There are 5 problems. Each problem is worth 25 points. You must do 4 of them. Please mark which 4 you choose – only 4 will be graded. A sheet of convenient formulae is provided.

1. Let \( u(x,t) \) solve the first-order equation for \( t > 0 \) and \( x > 0 \),

\[
\partial_t u + u \partial_x u + u^2 = 0,
\]

subject to boundary and initial conditions:

\[
\begin{align*}
i) & \quad \text{on } x = 0, \quad 0 \leq t, \quad u = 1; \\
ii) & \quad \text{on } t = 0, \quad 0 < x < 1, \quad u = 1 + x; \\
& \quad \text{on } t = 0, \quad 1 \leq x, \quad u = 2.
\end{align*}
\]

a) Write down the characteristic equations (ODEs).

b) Find \( u(x,t) \) for \( 0 \leq x, 0 \leq t \). Your answer might be implicit (with \( u(x,t) \) defined in terms of other variables), but be sure that everything is defined.

c) Do the characteristics ever cross in the region \( 0 \leq x < \infty, 0 \leq t < \infty \)? If so, find a point \( (x,t) \) where two (or more) characteristics cross. If not, evaluate \( u(x,t) \) at \( x = 1, t = 2 \).

d) Is the solution bounded in the region \( 0 \leq x < \infty, 0 \leq t < \infty \)? If not, find a point \( (x,t) \) where \( u(x,t) \) is unbounded. If so, find a lower bound for \( u(x,t) \) in the region \( 0 \leq x < \infty, 0 \leq t < \infty \).

2. Decide whether each of the following statements is always true (T), or not necessarily true (F). If the statement is always true, write T. If it is not necessarily true, write F, and give a counterexample that shows that the statement can be false.

a) Let \( f(x) \) be defined and real-valued on \(-L < x < L\). If \( f(x) \) and \( f'(x) \) are each piecewise continuous on \(-L < x < L\), and if \( f(x) \) is continuous on \(-L < x < L\), then \( f(x) \) has a Fourier series that converges absolutely and uniformly in \( x \).

b) Let \( f(x) \) be defined on \(-\pi < x < \pi\), and have a convergent Fourier series representation:

\[
f(x) = \sum_{n=-\infty}^{\infty} a_ne^{inx}.
\]

If \( f(x) \) is real-valued, then \( a_{-n} = a_n \).

c) Let \( f(x) \) be defined, real-valued and continuous on \(-\infty < x < \infty\), with \( |f(x)| \to 0 \) as \(|x| \to \infty\). Define \( \hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx} \, dx \). Then \( \hat{f}(k) \) is defined and bounded for all real \( k \), and \( \hat{f}(k) \to 0 \) as \(|k| \to \infty\).

d) Let \( f(x) \) be defined and real-valued on \(-\infty < x < \infty\), with \( \int_{-\infty}^{\infty} |f(x)| \, dx = N < \infty \).

Let \( \hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx} \, dx \). If \( f(-x) = -f(x) \), then \( \hat{f}(-k) = -\hat{f}(k) \).

e) Let \( f(x) \) be defined on \(-\infty < x < \infty\). If \( \int_{-\infty}^{\infty} |f(x)| \, dx < \infty \), then \( \int_{-\infty}^{\infty} f(x) \, dx < \infty \).
3. The electromagnetic field in a medium obeys the wave equation with a speed \( c \) that depends on the medium. Suppose the medium has a jump discontinuity at \( x = 0 \), with speed \( c_1 \) for \( x < 0 \), and with speed \( c_2 \) for \( x > 0 \):

\[
\begin{align*}
\partial_t^2 E - c_1^2 \partial_x^2 E &= 0, \quad x < 0, \ t > 0, \\
\partial_t^2 E - c_2^2 \partial_x^2 E &= 0, \quad x > 0, \ t > 0.
\end{align*}
\]

The electric field and its spatial derivative are continuous across \( x = 0 \):

\[
E(0_-,t) = E(0_+,t) \quad \text{and} \quad \partial_x E(0_-,t) = \partial_x E(0_+,t)
\]

As \( |x| \to \infty \), both \( E(x,t) \to 0 \) and \( \partial_x E(x,t) \to 0 \). At the initial time \( (t = 0) \),

\[
E(x,0) = F(x), \quad \partial_t E(x,0) = G(x),
\]

where \( F(x) \) and \( G(x) \) are twice-differentiable functions that vanish outside a finite interval. Prove that this problem has a unique solution. Your proof should identify the space of functions in which you prove uniqueness.

4. Consider the modified heat equation

\[
\begin{align*}
\partial_t w &= \partial_x^2 w - w, \quad 0 < x < 1, \ 0 < t \leq T \\
w(x,0) &= f(x), \quad 0 < x < 1 \\
w(0,t) &= 1, \quad 0 < t \leq T \\
w(1,t) &= 0, \quad 0 < t \leq T
\end{align*}
\]

for some \( T > 0 \) and \( L > 0 \).

a) Show that, under certain differentiability conditions which you should state, \( w \) cannot have a positive maximum in the region \( R = \{(x,t): 0 < x < 1, \ 0 < t \leq T\} \).

b) Find the steady state solution: \( w(x,t) = w_{ss}(x) \).

c) Use an energy argument on the function \( u(x,t) = w(x,t) - w_{ss}(x) \) to make a statement about the \( t \to \infty \) behavior of \( w \).
5. The one-dimensional wave equation

\[ \partial_t^2 u = c^2 \partial_x^2 u, \quad c^2 > 0, \]  

(A)

describes approximately the longitudinal vibrations of a stretched string. A more accurate model would take into account the fact that longitudinal vibrations change the cross-sectional area of the string, which then affects the motion. Based on this idea, Pochhammer and Chree derived the following improvement on the wave equation:

\[ \partial_t^2 u = c^2 \partial_x^2 u + \alpha \partial_t \partial_x^2 u, \]  

(B)

where \( c^2 > 0 \), and \( \alpha \) is a small positive constant.

Consider both (A) and (B) on \( 0 < x < L, t > 0 \), with

\[ u(0,t) = 0, \quad u(L,t) = 0, \]

\[ u(x,0) = U(x), \quad \partial_t u(x,0) = V(t) \quad 0 < x < L, \]

where \( U(x) \) and \( V(x) \) are both given and continuous.

a) Find formal series representations for the solutions of each of these problems. [You need not discuss the convergence of the series.]

b) Compare the frequency of vibration of the lowest (i.e., slowest and longest) mode for a string described by (B) to the corresponding frequency in (A). Does the added term in (B) cause this frequency to increase or decrease?

c) Show that for any choice of \( U(x) \) and \( V(x) \) that are smooth enough, the solution of (A) is periodic in time, and that the period of the motion is the period of the lowest mode (i.e., the one you discussed in (b)). Is this also true for (B)? Why or why not?