APPM 2360 Lab #1: Mortgage 101

Due: October 5, 2015

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1 Introduction

A pair of close friends are currently on the market to buy a house in Boulder. Both have obtained engineering degrees from CU and have an understanding of differential equations but lack the skills necessary to decide on what type of mortgage to take out on their new house. In the following sections, you'll find some research they have done on commonly used mortgage structures. However, they have become busy and cannot continue their analysis. Since you are in Differential Equations this semester, they have decided to enlist your help in writing a report to understand their options.
Your task is to read the following sections and complete the tasks found in Section 3. Your friends were nice enough to list various problems they would like solved. However, you should be careful to write your responses in a cohesive report.

2 Background Information

In this section you will find information on some of the most prevalent mortgage structures.

2.1 Interest Compounding

Often, the interest of a loan is expressed as the annual rate, that is, the percent of the outstanding balance that is charged as interest over a year. However, the frequency with which the rate is applied to the current balance may vary. This frequency is how often the loan *compounds*. If the interest is compounded annually, the formula to calculate the amount of money owed after the first year is

$$y(1) = (1 + r)y(0),$$

where $y(t)$ is the outstanding balance after t years, and r is the annual interest rate.
How would this change if, instead, the loan compounded semiannually? Then, half the interest rate would be applied to the loan value every 6 months.

$$y(.5) = \left(1 + \frac{r}{2}\right) y(0)$$

$$y(1) = \left(1 + \frac{r}{2}\right) y(.5) = \left(1 + \frac{r}{2}\right)\left(1 + \frac{r}{2}\right) y(0) = \left(1 + \frac{r}{2}\right)^2 y(0)$$

This pattern continues for any frequency of compounding. That is, if a loan is compounded n times per year, the value of the loan after 1 year is

$$y(1) = \left(1 + \frac{r}{n}\right)^n y(0)$$

The more frequently that a loan compounds, the higher the value at the end of the year. However, there is a limit as n goes to infinity. The limit, which models a *continuously compounding* loan, is:

$$y(t) = y(0)e^{rt}$$

This model can also be expressed as a differential equation:

$$y'(t) = ry(t), y(0) = y_0$$

If the borrower makes a monthly payment of p dollars, the model becomes:

$$y'(t, y) = ry(t) - 12p, y(0) = y_0 \tag{1}$$

2.2 Adjustable vs. fixed rate mortgages

Some mortgages use a fixed annual rate, the r in the model. The interest rate which a borrower receives depends on his/her credit score, the value of the house, the borrower's income, the duration of the loan, and several other factors. Fixed rate loans are sold with a minimum monthly payment, which ensures that the loan will be paid off within the duration. The bank will get all of the money back sooner with a shorter duration, so the rate is lower for this type of loan. Adjustable rate mortgages start at a lower rate than a fixed rate loan, but after a certain period of time, the interest rate increases and becomes tied to one of several public indexes. This results in the interest rate increasing above the rate that a fixed-rate mortgage would have for the second portion of the loan. Common fixed rate periods are 3,5,7, or 10 years. Mathematically, this results in r being a function of time $r(t)$ instead of a constant.

3 Questions

Your friends recently found a house on the market and needed to borrow **$500,000** from the bank in order to purchase. Your friends are very careful when reading mathematics so in your analysis you should include as much relevant work as necessary so your friends can follow. They provided you a list of possible questions that they need answered.

3.1 Analysis of Fixed Rate Mortgages

Your friends have various options when choosing a mortgage structure. The following points should help you know what to include in your analysis.

1. First, examine the effect of continuous compounding on the value of a loan. Assuming that $r = .05$ and the original balance is $\$500,000$, compute the total cost of the loan after 5 years for loans compounded $1, 2, 4, 12$ times per year, without any payments. Compare these values to the cost of a loan compounded continuously.

2. Next, gain a broad understanding of the behavior of the loan value by determining whether there any equilibrium solutions to (1). If so, what are they, and what is their stability? What do these equilibria represent in real-word terms?

3. Determine the exact behavior of the loan in your friends' situation by solving (1) using separation of variables, with $y(0) = 500,000$, $r = .05$.

4. The size of the monthly payment p that your friends are willing to make plays a large role in deciding the type of loan they should choose. Use the solution to (1) to find the correct p to pay off a 10 year fixed rate mortgage with rate 3% and initial debt of $500,000$. Do the same for a 30 year fixed-rate mortgage with rate 5%.

5. While having a low monthly payment is nice, you should warn your friends that there is quite literally a price to pay for this convenience. We can determine the total paid by summing each monthly payment over the duration of the loan. How much interest is paid in the 30 year fixed rate mortgage? The 10 year?

6. Buyers often choose to pay as much of the cost as they can up front so that they don't have to borrow quite so much. Might this option be worth it for your friends? How much money would the borrower save in each case if he/she paid $50,000 down on the house? (That is, the mortgage began at $450,000.)

7. What are the advantages and disadvantages of taking out a 30 year fixed rate mortgage as opposed to a 10 year mortgage?

3.2 Programming Euler's method

Often times, the differential equations we wish to solve will be difficult to solve by hand so we enlist the help of a numerical scheme. Here we will utilize MATLAB to perform Euler's method.

1. First we want to create each piece we need to calculate $y'(t)$. Create functions $r(t)$ and $p(t)$ which return a constant interest rate of 5%, and monthly payment $8,000$.

2. Now, we put the pieces together. Create a function $y'(t, y)$, which takes in t and y as input arguments, and outputs the value of y' at that point. Note that we can put the functions r and p inside the function y'. This is known as a *subfunction*.

3. Finally, we must program Euler's method. Use a step size of .01, and use a *while* loop to run the method until the mortgage is paid off ($y = 0$). When will a mortgage initial for $500,000 be paid off, with a constant interest rate of 5% and a monthly payment $8,000? Include a plot of the solution $y(t)$.

3.3 Analysis of Adjustable Rate Mortgages

Now we turn to the adjustable rate mortgages. Use the Euler's method code for these problems. You may also find the *find* command helpful. Type 'help find' into the command window to learn more. Suppose that a bank offers an adjustable rate mortgage, with the rate fixed at 3% for the first 3 years, and then the rate follows an index which behaves as $r(t) = .05 + .002t + .003\cos(\frac{\pi}{5}t)$.

1. Suppose the borrower pays $3000 per month. How long will it take him/her to pay off the mortgage?

2. What about if he/she pays $4500 per month?

3. How much interest is paid in each case?

4. Plot both scenarios on the same graph. Explain what is going on in it. How does the interest rate affect the graph?

3.4 Conclusion

1. What do you think accounts for the rise in popularity of adjustable rate mortgages over fixed rate mortgages? Which type of mortgage would you recommend to your friends who are buying a house, and why?

4 Items to Remember

- Your friends want a complete report so you should write full sentences explaining the questions posed and your responses. Don't simply number your responses to individual questions.
- All reports must be submitted to D2L by the due date in a .pdf format. Failure to do so will result in a penalty. Any MATLAB code must also be submitted to D2L.
- MATLAB code is bulky and hard to read. If you would like to include it in your report, do so in an appendix.
- Your friends are sticklers for the rules, so they really want you to follow all of the project guidelines on the APPM 2360 webpage. Be sure to read these carefully before starting your project!