APPM 5440: Applied Analysis I

Solutions to Problem Set One

1. (i) d_1 a metric $\Rightarrow d_1(x_1, y_1) \geq 0$ and $d_1(x_2, y_2) \geq 0$ which implies that

$$d(x,y) = \underbrace{d_1(x_1,y_1)}_{\geq 0} + \underbrace{d_1(x_2,y_2)}_{\geq 0} \geq 0 \quad \surd$$

Also, d_1 a metric $\Rightarrow d1 \geq 0 \Rightarrow d(x,y) = 0$ if and only if both $d_1(x_1, y_1) = d_1(x_2, y_2) = 0$ and d_1 a metric $\Rightarrow d_1(x_1, y_1) = 0$ and $d_1(x_2, y_2) = 0$ if and only if $x_1 = y_1$ and $x_2 = y_2$ which happens if and only if $(x_1, y_1) = (x_2, y_2)$. Thus

$$d(x,y) = 0 \qquad \Leftrightarrow \qquad x = y. \quad \surd$$

(ii)

$$d(y,x) = d_1(y_1,x_1) + d_1(y_2,x_2) \overset{d_1\, metric}{=} d_1(x_1,y_1) + d_1(x_2,y_2) = d(x,y) \quad \surd$$

(iii) For $z = (z_1, z_2)$,

$$\begin{aligned} d(x,y) &= \underbrace{d_1(x_1,y_1)}_{\leq d_1(x_1,z_1)+d_1(z_1,y_1)} + \underbrace{d_2(x_2,y_2)}_{\leq d_1(x_2,z_2)+d_1(z_2,y_2)} \\ &\leq [d_1(x_1,z_1) + d_1(x_2,z_2)] + [d_1(z_1,y_1) + d_1(z_2,y_2)] \\ &= d(x,z) + d(z,y) \quad \surd \end{aligned}$$

---

2. (i)

$$||x||_{max} = \max\{|x_1|, |x_2|, \ldots, |x_n|\} \geq 0$$

since $|x_i| \geq 0$ for all $i = 1, 2, \ldots$.
Furthermore, the only way for $\max\{|x_1|, |x_2|, \ldots, |x_n|\} = 0$ is to have $x_1 = x_2 = \cdots, x_n = 0$, or, equivalently, $x = (0, 0, \ldots, 0)$. $\surd$

(ii) For $\lambda \in \mathbb{R}$, $x = (x_1, x_2, \ldots, x_n) \Rightarrow \lambda x = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n)$. Thus,

$$\begin{aligned} ||\lambda x||_{max} &= \max\{|\lambda x_1|, |\lambda x_2|, \ldots, |\lambda x_n|\} \\ &= \max\{|\lambda| \cdot |x_1|, |\lambda| \cdot |x_2|, \ldots, \cdot|\lambda||x_n|\} \\ &= |\lambda| \max\{|x_1|, |x_2|, \ldots, |x_n|\} \\ &= |\lambda| \cdot ||x||_{max} \quad \surd \end{aligned}$$

(iii) $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ imply that

$$x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n).$$

So,

$$\begin{aligned} ||x+y||_{max} &= \max\{|x_1+y_1|, |x_2+y_2|, \ldots, |x_n+y_n|\} \\ &\leq \max\{|x_1|+|y_1|, |x_2|+|y_2|, \ldots, |x_n|+|y_n|\} \\ &= \max\{|x_1|, |x_2|, \ldots, |x_n|\} + \max\{|y_1|, |y_2|, \ldots, |y_n|\} \\ &= ||x||_{max} + ||y||_{max} \ \checkmark \end{aligned}$$

---

3. By the triangle inequality,

$$d(x,z) \leq d(x,y) + d(y,z).$$

Thus,

$$d(x,z) - d(y,z) \leq d(x,y). \tag{1}$$

On the other hand,

$$d(y,z) \leq d(y,x) + d(x,z) = d(x,y) + d(x,z)$$

implies that

$$-d(x,y) \leq d(x,z) - d(y,z) \tag{2}$$

(1) and (2) together imply that

$$|d(x,z) - d(y,z)| \leq d(x,y),$$

as desired.

