There are 5 problems. Each problem is worth 25 points. You must do 4 of them. Please mark which four you choose—only four problems will be graded.

1. Let $u(x, t)$ satisfy

$$
\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad 0 < x < \infty, \ t > 0
$$

$$
u(0, t) = f(t) \quad t > 0
$$

$$
u(x, 0) = 0, \ \partial_t u(x, 0) = 0 \quad 0 < x < \infty
$$

where $f(t)$ is twice continuously differentiable, and $f(0) = 0$.

a) Find $u(x, t)$. [Hint: This problem can be done in two different ways. The easiest approach is probably to use D’Alembert’s solution, to reverse the usual roles of $\{x, t\}$, and to extend: $f(t) \equiv 0, \ t < 0$.]

b) Show that if $|f(t)|$ is bounded, then so is $u(x, t)$.

2. Show that the following problem has at most one solution with continuous first partial derivatives:

$$
\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad 0 < x < \infty, \ t > 0
$$

$$
u(0, t) = f(t) \quad t > 0
$$

$$
u(x, 0) = U(x), \ \partial_t u(x, 0) = V(x) \quad 0 < x < \infty
$$

State clearly conditions that you require on $f$, $U$, and $V$ for your theorem. Note: Problem 2 can be answered independently of problem 1.

3. Let $D$ represent the interior of the three-dimensional ellipsoid

$$
\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1
$$

and $\partial D$ be its boundary. Let $u(x, y, z)$ represent a solution of the following problem, if one exists:

$$\Delta u + e^u = 0, \quad (x, y, z) \in D$$

$$u = 0, \quad (x, y, z) \in \partial D$$

Show that any solution of this problem is necessarily non-negative in $D$. [If you use the maximum principle to show this, you must prove the maximum principle.]
4. (50 points) Consider the Fourier series on \([-\pi, \pi]\) given by

\[ f(x) = \sum_{n=0}^{\infty} \frac{n}{1+n^2} \cos(nx), \quad g(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \sin(2^n x) \]

a) What is the Fourier series for \( F(x) = \int_{0}^{x} f(\xi) d\xi \)?

b) For each function \( f, g \) and \( F \) can you say if the function is
   i) square integrable?
   ii) continuous?
   Why or why not?

c) Using the facts that \( \frac{n}{1+n^2} = \frac{1}{n} - \frac{1}{(1+n^2)n} \) and \( \frac{x}{L} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin\left(\frac{n\pi x}{L}\right) \), \( 0 < x < L \) find the discontinuities in the function \( f \).

5. (50 points) Consider the equation

\[(x + y) \left( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \right) = u - 1 \]

a) Find and sketch the characteristics in the \((x,y)\) plane.

b) Solve for \( u(x,y) \) assuming that \( u(x,1) = e^{-x^2}, \infty < x < \infty \).

c) For what values of \((x,y)\) is your solution valid?

d) Find the limits \( a = \lim_{x \to -1^+} u(x,1) \) (from the right) and \( b = \lim_{y \to 1^+} u(-1,y) \) (from the top).

Does \( a = b \)? Why or why not?