN 5740 – Spring 2001

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1. *(3 pts.)* For each of the following analytical methods, state whether the independent or dependent variable(s) are transformed into a new variable(s):

   a.) separation of variables

   b.) Laplace transform

   c.) similarity solution

2. *(5 pts.)* True/False

   _____ Laplace transform methods are restricted to differential equations with constant coefficients.
   _____ Laplace transform methods are restricted to governing equations in which at least one of the independent variables is in the time domain.
   _____ Separation of variables cannot be applied to a PDE with three independent variables.
   _____ The PDE toolbox cannot be applied to a PDE with three independent variables.
   _____ Suppose a PDE can be solved using a similarity solution. The original dependent variable can then be expressed as a function of a new, combined independent variable only (i.e., it is no longer a function of the original independent variables).

3. *(2 pts.)* Suppose a 2^{nd} ODE has a solution of the form $y = A\sin(\lambda x) + B\cos(\lambda x)$, where A and B are integration constants that can be evaluated using the boundary conditions $y(x=0)=0$ and $y(x=1)=1$. Does this represent a characteristic value problem? Explain.

4. *(5 pts.)* An unnamed faculty member just purchased a home with a built-in hot tub in the backyard. Having never owned a hot tub before, the faculty member does not have experience with the time constant associated with heating the water to the desired temperature. Being more of a theoretician than an experimentalist, she decides to formulate a mathematical model of the system using the following assumptions:

   - the hot tub is approximately rectangular (length = l, width = w, depth = d), with the top surface exposed to the ambient air, and the remaining surfaces in the ground;
   - the ambient temperature varies approximately in a sinusoidal fashion with the nighttime low of T_{low} (2AM) and the daytime high of T_{high} (2PM);
   - the ground temperature remains approximately constant;
   - the heat transfer between a given surface of the hot tub and its surrounds can be characterized by a constant heat transfer coefficient;
   - the hot tub is well-stirred, and thus exhibits a relatively constant temperature throughout; and
   - the heating element provides a constant heat rate to the hot tub.

   Based on these assumptions, what is the equation governs the transient nature of the hot-tub temperature? (*Note*: Be sure to indicate the meaning and dimensions of any nomenclature that you introduce.)

5. *(4 pts.)* Consider transient heat transfer in a very long, solid cylinder (i.e., end effects are negligible). Initially, the entire cylinder is at temperature T_o. At time equal zero, the pipe is immersed into a well-stirred bath. While in this bath, the temperature of the surface remains approximately constant at the bath temperature, T_{bath}. The energy balance for this system takes the form:

$$\frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)$$

a.) Comment on whether each of the following boundary conditions are legitimate boundary conditions (i.e., label each as 'legitimate' or 'not legitimate').

(i) $T(t=0) = T_o$

(ii) $T(t=\infty) = T_{bath}$

(iii) $\left.\frac{dT}{dr}\right|_{r=0} = 0$

(iv) $T(r=R) = T_{bath}$

b.) Can the governing equations be solved using boundary conditions (i), (ii), (iii), and (iv)? Explain.

c.) Can the governing equations be solved using boundary conditions (i), (ii), and (iv)? Explain.

d.) Can the governing equations be solved using boundary conditions (i), (iii), and (iv)? Explain.

6. *(2 pts.)* Can separation of variables be used to directly solve a non-homogeneous PDE? Explain.

7. *(4 pts.)* Consider heat transfer occurring in turbulent tube flow. For the case of negligible axial conduction, the dimensionless form of the energy balance takes the form:

$$y^{1/6}\frac{\partial \theta}{\partial z} = \alpha \frac{\partial^2 \theta}{\partial y^2}$$

An examination of the known boundary conditions indicates that this PDE can be solved using a similarity solution method if the following transformations are implemented:

$$\theta = A\,f(\eta) \qquad \text{and} \qquad \eta = \frac{By}{z^m}$$

What value of m should be used?

Final Exam – Part II: Open Crib Sheet (75%)

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Potentially useful properties:

$$\int_0^1 \left(x^2-1\right) x\, J_0(\alpha x)\, dx = -\frac{4J_1\alpha}{\alpha^3} \qquad J_{-k}(\alpha x) = (-1)^k J_k(\alpha x)$$

$$I_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\ \sinh(x) \qquad I_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\ \cosh(x)$$

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\ \sin(x) \qquad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\ \cos(x)$$

$$J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k+p}}{k!(k+p)!} \qquad J_{-p}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k-p}}{k!(k-p)!}$$

Problem 1 (15 points)
Two solutions to the equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$$

are $y = x$ and $y = 1/x$. What is the most complete form of the solution to the equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x$$

for $x \neq 0$?

Problem 2 (20 points)
A tubular heat exchanger operates with turbulent flow a constant wall temperature T_w. Initially, the heat exchanger is at steady-state with temperature profile $T_o(z)$, where z is the axial length along the tube. Suddenly, the inlet temperature increases by the amount α.. The governing energy balance for this system takes the form

$$\rho c_p \frac{\partial T}{\partial t} + V_o \rho c_p \frac{\partial T}{\partial z} + \left(\frac{2h}{R}\right)(T - T_w) = 0$$

where it has been assumed that all fluid properties are constant.

a.) List each of the methods which were covered in class that can be used to solve PDEs, and indicate whether or not they can be applied to solve the above PDE. If not applicable, explain why they cannot be used.

b.) For the design of control systems, it is of more interest to determine the behavior of the *deviation of temperature* from steady-state, rather than the steady-state value itself. What is the differential equation and boundary conditions that govern the temperature deviation?

c.) Solve the system determined in part b using the Laplace transform method.

Problem 3 (40 points)
A very thin thermistor probe is placed into the center of a spherically shaped, unknown metal for the purposes of deducing thermal diffusivity (α). The ball is held in an oven overnight and reaches an initially uniform temperature T_s. It is then placed in the middle of a stream of very fast flowing, cool liquid of temperature T_f. The ball begins to cool and the thermistor records the centerline temperature. An energy balance on the sphere yields

$$\frac{\partial T}{\partial \tau} = \frac{1}{\xi^2} \frac{\partial}{\partial \xi}\left(\xi^2 \frac{\partial T}{\partial \xi}\right)$$

where T is temperature
ξ is dimensionless radius (r / R)
τ is dimensionless time ($\alpha t / R^2$)

Solve for $T(\xi, \tau)$ using separation of variables.