---

4. First note that

- $$1 = |1| = |1 - x + x| \leq |1-x| + |x| \qquad \Rightarrow \qquad 1 - |x| \leq |1-x|.$$

  and

- $$\sum_{k=0}^{n} x^k = \frac{1 - x^{k+1}}{1 - x.}$$

So,

$$\left| \frac{1-x^{n+1}}{1-x} - \frac{1}{1-x} \right| = \left| \frac{x^{n+1}}{1-x} \right| = \frac{|x^{n+1}|}{|1-x|} = \frac{|x|^{n+1}}{|1-x|} \leq \frac{|x|^{n+1}}{1-|x|} \overset{\text{want}}{<} \varepsilon.$$

That is, we want

$$|x|^{n+1} < \varepsilon(1 - |x|)$$

for large enough n.

Since $|x| < 1$, we know that $|x|^{n+1}$ goes to 0 as $n \to \infty$. Therefore, $\exists\ N \in \mathbb{N}$ s.t.

$$|x|^{n+1} < \varepsilon(1 - |x|) \qquad \forall\ n \geq N,$$

as desired.

---

5. Since the sequence $(d(x_n, y_n))$ lives in the reals, we only need to show that it is a Cauchy sequence. Then, by completeness of $\mathbb{R}$ we are done!

   Let $\varepsilon > 0$.

   To show that $(d(x_n, y_n))$ is Cauchy sequence, we want to show that $\exists\ N \in \mathbb{N}$ s.t.

$$|d(x_n, y_n) - d(x_m, y_m)| < \varepsilon$$

   whenever $m, n \geq N$.

   Since (x_n) and (y_n) are Cauchy sequences, we know that we can get $d(x_n, x_m)$ and $d(y_n, y_m)$ as small as we want for large enough m and n.

   Now, by two applications of the traingle inequality, we get

$$\begin{aligned} d(x_n, y_n) &\leq d(x_n, x_m) + d(x_m, y_n) \\ &\leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n) \end{aligned}$$

   so

$$d(x_n, y_n) - d(x_m, y_m) \leq d(x_n, x_m) + d(y_m, y_n). \tag{3}$$

   On the other hand,

$$\begin{aligned} d(x_m, y_m) &\leq d(x_m, x_n) + d(x_n, y_m) \\ &\leq d(x_m, x_n) + d(x_n, y_n) + d(y_n, y_m) \end{aligned}$$

   so

$$d(x_m, y_m) - d(x_n, y_n) \leq d(x_m, x_n) + d(y_n, y_m)$$

   which is equivalent to

$$d(x_n, y_n) - d(x_m, y_m) \geq -[d(x_m, x_n) + d(y_n, y_m)]. \tag{4}$$

   Now (3) and (4)) imply that

$$|d(x_n, y_n) - d(x_m, y_m)| \leq d(x_n, x_m) + d(y_n, y_m)$$

   (x_n) Cauchy implies $\exists\ N_1 \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon/2$ for all $m, n \geq N_1$.

   (y_n) Cauchy implies $\exists\ N_2 \in \mathbb{N}$ such that $d(y_n, y_m) < \varepsilon/2$ for all $m, n \geq N_y$.

   Take $N = \max\{N_1, N_2\}$. Then both $d(x_n, x_m) < \varepsilon/2$ and $d(y_n, y_m) < \varepsilon/2$ will hold for all $m, n \geq N$ and thus

$$|d(x_n, y_n) - d(x_m, y_m)| \leq d(x_n, x_m) + d(y_n, y_m) < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

   for all $m, n \geq N$.

   Therefore, $(d(x_n, y_n))$ is a Cauchy sequence.

6. (Note: I'm going to change $\mathbb{R}^n$ to $\mathbb{R}^k$ so i can use the usual m's and n's for our Cauchy sequences.) As mentioned in class, you do not need to verify that $\mathbb{R}^k$ is a linear space and that the given norms are proper norms. All you need to do is verify, for each norm $||\cdot||$, that the space is complete with respect to the induced metric $d(x,y) = ||x-y||$.

   Let (x_n) be a sequence in $\mathbb{R}^k$. That is, each x_n is a vector $x_n = (x_{n1}, x_{n2}, \ldots, x_{nk})$.

   (a) Suppose that (x_n) is any Cauchy sequence with respect to the metric $d(x,y)$ induced by the Euclidean norm, $||\cdot||$.
   Then for any $\varepsilon > 0$, $\exists\ N \in \mathbb{N}$ s.t.

$$d(x_n, x_m) < \varepsilon \qquad \forall\ m, n \geq N.$$

   But,

$$\begin{aligned} d(x_n, x_m) &= ||x_n - x_m|| = \sqrt{(x_{n1}-x_{m1})^2 + (x_{n2}-x_{m2})^2 + \cdots, (x_{nk}-x_{mk})^2} \\ &\leq \sqrt{(x_{n1}-x_{m1})^2} + \sqrt{(x_{n2}-x_{m2})^2} + \cdots + \sqrt{(x_{nk}-x_{mk})^2} \qquad (\Delta\text{-ineq}) \\ &= |x_{n1}-x_{m1}| + |x_{n2}-x_{m2}| + \cdots + |x_{nk}-x_{mk}| \end{aligned}$$

   So

$$|x_{n1}-x_{m1}| + |x_{n2}-x_{m2}| + \cdots + |x_{nk}-x_{mk}| < \varepsilon \qquad \forall\ m, n \geq N.$$

   This implies that each $|x_{ni} - x_{mi}| < \varepsilon$ for all $m, n \geq N$ for $i = 1, 2, \ldots, k$.
   Thus, the k components of (x_n) form k one-dimensional Cauchy sequences in $\mathbb{R}$. By completeness of $\mathbb{R}$, each of these component sequences converge, say $\lim_{n\to\infty} x_{ni} = x_i \in \mathbb{R}$ for $i = 1, 2, \ldots, k$.
   Thus, (x_n) converges in $\mathbb{R}^k since$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} (x_{n1}, x_{n2}, \ldots, x_{nk}) = (x_1, x_2, \ldots, x_k) =: x \in \mathbb{R}.$$

   Therefore $\mathbb{R}^k$ is complete with respect to the metric induced by $||\cdot||$ and is a Banach space.

   (b) Suppose that (x_n) is any Cauchy sequence with respect to the metric $d(x,y)$ induced by the 1-norm, $||\cdot||_1$.
   Then for any $\varepsilon > 0$, $\exists\ N \in \mathbb{N}$ s.t.

$$d(x_n, x_m) < \varepsilon \qquad \forall\ m, n \geq N.$$

   But,

$$d(x_n, x_m) = ||x_n - x_m||_1 = |x_{n1}-x_{m1}| + |x_{n2}-x_{m2}| + \cdots, |x_{nk}-x_{mk}|$$

   so

$$|x_{n1}-x_{m1}| + |x_{n2}-x_{m2}| + \cdots, |x_{nk}-x_{mk}| < \varepsilon \qquad \forall\ m, n \geq N.$$

   The remainder of the problem is identical to part (a).

(c) Finally, suppose that (x_n) is any Cauchy sequence with respect to the metric $d(x, y)$ induced by the max norm, $||\cdot||_{max}$. Then for any $\varepsilon > 0$, $\exists\ N \in \mathbb{N}$ s.t.

$$d(x_n, x_m) < \varepsilon \qquad \forall\ m, n \geq N.$$

But,

$$d(x_n, x_m) \quad = \quad ||x_n - x_m||_{max} = \max\{|x_{n1} - x_{m1}|, |x_{n2} - x_{m2}|, \cdots, |x_{nk} - x_{mk}|\}$$

so

$$\max\{|x_{n1} - x_{m1}|, |x_{n2} - x_{m2}|, \cdots, |x_{nk} - x_{mk}|\} < \varepsilon \forall\ m, n \geq N$$

which implies that each of $|x_{ni} - x_{mi}| < \varepsilon\ \forall\ m, n \geq N$ for each of $i = 1, 2, \ldots, k$. Again, the remainder of the problem is identical to part (a).